Question One (12 marks) **Marks**

a) Find $\int \frac{dx}{9+x^2}$ [1]

b) Given $f(x)=\frac{1}{2}\sin^{-1}2x$:

i) State the range of $f(x)$ [1]

ii) State the domain of $f(x)$ [1]

iii) Sketch $f(x)$ [1]

c) Solve $\frac{5}{x-4}<1$ [2]

d) Evaluate $\int_0^1 x\sqrt{1-x^2}\,dx$ using the substitution $u=1-x^2$ [3]

e) i) Show that there is a solution to $x^3=x+1$ between $x=1$ and $x=2$ [1]

ii) Use one application of Newton's method and $x=1.5$ to find a further approximation correct to one decimal place. [2]

Question Two (12 marks) **Marks**

a) The point $P(2,3)$ divides the interval AB internally in the ratio $2:3$. [2]
If A has coordinates $(-1,6)$ find the coordinates of B.

b) A function is defined $f(x)=\frac{3}{x}-4$.

i) Find f^{-1} [1]

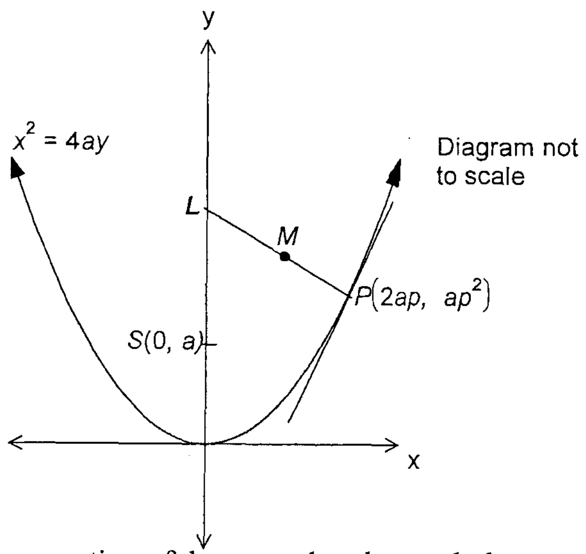
ii) Evaluate $f^{-1}(4)$ [1]

c) Given $P(x)=2x^3-17x^2+7x+8$:

i) Show that $(x-1)$ is a factor of $P(x)$ [1]

ii) Hence fully factorise $P(x)$ [2]

d) The diagram shows the parabola $x^2=4ay$. The point $P(2ap,ap^2)$ where $p\neq 0$ lies on the parabola. The normal at P cuts the y-axis at L. M is the midpoint of LP.



i) Show that the equation of the normal to the parabola at P is $x+py=ap^3+2ap$. [2]

ii) Find the coordinates of L, the point where the normal cuts the y-axis. [1]

iii) Show that SM is parallel to the tangent at P. [2]

Question Three (12 marks) **Marks**

a) Find the size of the acute angle between the lines whose equations are [3]

$x-2y-1=0$ and $x+3y+2=0$

b) A tower TO is due north of an observer at A. The angle of elevation from A to the top of the top of the tower T is 10°. From a point B due east of the tower, the angle of elevation to the top of the tower is 15°. The distance from A to B is 400m

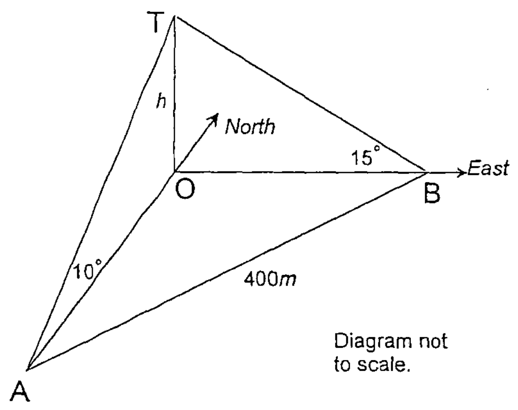


Diagram not to scale.

i) Find an expression for AO in terms of h. [1]
ii) Calculate the height h of the tower. [3]
iii) Find the bearing of A from B [2]

c) The polynomial $P(x)$ is defined as $P(x)=x^3+ax^2+2ax+b$ [3]
where a and b are constants. The zeros of $P(x)$ are $2,-3$ and γ.
Find the values a,b and γ

