Sydney Girls High School

2019	TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations. A correct answer without working will be awarded a maximum of 1 mark.

Total marks: 70

Section 1 – 10 marks (pages **3 – 6**)

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II – 60 marks (pages **7 – 13**)

- Attempt Questions 11 – 14
- Answer on the blank paper provided
- Begin a new page for each question
- Allow about 1 hour and 45 minutes for this section

Name: …………………………………………………………...	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2018 HSC Examination Paper in this subject.
Teacher: …………………………………………………………	

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1 – 10

1. What is the value of $\lim_{x \to 0}\left(\dfrac{\sin 4x}{3x}\right)$?

   (A) 1

   (B) 0

   (C) $\dfrac{4}{3}$

   (D) $\dfrac{3}{4}$

2. After t years, the number of individuals in a population is given by N where $N = 300 + 100e^{-0.2t}$. What is the difference between the initial population and the limiting population size?

   (A) 100

   (B) 300

   (C) 350

   (D) 400

3. The point *P* divides the interval from $A(2,3)$ to $B(6,1)$ **externally** in the ratio 4:5. What is the *x* coordinate of *P*?

   (A) -14

   (B) 8

   (C) -12

   (D) $\dfrac{34}{9}$

4. The velocity $v\ ms^{-1}$ of a particle moving in simple harmonic motion along the x-axis is given by $v^2 = 32 + 8x - 4x^2$. What is the amplitude, A, and the period, T, of the motion?

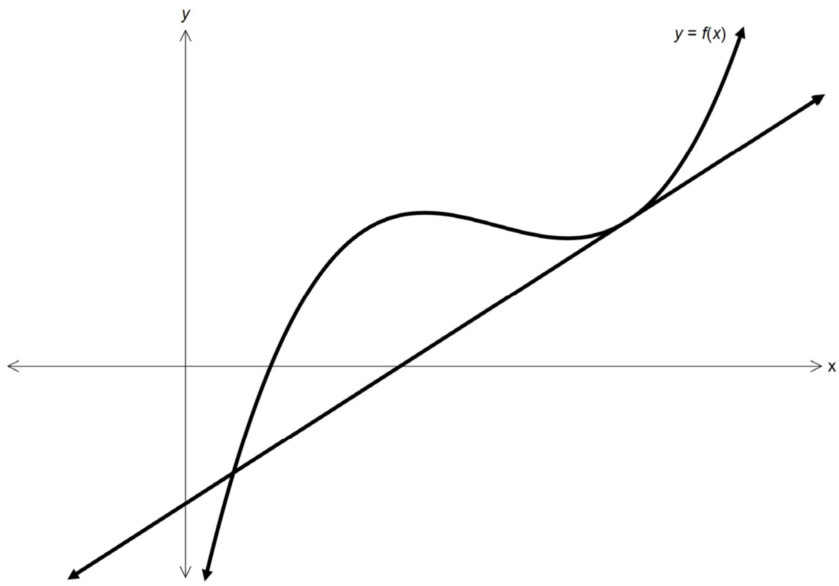
(A) A = 2 and $T = \frac{\pi}{2}$

(B) A = 2 and $T = \pi$

(C) A = 3 and $T = \frac{\pi}{2}$

(D) A = 3 and $T = \pi$

5.



The diagram shows the curve $y = f(x)$. The tangent to the curve at the point $x = 3$ cuts the x-axis at $x = \frac{7}{5}$. Which of the following is the value of $\frac{f(3)}{f'(3)}$?

(A) $-\frac{8}{5}$

(B) $-\frac{5}{8}$

(C) $\frac{5}{8}$

(D) $\frac{8}{5}$

6. Express $3\sin\theta + 4\cos\theta$ in the form $R\sin(\theta+\alpha)$, where α is in radians.

(A) $7\sin\left(\theta + \tan^{-1}\left(\frac{4}{3}\right)\right)$

(B) $5\sin\left(\theta + \tan^{-1}\left(\frac{4}{3}\right)\right)$

(C) $7\sin\left(\theta + \tan^{-1}\left(\frac{3}{4}\right)\right)$

(D) $5\sin\left(\theta + \tan^{-1}\left(\frac{3}{4}\right)\right)$

7. The Cartesian equation of the tangent to the parabola $x = t - 3,\ y = t^2 + 2$ at $t = -3$ is:

(A) $6x - y - 47 = 0$

(B) $2x + 3y + 9 = 0$

(C) $6x + y + 25 = 0$

(D) $3x - 2y + 11 = 0$

8. A group of 4 women and 8 boys include a mother and son. From this group, a team consisting of 2 women and 2 boys is to be chosen. How many ways can the team be chosen if the mother and son cannot be on the team together?

