Sydney
Girls
High
School

2018

TRIAL
HIGHER
SCHOOL
CERTIFICATE
EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks : 70

Section 1 – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II – 60 marks (pages **7 – 14**)

- Attempt Questions **11 – 14**
- Answer on the blank paper provided
- Begin a new page for each question
- Allow about 1 hour and 45 minutes for this section

Name: …………………………………………………………... Teacher: …………………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2018 HSC Examination Paper in this subject.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1. When the polynomial $P(x) = 3x^3 - 2x^2 + cx - 5$ is divided by $(x+1)$, the remainder is -14. What is the value of c?

   (A) -10

   (B) -8

   (C) -4

   (D) 4

2. The acute angle between the lines $y = 3x$ and $y = mx$ is equal to $\frac{\pi}{4}$.

   What is the value of m ?

   (A) -2 or $-\frac{1}{2}$

   (B) -2 or $\frac{1}{2}$

   (C) 2 or $-\frac{1}{2}$

   (D) 2 or $\frac{1}{2}$

3. The parabola is defined by the parametric equations $x = 2t$, $y = (t-2)^2$.

   Which of the following is the vertex of the parabola?

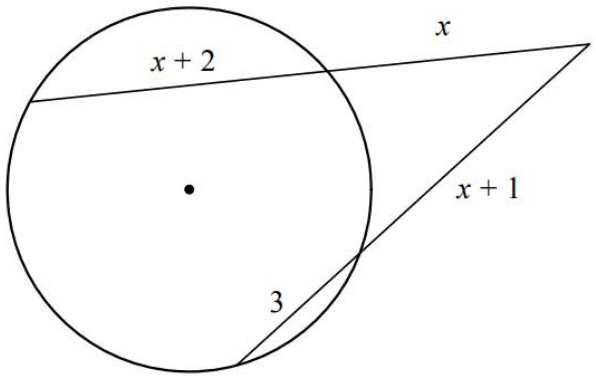
   (A) $(0,2)$

   (B) $(2,0)$

   (C) $(2,1)$

   (D) $(4,0)$

4. Two secants from an external point, cut off intervals on a circle as shown below.



What is the value of x?

(A) $\dfrac{1+\sqrt{14}}{2}$

(B) 4

(C) $\dfrac{-3+\sqrt{73}}{4}$

(D) 5

5. Let α, β and γ be the roots of the equation $x^3 + px^2 + q = 0$.

Express $\dfrac{1}{\alpha\beta}+\dfrac{1}{\beta\gamma}+\dfrac{1}{\gamma\alpha}$ in terms of p and q.

(A) pq

(B) $-pq$

(C) $-\dfrac{p}{q}$

(D) $\dfrac{p}{q}$

6. What is the exact value of the definite integral $\int_0^{\frac{\pi}{8}} \cos^2 x \, dx$?

(A) $\dfrac{\pi - 2\sqrt{2}}{8}$

(B) $\dfrac{\pi - 2\sqrt{2}}{16}$

(C) $\dfrac{\pi + 2\sqrt{2}}{8}$

(D) $\dfrac{\pi + 2\sqrt{2}}{16}$

7. A mixed male and female netball team of 7 players is to be formed from a group of 16 students. Among the 16 students are 5 females.

What is the number of possible teams containing at least 3 females?

(A) 4180

(B) 7150

(C) 8800

(D) 11 440

8. A particle is moving in Simple Harmonic Motion with its velocity given by $v^2 = 144(25 - x^2)$ where x is the displacement.

What is the amplitude A, and the period T, of the motion?

(A) $A = 12$ and $T = \frac{\pi}{6}$

(B) $A = 12$ and $T = \frac{\pi}{12}$

(C) $A = 5$ and $T = \frac{\pi}{6}$

(D) $A = 5$ and $T = \frac{\pi}{12}$

9. The area of the region defined by the set of points (x, y) satisfying simultaneously $|2x-3y| \le 12$ and $|2x+3y| \le 12$ is:

(A) 96

(B) 48

(C) 72

(D) 24

10. A particle is projected horizontally with speed V ms^{-1} from the top of a vertical tower of height h metres which stands on horizontal ground. At time t seconds its horizontal and vertical displacements from the foot O of the tower, x metres and y metres respectively, are given by $x = Vt$ and $y = h - \frac{1}{2}gt^2$ where the acceleration due to gravity is g ms^{-2}.

What is the angle of inclination to the horizontal at which the particle hits the ground?

(A) $\tan^{-1}\left(\frac{V}{\sqrt{V^2+2gh}}\right)$

(B) $\tan^{-1}\left(\frac{\sqrt{2gh}}{\sqrt{V^2+2gh}}\right)$

(C) $\tan^{-1}\left(\frac{\sqrt{2gh}}{V}\right)$

(D) $\tan^{-1}\left(\frac{V}{\sqrt{2gh}}\right)$

Section II

90 marks

Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer on the blank paper provided. Begin a new page for each question.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Solve the inequality: $\dfrac{x}{x-3} < 4$ **3**

(b) Consider the function $y = \cos^{-1}(2x) - \dfrac{\pi}{2}$.

(i) State the domain of this function. **1**

(ii) State the range of this function. **1**

(iii) Sketch the graph of this function. **1**

(c) Let $A(-3,6)$ and $B(1,10)$ be points on the number plane. **2**

Find the coordinates of the point C, which divides the interval AB externally in the ratio $5:3$.

(d) Find: $\lim\limits_{\theta \to 0} \dfrac{\theta + \sin 2\theta}{7\theta}$. **2**

(e) The equation $x^3 + 3x^2 + 2x + 1 = 0$ has roots α, β and γ. **2**
Find the value of $(\alpha - 1)(\beta - 1) + (\beta - 1)(\gamma - 1) + (\gamma - 1)(\alpha - 1)$.