Question Four (12 marks) **Marks**

a) Find $\int 2\cos^2 4x\,dx$ [2]

b) The velocity of a particle is given by $\dot{x} = 2 - 3e^{-t}$ where x is the displacement in metres and t is the time in seconds. Initially the particle is at the origin.

i) Find an expression for the acceleration $\ddot{x}$ of the particle at any time t. [1]

ii) Find an expression for the displacement x of the particle at any time t. [2]

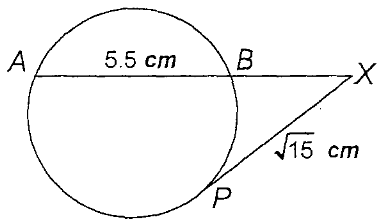
iii) Find the time when the particle is next at rest (give exact answer). [2]

iv) Explain what happens to the acceleration and hence the velocity as t becomes very large. [2]

c) Prove by mathematical induction that $2 \times 5^{n-1} + 12^n$ is divisible by 7 for all integers $n \geq 1$ [3]

Question Five (12 marks) **Marks**

a) In the diagram below the tangent at P meets AB at X.



If $AB = 5.5cm$ and $PX = \sqrt{15}cm$ find the length of BX. [2]

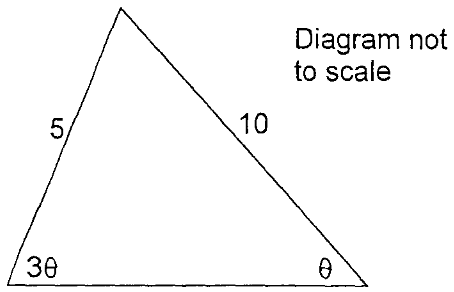
b) Evaluate $\lim_{x\to\infty} \frac{2x^2 - 3x + 1}{5x^2 + x - 4}$ [1]

c) i) Write $\sqrt{12}\sin x + 2\cos x$ in the form $r\sin(x+\alpha)$ [2]

ii) Hence or otherwise solve $\sqrt{12}\sin x + 2\cos x = -2\sqrt{2}$ for $0 \le x \le 2\pi$ [3]

d) i) Prove $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ [2]

ii)



Hence find the value of θ in the triangle [2]

Question Six (12 marks) **Marks**

a) If the displacement of a particle is given by $x = 2\sin 2t + 3\cos 2t$, show that the motion of the particle is simple harmonic. [2]

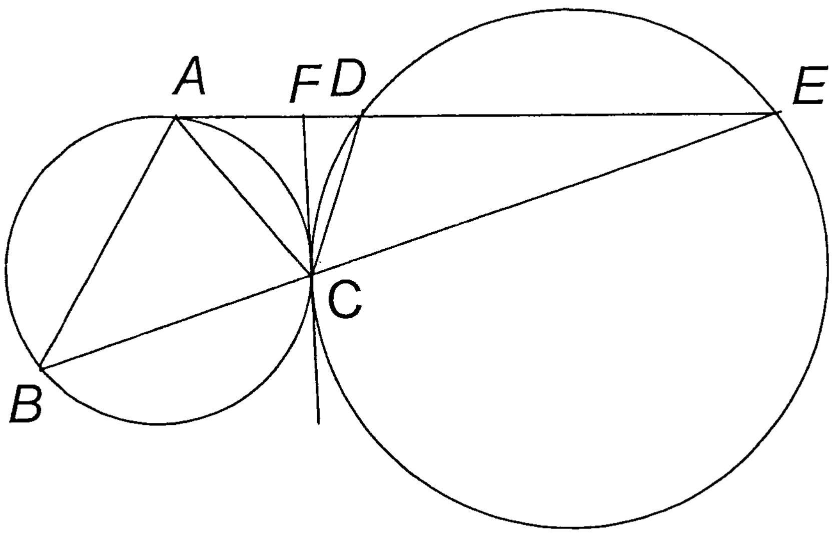
b) Jane is inflating balloons for the Year 12 Formal. Each empty balloon is being inflated so that its volume increases at the rate of $8cm^3/s$.

i) Show that the radius at any time t is $r = \sqrt[3]{\frac{6t}{\pi}}$ [2]

ii) Find the rate of increase of the surface area after 4 seconds [2]

iii) The balloon will burst when the surface area reaches $3000cm^2$. After how many seconds should Jane cease inflation? [3]

c) Two circles touch each other externally at C. The tangent to the smaller circle at A meets the larger circle at D and E. EC meets the smaller circle at B. FC is the common tangent to both circles. Copy or trace the diagram.



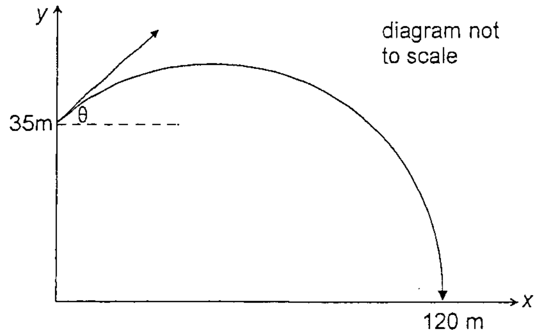
i) Prove $\angle FAC = \angle FCA$ [2]

ii) Prove $\angle ACD = \angle ACB$ [2]

Question Seven (12 Marks) **Marks**

a) Find the gradient of the tangent to $y = \sin^{-1}(\tan x)$ at $x = 0$. [2]

b)



A particle is projected from the top of a tower with a velocity of $30ms^{-1}$ to hit an object that is 120 metres away in the horizontal direction and 35 metres below in the vertical direction (as shown above). The components of its displacement after t seconds are:

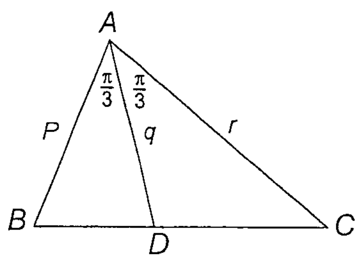
$$\left.\begin{array}{l} x = 30t\cos\theta \\ y = 30t\sin\theta - 5t^2 + 35 \end{array}\right\} \text{Do not prove these.}$$

i) If the particle hits the object prove $80\sec^2\theta - 120\tan\theta - 35 = 0$ [3]

ii) Find the angle of projection to the nearest minute [3]

iii) Find the time taken for the particle to reach the object [2]

c) In triangle ABC below $AB = p,\ AD = q,\ AC = r,\ \angle BAD = \frac{\pi}{3} = \angle DAC$ [2]



Show that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$

End of paper

Extension One Mathematics
TRIAL HSC 2011 SOLUTIONS

SGHS

Question One:

a) $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$ ✓

b) Let $y = \frac{1}{2}\sin^{-1}2x$

$\therefore 2y = \sin^{-1}2x$.

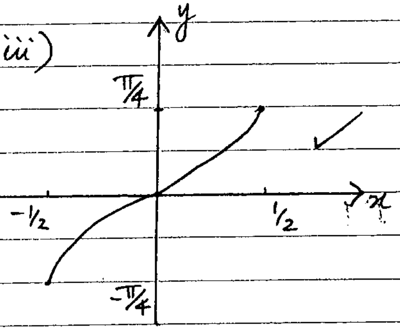
i) $-\frac{\pi}{2} \le 2y \le \frac{\pi}{2}$

$\therefore -\frac{\pi}{4} \le y \le \frac{\pi}{4}$ ✓

ii) $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$ ✓

iii)



✓

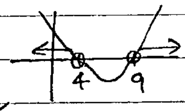
c) $\frac{5}{x-4} < 1$

$5(x-4) < (x-4)^2$

$5x - 20 < x^2 - 8x + 16$

$0 < x^2 - 13x + 36$

$0 < (x-4)(x-9)$



$x < 4$ or $x > 9$ ✓✓

d) $\int_0^1 x\sqrt{1-x^2}\,dx$

$u = 1 - x^2$

$\frac{du}{dx} = -2x$

$\frac{du}{-2x} = dx$.