(f) Use the substitution $u = 2x - 1$ to evaluate $\displaystyle\int_0^1 x(2x-1)^4\, dx$. **3**

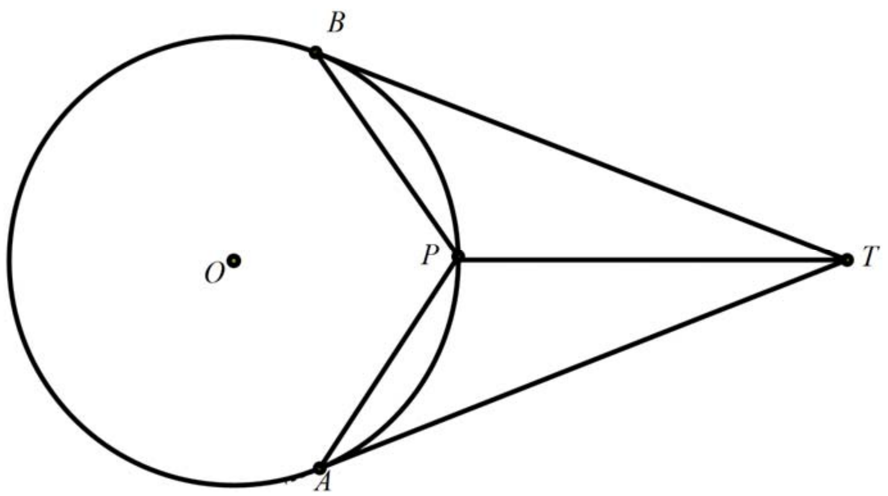
End of Question 11

Question 12 **(15 marks)** Begin a new page.

(a) Differentiate $y = \log_e\left(\sin^{-1} 2x\right)$. **2**

(b) In the diagram below, TA and TB are tangents to the circle, centre O.

P is a point on the circumference of the circle such that $\angle BTA$ is bisected by the line PT. Let $\angle BTP = x^\circ$ and $\angle PAT = y^\circ$.



(i) Prove that the chords BP and AP are equal. **2**

(ii) Hence, by considering ΔAPB, or otherwise, prove that **2**

$x + 2y = 90^\circ$.

Question 12 continues on the next page.

Question 12 (continued)

(c) A function is defined as $f(x) = 1 + e^{2x}$.

(i) Find the range of the function. **1**

(ii) Show that the inverse function can be defined as **1**

$$f^{-1}(x) = \frac{1}{2}\ln(x-1).$$

(iii) On the same set of axes, sketch the graphs of $y = f(x)$ **2**

and $y = f^{-1}(x)$ clearly showing any features.

(iv) Show that the equation of the normal to $y = f^{-1}(x)$ at the point **2**

where $f^{-1}(x) = 0$ is $2x + y - 4 = 0$.

(v) Show that the x-coordinate of the point of intersection of this **1**

normal and $y = f(x)$ is a solution to the equation $e^{2x} + 2x = 3$.

(vi) By taking $x = 0.4$ as the first approximation of the root to **2**

$e^{2x} + 2x = 3$, use one application of Newton's Method to find a

better approximation of the root, correct to 3 significant figures.

End of Question 12

Question 13 **(15 marks)** Begin a new page.

(a) Use the method of mathematical induction to show that for all positive integers n: **2**

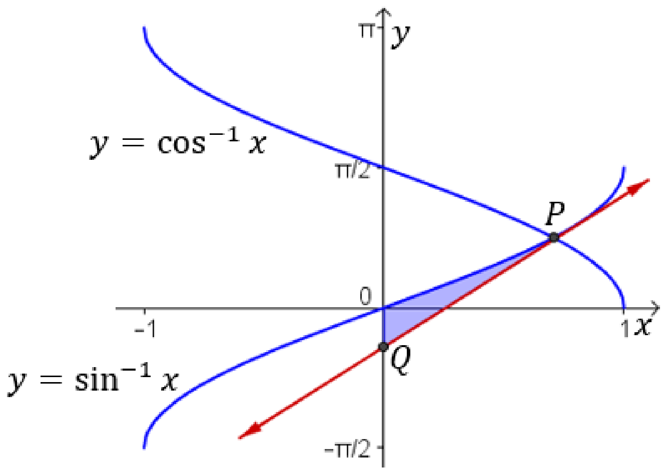
$$1\times1! + 2\times2! + 3\times3! + \ldots + n\times n! = (n+1)! - 1.$$

(b) Triangle PQR is isosceles, having $PQ = PR = 6$ cm and $\angle RPQ = \theta$, where θ is in radians.

(i) Find the area of triangle PQR in terms of θ. **1**

(ii) As θ changes, the area of triangle PQR is increasing at the rate of 3cm^2 per second. Determine the rate at which $\angle RPQ$ is changing at the instant the area of triangle PQR is 9cm^2. **2**

(c)



The diagram above shows the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ and their point of intersection P. PQ is a tangent to $y = \sin^{-1} x$ at P.

(i) Show that the coordinates of P are $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$. **1**

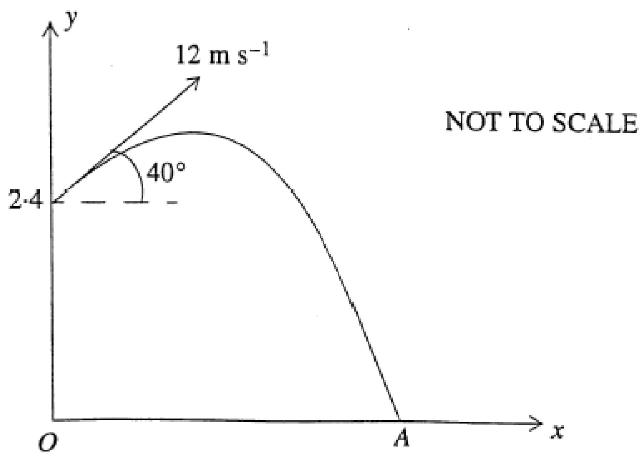
(ii) Find the equation of the tangent at P. **1**

(iii) Hence find the shaded area enclosed by this tangent, the y-axis and the curve $y = \sin^{-1} x$. **3**

Question 13 continues on the next page.

Question 13 (continued)

(d)



In an Olympic trial, a shot putter releases the shot from a height of 2.4 metres above ground level at an angle of 40° to the horizontal, and with a speed of 12 metres per second.

Take the origin O at a point on the ground directly under the point of release of the shot.

(i) Using calculus, show that the position of the shot at time t seconds is given by $x = 12\cos 40^\circ t$ and $y = 2.4 + 12\sin 40^\circ t - \frac{1}{2}gt^2$. **3**

(ii) The shot lands at a point A on the ground. Find the length of OA in metres, correct to one decimal place. (take $g = 9.8$). **2**

End of Question 13

Question 14 **(15 marks)** Begin a new page.

(a) A teacher organising an examination timetable has to schedule seven examinations. Of these examinations, one is English and two are Mathematics.

(i) If the two Mathematics examinations are not to be scheduled consecutively, in how many ways can the seven examinations be scheduled ? **1**

(ii) If the English examination is scheduled first, find the probability that one Mathematics examination will be scheduled second and the other Mathematics examination scheduled last. **1**

(b) Assuming that $\sin\left(k\pi+\phi\right)=\left(-1\right)^k\sin\phi$ is true for some positive integer k, prove that **2**

$$\sin\left(\left(k+1\right)\pi+\phi\right)=\left(-1\right)^{k+1}\sin\phi.$$

(c) A particle moving in a straight line is performing Simple Harmonic Motion about a fixed point O on the line. At time t seconds the displacement x metres of the particle from O is given by:

$$x=a\cos nt,\ \text{where } a>0 \text{ and } \pi<n<2\pi.$$

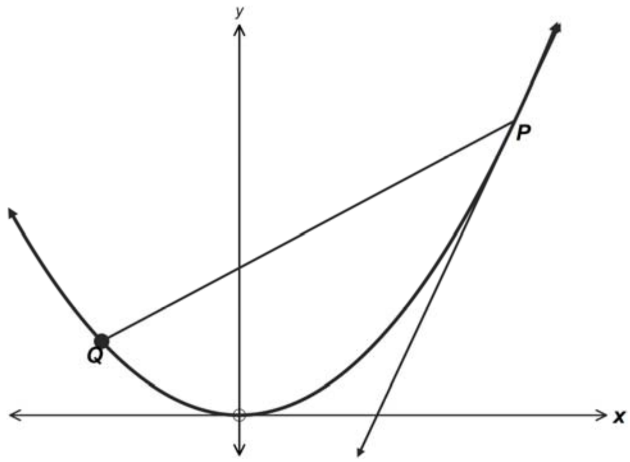
After 1 second the particle is 1 metre to the right of O, and after another 1 second the particle is 1 metre to the left of O.

Find the value of n. **3**

Question 14 continues on the next page.

Question 14 (continued)

(d)



The diagram above shows the variable points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ on the parabola $x^2 = 4ay$.

M is the mid-point of the chord PQ.

P and Q move such that the gradient of the tangent at P is four times the gradient of the chord PQ.

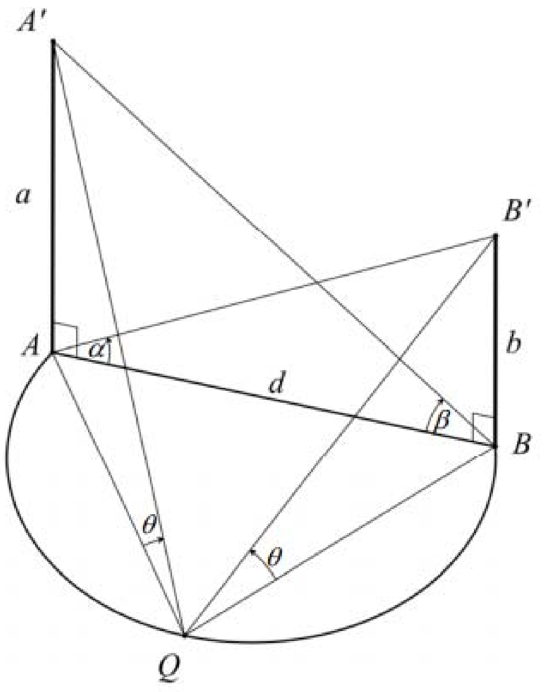
(i) Show that $q = -\frac{1}{2}p$. **2**

(ii) Show that, as p varies, M moves on a parabola. **2**

Question 14 continues on the next page.

Question 14 (continued)

(e)



In the diagram, AQB is a semicircle in the horizontal plane with diameter AB of length d metres.

There are two vertical posts AA' and BB' of heights a and b, respectively.

From Q, the angle of elevation to the tops of both posts A' and B' is $\theta°$ where $0 < \theta < 90$. From A the angle of elevation to B' is α and from B the angle of elevation to A' is β.

(i) Prove giving reasons that: **3**

$$d^2 = \frac{a^2 + b^2}{\tan^2 \theta}$$

(ii) Hence, or otherwise, find an expression for θ in terms of α and β only. **1**

END OF PAPER

Sydney Girls High School

Mathematics Faculty

Multiple Choice Answer Sheet
Trial HSC Mathematics Extension 1 (2018).

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2+4=?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ✗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ✗ B ✗ C ○ D ○

Student Number: ANSWERS.

Completely fill the response oval representing the most correct answer.

1. A ○ B ○ C ○ D ●
2. A ○ B ● C ○ D ○
3. A ○ B ○ C ○ D ●
4. A ○ B ● C ○ D ○
5. A ○ B ○ C ○ D ●
6. A ○ B ○ C ○ D ●
7. A ● B ○ C ○ D ○
8. A ○ B ○ C ● D ○
9. A ○ B ● C ○ D ○
10. A ○ B ○ C ● D ○

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MULTIPLE CHOICE:

(1) D.

$$P(-1) = 3(-1)^3 - 2(-1)^2 + c(-1) - 5$$
$$-14 = -3 - 2 - c - 5$$
$$-14 = -10 - c$$
$$\therefore \underline{c = 4}.$$

(2) B.

$$\tan\left(\frac{\pi}{4}\right) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$1 = \left|\frac{m - 3}{1 + 3m}\right|$$

$\therefore 1 = \frac{m-3}{1+3m}$ or $-1 = \frac{m-3}{1+3m}$

$1 + 3m = m - 3$; $-1 - 3m = m - 3$

$2m = -4$; $2 = 4m$

$m = -2$; $m = \frac{1}{2}$.

(3) D.

$x = 2t$

$y = (t-2)^2$ $\Rightarrow$ $y = \left(\frac{x}{2} - 2\right)^2$

$$y = \frac{x^2}{4} - 2(2)\frac{x}{2} + 4$$
$$4y = x^2 - 8x + 16$$
$$4y = (x-4)^2$$
$$\therefore \text{vertex} = (4, 0).$$

MULTIPLE CHOICE (Continued)

④ B. $(2x+2)(x) = (x+4)(x+1)$

$2x^2 + 2x = x^2 + x + 4x + 4$

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$\therefore x = -1 \quad \text{or} \quad x = 4$

$\underline{x = 4} \quad (x > 0)$

⑤ D. $x^3 + px^2 + 0x + q = 0$

$a \quad -b \quad +c \quad -d = 0$

$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$

$= \frac{\gamma}{\alpha\beta\gamma} + \frac{\alpha}{\alpha\beta\gamma} + \frac{\beta}{\alpha\beta\gamma}$

$= \frac{(\alpha + \beta + \gamma)}{\alpha\beta\gamma}$

$= \frac{-p}{1} \div \frac{-q}{1}$

$= -p \times \frac{1}{-q}$

$= \frac{p}{q}$.

MULTIPLE CHOICE (Continued).

⑥ D. $\int_0^{\pi/8} \cos^2 x \, dx$.