when $x = 0$, $u = 1$

when $x = 1$, $u = 0$

$= \int_0^1 x.\sqrt{u}\,.\frac{du}{-2x}$ ✓

$= -\frac{1}{2}\int_1^0 u^{1/2}\,du$

$= -\frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_1^0$ ✓

$= -\frac{1}{2}\left[0 - \frac{2}{3}\right]$

$= \frac{1}{3}$ ✓

e) i) Let $y = x^3 - x - 1$

when $x = 1$, $y = -1 < 0$

when $x = 2$, $y = 5 > 0$

Since there is a change of sign ✓

then there is a solution between $x = 1$ and $x = 2$

ii) $x_1 = x - \frac{f(x)}{f'(x)}$

$= 1.5 - \frac{f(1.5)}{f'(1.5)}$ ✓

$= 1.5 - \frac{0.875}{3.5}$

$= 1.25$

$\therefore x_1 = 1.3$ (1 dec.pl.) ✓

Question Two (12 marks).

a) $P(2,3)$ $m:n$ $A(-1,6)$ $B(x_2, y_2)$

xy $2:3$ (x_1, y_1)

$x = \frac{nx_1 + mx_2}{m+n}$ $y = \frac{ny_1 + my_2}{m+n}$

$2 = \frac{3(-1) + 2(x_2)}{2+3}$ $3 = \frac{3(6) + 2(y_2)}{2+3}$

$10 = -3 + 2x_2$ $15 = 18 + 2y_2$

$13 = 2x_2$

$\therefore x_2 = 6\frac{1}{2}$ $-3 = 2y_2$

$y_2 = -\frac{3}{2}$

$\therefore B(6\frac{1}{2}, -3/2)$. (2)

b) $f(x) = \frac{3}{x} - 4$

i) let $y = \frac{3}{x} - 4$

$\therefore f^{-1}$: $x = \frac{3}{y} - 4$

$x + 4 = \frac{3}{y}$

$y = \frac{3}{x+4}$ (1)

ii) Evaluate

$f^{-1}(4) = \frac{3}{x+4}$

$= \frac{3}{4+4}$

$f^{-1}(4) = 3/8$ (1)

c) i) $P(x) = 2x^3 - 17x^2 + 7x + 8$ $(x-1)$ is a factor

$\therefore P(1) = 0$

$= 2(1)^3 - 17(1)^2 + 7(1) + 8$

$= 2 - 17 + 7 + 8$

$P(1) = 0$ $\therefore$ $(x-1)$ is a factor (1)

c) ii) $x - 1 \overline{)\, 2x^3 - 17x^2 + 7x + 8}$

$2x^3 - 2x^2$

$-15x^2 + 7x + 8$

$-15x^2 + 15x$

$-8x + 8$

$-8x + 8$

0

$\therefore P(x) = (x-1)(2x^2 - 15x - 8)$

$= (x-1)(2x+1)(x-8)$ (2)

d) i) Gradient of tangent at $P = p$ $P(2ap, ap^2)$

Gradient of normal at $P = -\frac{1}{p}$

$\therefore$ Equation of normal at P: $y - y_1 = m(x - x_1)$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$\therefore x + py = ap^3 + 2ap$. (2)

ii) Coordinates of L at $x = 0$

$x + py = ap^3 + 2ap$

$py = ap^3 + 2ap$

$py = p(ap^2 + 2a)$

$y = ap^2 + 2a$

$\therefore L(0, ap^2 + 2a)$ (1)

iii) Midpoint $M\left(\frac{0 + 2ap}{2}, \frac{ap^2 + 2a + ap^2}{2}\right)$

$L(0, ap^2 + 2a)$

$2(2ap, ap^2)$ $M\left(ap, \frac{2ap^2 + 2a}{2}\right)$

$M(ap, ap^2 + a)$ $S(0, a)$

Gradient of SM $= \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{ap^2}{ap}$ (2)

$= \frac{ap^2 + a - a}{ap - 0}$ $m = p$

$\therefore$ Gradient of SM = Gradient of tangent at $P = p$

$\therefore$ SM // to tangent at P.