$= \frac{1}{2}\int_0^{\pi/8} (1+\cos 2x)\, dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/8}$

$= \frac{1}{2}\left[\left(\frac{\pi}{8}+\frac{1}{2\sqrt{2}}\right) - 0\right]$

$= \frac{\pi}{16} + \frac{1}{4\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\pi}{16} + \frac{\sqrt{2}}{8}$

$= \frac{\pi + 2\sqrt{2}}{16}$.

⑦ A. $^{16}C_7$ $\quad 16 = \begin{cases} 5 \text{ females} \\ 11 \text{ males.} \end{cases}$

n (at least 3 females)

$= {}^{16}C_7 - n\,[0, 1 \text{ or } 2 \text{ females}]$

$= {}^{16}C_7 - \left[{}^5C_0\,{}^{11}C_7 + {}^5C_1\,{}^{11}C_6 + {}^5C_2\,{}^{11}C_5\right]$

$= 11440 - [1 \times 330 + 5 \times 462 + 10 \times 462]$

$= 11440 - 7260$

$= 4180$.

MULTIPLE CHOICE (Continued).

(8) C. $v^2 = 144(25 - x^2)$

$\Rightarrow v^2 = n^2(a^2 - x^2)$

Period $= T = \frac{2\pi}{n}$

$\therefore a^2 = 25$ $\quad n^2 = 144$ $\quad T = \frac{2\pi}{12} = \frac{\pi}{6}$.

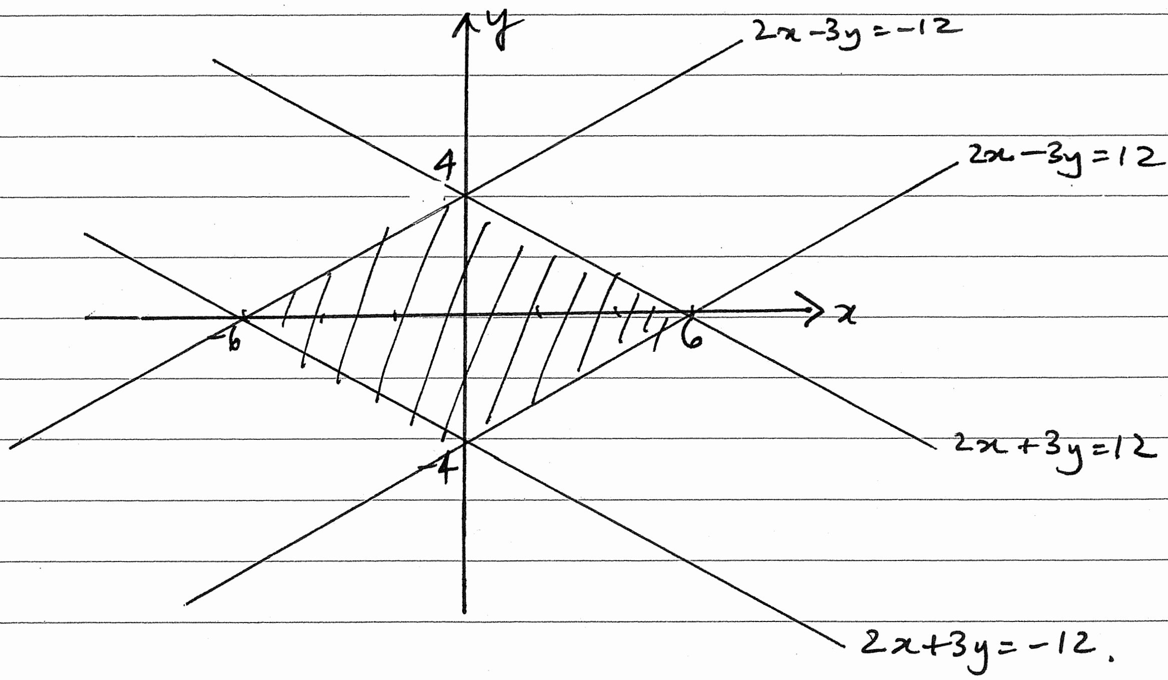
$\underline{a = 5}$ $\quad n = 12$

(9) B. $|2x - 3y| \leq 12$

$\Rightarrow 2x - 3y \leq 12$ or $2x - 3y \geq -12$

$|2x + 3y| \leq 12$

$\Rightarrow 2x + 3y \leq 12$ or $2x + 3y \geq -12$.



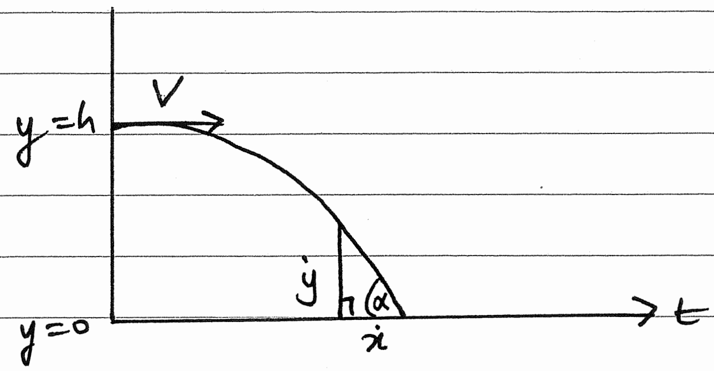
Shaded region satisfies all inequations:

Area $= 4 \times \frac{1}{2} \times 6 \times 4$

$= 48$ units^2.

MULTIPLE CHOICE (Continued).

(10) C.



$x = Vt$

$\dot{x} = V$

$y = h - \frac{1}{2}gt^2$

$\dot{y} = -gt$

$\dot{y} = -g.\sqrt{\frac{2h}{g}}$

when $y=0$

$0 = h - \frac{1}{2}gt^2$

$\frac{1}{2}gt^2 = h$

$t^2 = \frac{2h}{g}$

$t = \sqrt{\frac{2h}{g}}, \; t > 0.$

$\therefore \tan\alpha = \frac{\dot{y}}{\dot{x}}$

$= \frac{-g.\sqrt{\frac{2h}{g}}}{V}$

$= \frac{-\sqrt{\frac{2g^2h}{g}}}{V}$

$\tan\alpha = \frac{-\sqrt{2gh}}{V}$

$\therefore \alpha = -\tan^{-1}\left(\frac{\sqrt{2gh}}{V}\right)$

which has magnitude $\tan^{-1}\left(\frac{\sqrt{2gh}}{V}\right)$.

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Q. 11

a)

$$\frac{x}{x-3} < 4$$

$\times (x-3)^2$ on both sides

$$x(x-3) < 4(x-3)^2$$

Some students $\times (x-3)$, which is wrong as if $(x-3)$ is $-ve$, then it changes the inequality sign.

$$x(x-3) - 4(x-3)^2 < 0$$

$$(x-3)(x - 4(x-3)) < 0$$

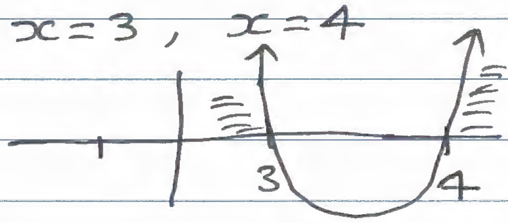
$$(x-3)(x - 4x + 12) < 0$$

$$(x-3)(-3x+12) < 0$$

$$-3(x-3)(x-4) < 0$$

$$(x-3)(x-4) > 0$$

$$x = 3, \quad x = 4$$



$$x < 3 \text{ and } x > 4$$

Some made mistake here and wrote the solution as $3 < x < 4$.

b)

$$y = \cos^{-1}(2x) - \pi/2$$

$$y + \pi/2 = \cos^{-1} 2x$$

$$\cos(\pi/2 + y) = 2x$$

$$-\sin y = 2x$$

$$\sin y = -2x$$

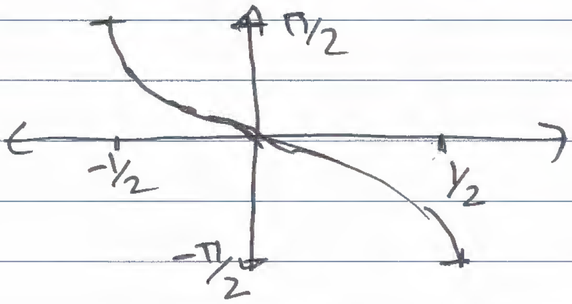
$$y = \sin^{-1}(-2x) = -\sin^{-1}(2x)$$

i) Domain $-\frac{1}{2} \le x \le \frac{1}{2}$

ii) Range $-\pi/2 \le y \le \pi/2$

Some students got the domain as $-1 \le x \le 1$ which is that of $y = -\sin^{-1}$

iii)



Some students got the shape of graph incorrect.

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11 c) $\left(\dfrac{-3\times3+1\times-5}{-5+3}, \dfrac{6\times3+10\times-5}{-5+3}\right)$

$(7, 16)$

Some students got $5+3$ at the denominator as if they are dividing AB internally. This is an external division.

11 d) $\lim\limits_{\theta\to0} \dfrac{\theta+\sin2\theta}{7\theta}$

$= \lim\limits_{\theta\to0} \dfrac{\theta+\sin 2\theta}{7\theta}$

$= \lim\limits_{\theta\to0} \dfrac{\theta}{7\theta} + \lim\limits_{\theta\to0} \dfrac{\sin 2\theta}{7\theta}$

$= \dfrac{1}{7} \lim\limits_{\theta\to0} \dfrac{\theta}{\theta} + \dfrac{2}{7}\lim\limits_{\theta\to0} \dfrac{\sin 2\theta}{2\theta}$

$= \dfrac{1}{7} + \dfrac{2}{7}$

$= \dfrac{3}{7}$

Some students wrote this step as $\dfrac{\theta}{7\theta} + \dfrac{\sin 2\theta}{2\theta}$ which is wrong. Some thought $\lim\limits_{\theta\to0} \dfrac{\theta}{\theta}$ is equal to zero.

11 e) $x^2+3x^2+2x+1=0$

$\alpha+\beta+\gamma = -\dfrac{3}{1}$

$\alpha\beta+\beta\gamma+\gamma\alpha = \dfrac{2}{1}$

$\alpha\beta\gamma = -1$

$(\alpha-1)(\beta-1)+(\beta-1)(\gamma-1)+(\gamma-1)(\alpha-1)$

$= \alpha\beta-\alpha-\beta+1+\gamma\beta-\beta-\gamma+1+\gamma\alpha-\gamma-\alpha+1$

$= (\alpha\beta+\gamma\beta+\gamma\alpha)-2(\alpha+\beta+\gamma)+3$

$= 2-2\times-3+3$

$= 2+6+3$

$= 11$

Some students got sum of roots with different number of roots taken at a time wrong. Some made the ~~rea~~ mistake at solving of equation algebraical at the final

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4 f)

let $u = 2x-1$

$du = 2dx$

$dx = \frac{du}{2}$

then $x = \frac{u+1}{2}$

Some student did not change limits with the change of x to u

when $x=0$, $u=-1$, when $x=1$, $u=1$

so $\int_0^1 x(2x-1)^4\,dx = \int_{-1}^{1} u^4 \frac{(u+1)}{2} \times \frac{du}{2}$

$\int_0^1 x(2x-1)^4\,dx = \frac{1}{4}\left[\int_{-1}^{1} u^5\,du + \int_{-1}^{1} u^4\,du\right]$

$= \frac{1}{4}\left\{\left[\frac{u^6}{6}\right]_{-1}^{1} + \left[\frac{u^5}{5}\right]_{-1}^{1}\right]$

$= \frac{1}{4}\left[\frac{1-1}{6} + \frac{1+1}{5}\right]$

$= \frac{1}{4}\left[0 + \frac{2}{5}\right] = \frac{1}{10}$

some students did not compute final limits correctly.

Q12

a) $y = \log_e(\sin^{-1} 2x)$

$$y' = \frac{\frac{2}{\sqrt{1-4x^2}}}{\sin^{-1} 2x} = \frac{2}{\sin^{-1} 2x \sqrt{1-4x^2}}$$

b)

i) In $\triangle BTP$ and $\triangle ATP$

PT : common side

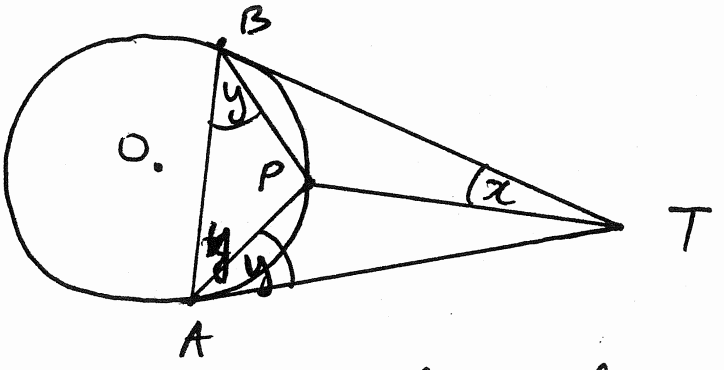
BT = AT (Two tangents to a circle from an external pt are equal)

$\angle BTP = \angle ATP$ (Given)

$\therefore \triangle BTP \equiv \triangle ATP$ (SAS)

Thus BP = AP (corresponding sides in congruent $\triangle$s)

ii)



A few students Assumed OPT is a Straight line without proving it.