Question 3

a) $x - 2y - 1 = 0 \Rightarrow y = \frac{1}{2}x - \frac{1}{2} \quad m_1 = \frac{1}{2}$ ✓

$x + 3y + 2 = 0 \Rightarrow y = -\frac{1}{3}x - \frac{2}{3} \quad m_2 = -\frac{1}{3}$

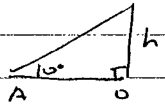
$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ ✓

$= \left|\left(\frac{1}{2} - \frac{-1}{3}\right) \div \left(1 - \frac{1}{6}\right)\right|$

$= |1|$

$\theta = 45^\circ$ or $\frac{\pi}{4}$ ✓ (3)

b) i)



$AO = \frac{h}{\tan 10^\circ}$ or $AO = h\cot 10^\circ$ ✓ (1)

ii) $OB = \frac{h}{\tan 15^\circ}$ or $OB = h\cot 15^\circ$

Now $OA^2 + OB^2 = 400^2$ ✓

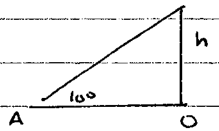
$\frac{h^2}{\tan^2 10^\circ} + \frac{h^2}{\tan^2 15^\circ} = 400^2$

$h^2\left(\frac{1}{\tan^2 10^\circ} + \frac{1}{\tan^2 15^\circ}\right) = 400^2$ ✓

$h = 400 \div \sqrt{\frac{1}{\tan^2 10^\circ} + \frac{1}{\tan^2 15^\circ}}$ (3)

$= 58.9\text{m}$ ✓

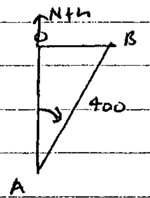
iii)



$\frac{h}{AO} = \tan 10^\circ$

$AO = \frac{h}{\tan 10^\circ}$

$= 334.03$



$\cos\angle OAB = \frac{OA}{400}$

$= \frac{334.03..}{400}$

$\angle OAB = 33^\circ$ or $\angle OOA =$ (2) ✓

$\therefore$ Bearing $= 180^\circ + 33^\circ$

$= 213^\circ$ ✓

c) $P(x) = x^3 + ax^2 + 2ax + b$

when $x = 2$, $\quad 8 + 4a + 4a + b = 0$

$8a + b = -8$ ①

when $x = -3$, $\quad -27 + 9a - 6a + b = 0$

$3a + b = 27$ ②

① − ② $\quad 5a = -35$

$a = -7$ ✓

$b = 48$ ✓ (the constant term)

Now $(x-2)(x+3)(x-\alpha) = 0$

$\therefore 6\alpha = 48$

$\alpha = 8$ ✓ (3)

2011 Ext 1 Trial – Solution to Question 4

(a)

$\cos 2x = 2\cos^2 x - 1 \quad \Rightarrow 2\cos^2 4x = \cos 8x + 1$

$$\int 2\cos^2 4x\,dx = \int (\cos 8x + 1)\,dx$$
$$= \frac{\sin 8x}{8} + x + C$$

(b)(i)

$\dot{x} = 2 - 3e^{-t} \qquad \therefore \ddot{x} = 3e^{-t}$

(b)(ii)

$\dot{x} = 2 - 3e^{-t} \qquad x = \int (2 - 3e^{-t})\,dt = 2t + 3e^{-t} + C$

$x = 0$ when $t = 0 \qquad 0 = 2(0) + 3e^{0} + C \Rightarrow C = -3$

$\therefore x = 2t + 3e^{-t} - 3$

(b)(iii)

$\dot{x} = 2 - 3e^{-t} \qquad \dot{x} = 0 \qquad \Rightarrow 0 = 2 - 3e^{-t}$

$3e^{-t} = 2 \qquad e^{-t} = \frac{2}{3} \qquad \Rightarrow -t = \ln\frac{2}{3}$

$\therefore t = -\ln\frac{2}{3} = \ln\frac{3}{2}$ s

(b)(iv)

as $t \to \infty \quad \ddot{x} \to 0^{+}$ and $\dot{x} \to 2^{-}$ **i.e. acceleration approaches 0 and velocity approaches 2 m/s**

(c)

Step 1 : Prove true for $n = 1$.