$\angle PAT = \angle ABP = y$ ($\angle$'s in alt segment)

$\angle BAP = \angle ABP = y$ (Base $\angle$'s of Isos $\triangle ABP$)

$\angle ABT = \angle BAT = 2y$ (Base $\angle$'s of Isos $\triangle$ ~~ABT~~)

$2x + 2y + 2y = 180°$ ($\angle$'s sum of $\triangle ABT$)

$2x + 4y = 180$

$x + 2y = 90$.

12/c

i) Range: $y > 1$

ii) $f(x) = 1 + e^{2x}$

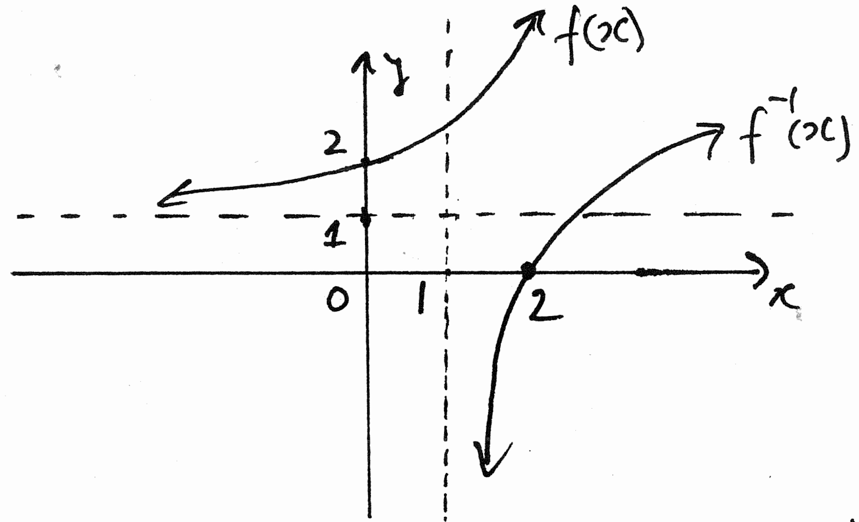
$x = 1 + e^{2y}$

$e^{2y} = x - 1$

$2y = \ln(x-1)$

$y = \frac{1}{2}\ln(x-1) : f^{-1}(x)$

iii)



iv) $\frac{d[f^{-1}(x)]}{dx} = \frac{1}{2x-2}$

$f^{-1}(x) = 0$

$\therefore \ln(x-1) = 0$

$x = 2, \; y = 0$

At $x = 2 \therefore \frac{d[f^{-1}(x)]}{dx} = \frac{1}{2(2)-2} = \frac{1}{2} \; (m_T)$

The grad of the Normal of $f^{-1}(x)$: $m_N = -2$

$y - y_1 = m_N(x - x_1)$

$y - 0 = -2(x - 2)$

$y = -2x + 4$

OR $2x + y - 4 = 0$

12/c

v) $2x + y - 4 = 0$ ①

$y = 1 + e^{2x}$ ②

Subs ② into ①

$2x + 1 + e^{2x} - 4 = 0$

$e^{2x} + 2x = 3$

vi) $e^{2x} + 2x = 3$

Let $g(x)$ be $e^{2x} + 2x - 3$

$g'(x) = 2e^{2x} + 2$

$$x_2 = x_1 - \frac{g(x_1)}{g'(x_1)}$$

$$= 0.4 - \frac{e^{0.8} + 0.8 - 3}{2 \times e^{0.8} + 2}$$

$$= 0.396$$

Question 13

(a) Prove for $n = 1$

$$LHS = 1 \times 1! = 1$$

$$RHS = (1+1)! - 1 = 2 - 1 = 1 = LHS$$

Some students assumed that only $k \times k! = (k+1)! - 1$, missing the rest of the sum.

Assume for $n = k$

$$1 \times 1! + 2 \times 2! + \cdots + k \times k! = (k+1)! - 1$$

Prove for $n = k + 1$

Required to prove

$$1 \times 1! + 2 \times 2! + \cdots + k \times k! + (k+1) \times (k+1)! = (k+2)! - 1$$

$$LHS = 1 \times 1! + 2 \times 2! + \cdots + k \times k! + (k+1) \times (k+1)!$$

$$= (k+1)! - 1 + (k+1) \times (k+1)! \qquad \text{by assumption}$$

$$= (k+1)!\big(1 + (k+1)\big) - 1$$

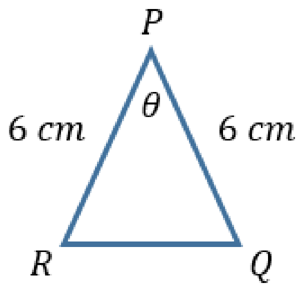
$$= (k+1)!\,(k+2) - 1$$

$$= (k+2)! - 1$$

$$= RHS$$

Therefore by the principles of mathematical induction this is true for all positive integers n.

(b)(i)



$$A = \frac{1}{2} \times 6^2 \times \sin\theta$$

$$A = 18\sin\theta \quad \text{cm}^2$$

(ii) When $A = 9$, $\theta = \frac{\pi}{6}$.

$$\frac{dA}{d\theta} = 18\cos\theta$$

$$\frac{dA}{d\theta} = 18\cos\frac{\pi}{6}$$

$$\frac{dA}{d\theta} = 9\sqrt{3}$$

$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$$

$$3 = 9\sqrt{3} \times \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{3\sqrt{3}}$$