$2 \times 5^{1-1} + 12^{1} = 2 + 12 = 14 = 2 \times 7$

$\therefore$ Divisible by 7 for $n = 1$

Step 2 : Assume true for $n = k$.

$2 \times 5^{k-1} + 12^{k} = 7A$ **where A is some integer.**

Step 3 : Prove true for $n = k + 1$.

i.e. prove $2 \times 5^{k+1-1} + 12^{k+1} = 7B$

$$\begin{aligned}
LHS &= 2 \times 5^{k+1-1} + 12^{k+1} \\
&= 5^{1} \times 2 \times 5^{k-1} + 12 \times 12^{k} \\
&= 5 \times (7A - 12^{k}) + 12 \times 12^{k} \\
&= 35A - 5 \times 12^{k} + 12 \times 12^{k} \\
&= 35A + 7 \times 12^{k} \\
&= 7(5A + 12^{k}) \\
&= 7B \quad \text{where } B = 5A + 12^{k}
\end{aligned}$$

$LHS = RHS$

Step 4 : If true for $n = k$, then proven true for $n = k + 1$. Since proven true for $n = 1$, must be true for $n = 2$. Since true for $n = 2$, must be true for $n = 3$, etc. Hence, proven true by mathematical induction for all positive integers.

Extension 1 Solutions Question 5

) let $BX = x$

$x(x+5.5) = (\sqrt{15})^2$

$x^2 + 5.5x - 15 = 0$

$2x^2 + 11x - 30 = 0$

$(2x+15)(x-2) = 0$

$\therefore x = -7.5, 2$

but $x > 0$

$\therefore x = 2$.

) $\lim_{x\to\infty} \frac{2x^2 - 3x + 1}{5x^2 + x - 4} = \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{1}{x^2}}{5 + \frac{1}{x} - \frac{4}{x^2}}$

$= \frac{2}{5}$

) i) $\frac{\sqrt{12}}{r}\sin x + \frac{2}{r}\cos x = \sin x\cos\alpha + \cos x \sin\alpha$

$\cos\alpha = \frac{\sqrt{12}}{r}$ $\sin\alpha = \frac{2}{r}$

$\cos^2\alpha = \frac{12}{r^2}$ $\sin^2\alpha = \frac{4}{r^2}$

$r^2 = 16$

$r = 4$

$\sin\alpha = \frac{2}{4}$

$\alpha = \frac{\pi}{6}$

$\therefore \sqrt{12}\sin x + 2\cos x = 4\sin\left(x + \frac{\pi}{6}\right)$

ii) $4\sin(x + \frac{\pi}{6}) = -2\sqrt{2}$

$\sin\left(x + \frac{\pi}{6}\right) = -\frac{\sqrt{2}}{2}$

$x + \frac{\pi}{6} = \frac{5\pi}{4}, \frac{7\pi}{4}$

$x = \frac{13\pi}{12}, \frac{19\pi}{12}$

d) i) $LHS = \sin 3\theta$

$= \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \sin\theta\cos 2\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta(1 - 2\sin^2\theta)$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta(1 - 2\sin^2\theta)$

$= 3\sin\theta - 4\sin^3\theta$

$= RHS$

ii) $\frac{\sin 3\theta}{10} = \frac{\sin\theta}{5}$

$\sin 3\theta = 2\sin\theta$

$3\sin\theta - 4\sin^3\theta = 2\sin\theta$

$\sin\theta - 4\sin^3\theta = 0$

$\sin\theta(1 - 4\sin^2\theta) = 0$

$\sin\theta = 0$ or $\sin^2\theta = \frac{1}{4}$

not a solution to this problem

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

however only $\theta = \frac{\pi}{6}$ satisfies the problem.