$$\frac{d\theta}{dt} = \frac{\sqrt{3}}{9} \quad radians/second$$

(c)(i) $\sin y = x \qquad \cos y = x$

$$\sin y = \cos y$$

$$\tan y = 1$$

$$y = \frac{\pi}{4}$$

$$x = \frac{1}{\sqrt{2}}$$

Note that: $\frac{\sin^{-1} x}{\cos^{-1} x} \neq \tan^{-1} x$

(ii)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

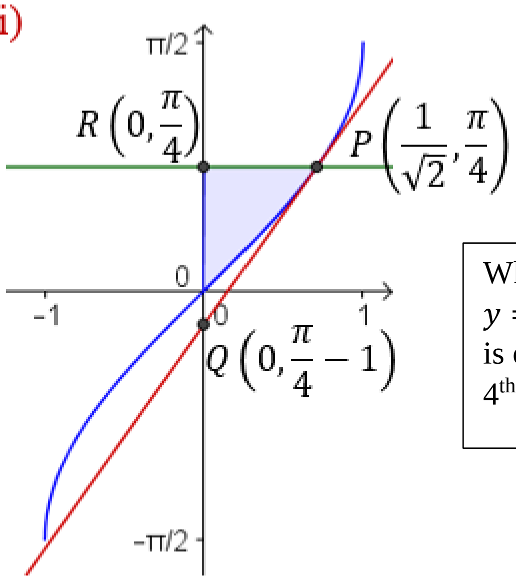
$$m_T = \frac{1}{\sqrt{\frac{1}{2}}} \quad \text{at} \quad x = \frac{1}{\sqrt{2}}$$

$$m_T = \sqrt{2}$$

$$y - \frac{\pi}{4} = \sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$y = \sqrt{2}x - 1 + \frac{\pi}{4}$$

(ii)



When integrating in the y direction the curve $y = \sin^{-1} x$ is not always under the tangent. This is only true in the 1st quadrant. The piece in the 4th quadrant must be treated separately.

$$Area\ \Delta RQP = \frac{1}{2} \times 1 \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \text{ units}^2$$

$$Area = Area\ \Delta RQP - \int_0^{\frac{\pi}{4}} \sin y\, dy$$

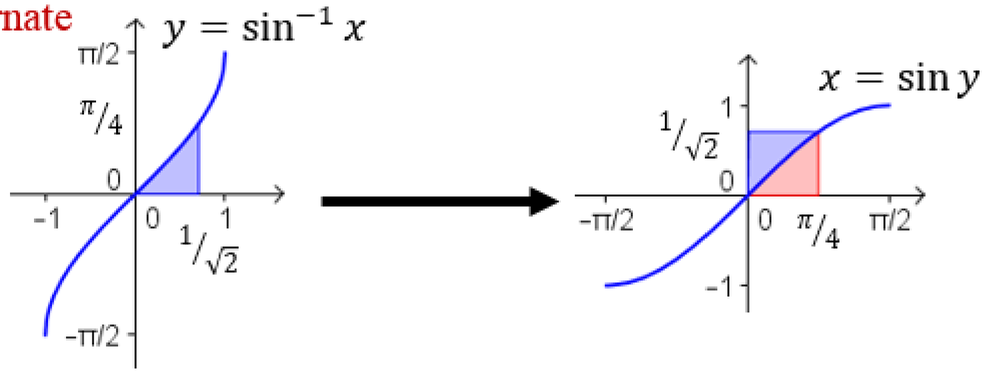
$$Area = \frac{1}{2\sqrt{2}} - \int_0^{\frac{\pi}{4}} \sin y\, dy$$

$$Area = \frac{1}{2\sqrt{2}} + [\cos y]_0^{\frac{\pi}{4}}$$

$$Area = \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

$$Area = \frac{3\sqrt{2}}{4} - 1 \quad \text{units}^2$$

(ii) alternate



$$Area = \int_0^{1/\sqrt{2}} \left(\sin^{-1} x - \left(\sqrt{2}x - 1 + \frac{\pi}{4} \right) \right) dx$$

$$Area = \int_0^{1/\sqrt{2}} \sin^{-1} x \, dx - \int_0^{1/\sqrt{2}} \left(\sqrt{2}x - 1 + \frac{\pi}{4} \right) dx$$

$$Area = \frac{\pi}{4\sqrt{2}} - \int_0^{\pi/4} \sin y \, dy - \int_0^{1/\sqrt{2}} \left(\sqrt{2}x - 1 + \frac{\pi}{4} \right) dx$$

$$Area = \frac{\pi}{4\sqrt{2}} - [-\cos y]_0^{\pi/4} - \left[\frac{\sqrt{2}}{2} x^2 - x + \frac{\pi}{4} x \right]_0^{1/\sqrt{2}}$$

$$Area = \frac{\pi}{4\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} + 1 \right) - \left(\frac{\sqrt{2}}{2} \times \frac{1}{2} - \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} \right)$$

$$Area = \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 - \frac{\sqrt{2}}{4} + \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}$$

$$Area = \sqrt{2} - \frac{\sqrt{2}}{4} - 1$$

$$Area = \frac{3\sqrt{2}}{4} - 1 \quad \text{units}^2$$

(d)(i) When $t = 0$

$x = 0 \qquad y = 2.4$

$\dot{x} = 12\cos 40° \qquad \dot{y} = 12\sin 40°$

$\ddot{x} = 0 \qquad \ddot{y} = -g$

$\dot{x} = 12\cos 40° \qquad \dot{y} = -gt + 12\sin 40°$

$x = 12\cos 40°\, t \qquad y = -\frac{1}{2}gt^2 + 12\sin 40°\, t + 2.4$

(ii) Find t when $y = 0$ and $g = 9.8$.

$$\frac{9.8}{2}t^2 - 12\sin 40°\, t - 2.4 = 0$$

$$t = \frac{12\sin 40° \pm \sqrt{12^2\sin^2 40° + 2 \times 9.8 \times 2.4}}{9.8}$$

$t \approx 1.84\ seconds$

Horizontal distance travelled $\qquad x = 12\cos 40° \times 1.84$

$OA = 16.9\ m$

Many students got very sloppy with their numbers. When calculations are complicated you must slow down to make sure you don't make silly errors.

Question 14.

a) 7 exams; 2 Maths; 1 English.

i)

- Maths exams can be together 2 ways
  M_1M_2 or M_2M_1
- If we treat M_1 and M_2 being together then there are $2\times 6!$ ways of ordering 7 exams.
- We don't want them together, so we need the complement of them being together.
- total number of scheduling 7 exams is $7!$

$\therefore$ M_1 and M_2 not together
$= 7! - (2\times 6!)$
$= 3600$ ways. ✓

ii) $\underline{E}\ \underline{M_1}\ \_\ \_\ \_\ \_\ \underline{M_2}$ or $\underline{E}\ \underline{M_2}\ \_\ \_\ \_\ \_\ \underline{M_1}$

- Two possibilities
- the remaining 4 exams can be arranged $4!$ ways
- total ways $\underline{E}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{6!}$

$\therefore P(\text{Maths second \& Maths last}) = \dfrac{2\times 4!}{6!} = \dfrac{48}{720}$
$= \dfrac{1}{15}$ ✓

* Some students incorrectly assumed that the 2 Maths exams were identical.