6 a) $\dot{x} = 4\cos 2t - 6\sin 2t$

$\ddot{x} = -8\sin 2t - 12\cos 2t$

$= -4x$ ✓✓

b) i) $V = \frac{4\pi r^3}{3}$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$8 = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dt}{dr} = \frac{\pi r^2}{2}$

$t = \frac{\pi r^3}{6}$

$\frac{6t}{\pi} = r^3$

$r = \sqrt[3]{\frac{6t}{\pi}}$ ✓✓

ii) when $t = 4$

$r = \sqrt[3]{\frac{24}{\pi}}$

$S = 4\pi r^2$

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$= 8\pi r \times \frac{2}{\pi r^2}$

$= \frac{16}{r}$

$= \frac{16}{\sqrt[3]{\frac{24}{\pi}}}$ ✓✓

iii) $4\pi r^2 = 3000$

$r^2 = \frac{750}{\pi}$

$r = \sqrt{\frac{750}{\pi}}$

$t = \frac{\pi \times \left(\sqrt{\frac{750}{\pi}}\right)^3}{6}$ ✓✓

c) i) $F\hat{A}C = \hat{B}$ (∠s in alternate segments)

$F\hat{C}A = \hat{B}$ (" " " ") ✓✓

ii) $A\hat{C}B = F\hat{A}C + \hat{E}$ (exterior ∠ of a Δ)

$F\hat{C}D = \hat{E}$ (∠s in alternate segment)

$\therefore A\hat{C}B = F\hat{A}C + F\hat{C}D$

$= A\hat{C}D$ ✓✓

Question 7:

a) let $u = \tan x$ $\qquad$ $y = \sin^{-1} u$

$\frac{du}{dx} = \sec^2 x$ $\qquad$ $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$

$$\frac{dy}{dx} = \sec^2 x \cdot \frac{1}{\sqrt{1-u^2}}$$

$$= \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} \quad (1)$$

when $x = 0$ $\quad m_T = \frac{\sec^2 0}{\sqrt{1-0^2}} = 1$ $\quad (1)$

b) When particle hits object, $x = 120$:

$120 = 30t\cos\theta$

$t = \frac{4}{\cos\theta}$ ① $\quad (1)$

When particle hits object, $y = 0$:

$0 = 30t\sin\theta - 5t^2 + 35$ ②

Sub ① into ②:

$0 = 30 \times \frac{4}{\cos\theta}\sin\theta - 5\left(\frac{4}{\cos\theta}\right)^2 + 35$ $\quad (1)$

$= 120\tan\theta - 80\sec^2\theta + 35$

$80\sec^2\theta - 120\tan\theta - 35 = 0$ $\quad (1)$

b) $80(1+\tan^2\theta) - 120\tan\theta - 35 = 0$ $\quad (1)$

$80\tan^2\theta - 120\tan\theta + 45 = 0$

$16\tan^2\theta - 24\tan\theta + 9 = 0$

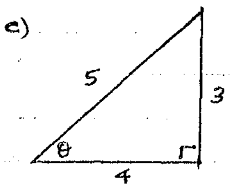
$(4\tan\theta - 3)^2 = 0$ $\quad (1)$

$4\tan\theta = 3$

$\tan\theta = \frac{3}{4}$

$\therefore \quad \theta = 36°52'$ (nearest minute) $\quad (1)$

e)



$x = 30t\cos\theta$

$120 = 30 \times t \times \frac{4}{5}$

$t = 5s.$ $\quad (2)$

c) In ΔABD:

Area $\Delta ABD = \frac{1}{2}pq\sin\frac{\pi}{3}$

In ΔADC:

Area $\Delta ADC = \frac{1}{2}qr\sin\frac{\pi}{3}$ $\quad (1)$

In ΔABC:

Area $\Delta ABC = \frac{1}{2}pr\sin\frac{2\pi}{3}$

Area ΔABC = Area ΔABD + Area ΔADC

$\frac{1}{2}pr\sin\frac{2\pi}{3} = \frac{1}{2}pq\sin\frac{\pi}{3} + \frac{1}{2}qr\sin\frac{\pi}{3}$

$pr\sin\frac{\pi}{3} = pq\sin\frac{\pi}{3} + qr\sin\frac{\pi}{3}$ $\quad (1)$

$pr = pq + qr$

$\therefore \quad \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$