Question 14. (Continued)

b) Given $\sin(k\pi + \phi) = (-1)^k \sin\phi$ (*)

$$\begin{aligned}
LHS &= \sin[(k+1)\pi + \phi] \\
&= \sin(k\pi + \pi + \phi) \\
&= \sin[(k\pi + \phi) + \pi] \\
&= \sin(k\pi+\phi)\cos\pi + \cos(k\pi+\phi)\sin\pi \checkmark \\
&= \sin(k\pi+\phi) \times (-1) + 0 \\
&= (-1)\sin(k\pi+\phi) \\
&= (-1)(-1)^k \sin\phi \quad (\text{given } (*)) \checkmark \\
&= (-1)^{k+1}\sin\phi \\
&= RHS.
\end{aligned}$$

* (There were many alternate ways to show this result.)

Question 14 (Continued)

c) $x = a\cos nt \qquad (a>0,\ \pi < n < 2\pi)$

when $t=1$, $x=1$: $\quad a\cos n = 1 \quad ①$

when $t=2$, $x=-1$: $\quad a\cos 2n = -1 \quad ②$

① + ②:

$$a\cos n + a\cos 2n = 0$$
$$\cos n + \cos 2n = 0$$
$$\cos n + 2\cos^2 n - 1 = 0$$
$$2\cos^2 n + \cos n - 1 = 0$$
$$(2\cos n - 1)(\cos n + 1) = 0$$

$\therefore \cos n = \frac{1}{2}$ or $\cos n = -1$

$n = \frac{\pi}{3}, \frac{5\pi}{3}$ $\qquad n = \pi$

$\therefore n = \frac{5\pi}{3} \quad (\pi < n < 2\pi)$.

* Students who lost marks in this question, did not use the given information to set up some equations.

Question 14. (continued)

d) i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

At $P(2ap, ap^2)$

$m_{tangent} = \frac{2ap}{2a} = p$

$m_{chord} = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{a(p-q)(p+q)}{2a(p-q)}$

$= \frac{p+q}{2}$. ✓

$m_{tangent} = 4 \times m_{chord}$.

$p = 4\left(\frac{p+q}{2}\right)$

$p = 2p + 2q$ ✓

$-p = 2q$

$\therefore q = -\frac{1}{2}p$, as required.

ii) $M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$ ... ①

$q = -\frac{1}{2}p$ ... ②

② → ① :

$M = \left(\frac{2ap + 2a\left(-\frac{p}{2}\right)}{2}, \frac{ap^2 + a\left(-\frac{p}{2}\right)^2}{2}\right)$.

$= \left(\frac{2ap - ap}{2}, \frac{ap^2 + \frac{ap^2}{4}}{2}\right)$. P.T.O.

Question 14 d) continued.

$$\therefore M = \left(\frac{ap}{2}, \frac{5ap^2}{8}\right)$$ ✓

Locus of M:

$$x = \frac{ap}{2} \Rightarrow p = \frac{2x}{a} \quad \ldots (1)$$

$$y = \frac{5ap^2}{8} \quad \ldots \quad (2)$$

Sub (1) → (2)

$$y = \frac{5a}{8} \times \left(\frac{2x}{a}\right)^2$$

$$= \frac{5a}{8} \times \frac{4x^2}{a^2}$$ ✓

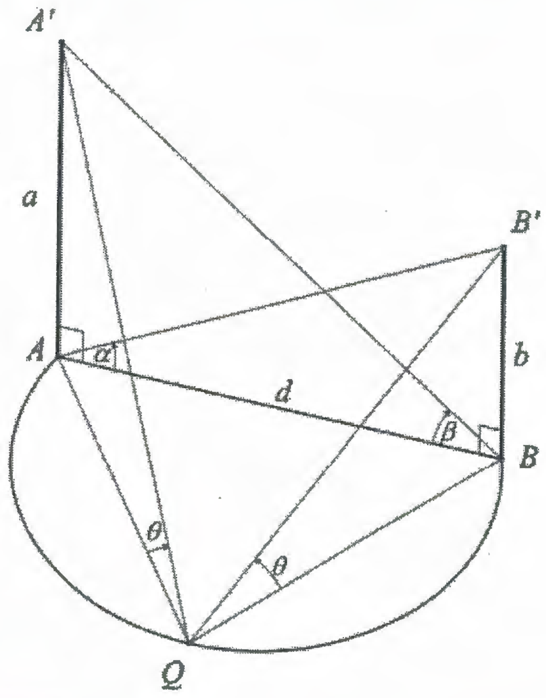
$$y = \frac{5x^2}{2a}$$, which is in the form of a parabola.

∴ M moves on a parabola.

* Students who did not eliminate the parameter from their final answer lost one mark.

Question 14 (continued)

e).



In $\Delta A'QA$:

$\tan\theta = \dfrac{a}{AQ}$

$\therefore AQ = \dfrac{a}{\tan\theta}$ ✓

In $\Delta B'QB$:

$\tan\theta = \dfrac{b}{BQ}$

$\therefore BQ = \dfrac{b}{\tan\theta}$.

i) $\angle AQB = 90^\circ$ (angle in a semi-circle) ✓

$\therefore d^2 = AQ^2 + BQ^2$ (Pythagoras Thm)

$= \left(\dfrac{a}{\tan\theta}\right)^2 + \left(\dfrac{b}{\tan\theta}\right)^2$ ✓

$= \dfrac{a^2}{\tan^2\theta} + \dfrac{b^2}{\tan^2\theta}$

$\therefore d^2 = \dfrac{a^2+b^2}{\tan^2\theta}$, as required

* Students lost one mark if they did not give a reason for $\angle AQB = 90^\circ$.

Question 14 (continued)

e) ii) In $\Delta ABA'$: $\tan\beta = \frac{a}{d}$

In $\Delta ABAB'$: $\tan\alpha = \frac{b}{d}$

From part i):

$$d^2 = \frac{a^2+b^2}{\tan^2\theta}$$

i.e: $\tan^2\theta = \frac{a^2+b^2}{d^2}$

$$\tan^2\theta = \left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2$$

$$= (\tan\beta)^2 + (\tan\alpha)^2$$

$$\therefore \tan^2\theta = \tan^2\alpha + \tan^2\beta$$ ✓

$\Rightarrow \therefore \tan\theta = \sqrt{\tan^2\alpha + \tan^2\beta}$ $\left(\begin{matrix} 0 < \theta < 90° \\ \tan\theta > 0 \end{matrix}\right)$.

$$\theta = \tan^{-1}\left(\sqrt{\tan^2\alpha + \tan^2\beta}\right)$$