(A) 147

(B) 168

(C) 120

(D) 63

9. What is the value of k such that $\int_0^k \frac{2dx}{\sqrt{4-25x^2}} = \frac{\pi}{5}$?

(A) $\frac{4}{5}$

(B) $\frac{2\pi}{3}$

(C) $\frac{\sqrt{3}}{5}$

(D) $\frac{2}{5}$

10. The derivative of a function $f(x)$ is given by $f'(x) = e^{\sin x} - \cos x - 1$ for $0 < x < 9$. On what interval is $f(x)$ decreasing?

(A) $0 < x < 0.633$ and $4.115 < x < 6.916$

(B) $0 < x < 1.947$ and $5.744 < x < 8.230$

(C) $0.633 < x < 4.115$ and $6.916 < x < 9$

(D) $1.947 < x < 5.744$ and $8.230 < x < 9$

Section II

60 marks

Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section
Answer on the blank paper provided. Begin a new page for each question.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 **(15 marks)** Begin a new page.

(a) Differentiate $y = \sin^{-1}(3x)$. **2**

(b) The acute angle between the lines $2x - 3y - 4 = 0$ and $y = mx - 3$ is 45°. Find the two possible values of m. **3**

(c) Sketch $y = 3\cos^{-1}(x-2)$, showing all key points. **2**

(d) Solve $\dfrac{x^2 - 4}{x} > 3$. **3**

(e) Evaluate $\displaystyle\int_0^2 \frac{5x}{(5x+2)^2}\,dx$ by using the substitution $u = 5x + 2$. **3**

Write your answer correct to 3 significant figures.

(f) Prove $\dfrac{\sin 2\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\cos\theta} = \sec\theta$. **2**

End of Question 11

Question 12 **(15 marks)** Begin a new page.

(a)

(i) How many different ways can the letters in the word 'YAMAHA' be arranged? **1**

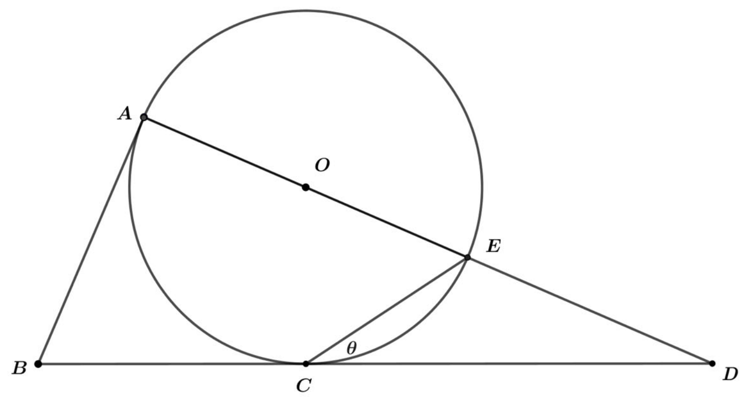
(ii) One of the different ways of arranging the letters of the word 'YAMAHA' is chosen at random. What is the probability that all the A's are together? **1**

(b) Evaluate $\tan\left[\sin^{-1}\left(-\frac{3}{5}\right)+\cos^{-1}\left(\frac{2}{3}\right)\right]$. **2**

(c) Evaluate $\int_0^2 \frac{dx}{16+4x^2}$. **2**

(d) Find the general solution of the equation $\sin 2\theta = \sin^2\theta$. **3**

(e)



The diagram above shows a circle with centre O and diameter AE.

BA and BCD are tangents to the circle and $\angle ECD = \theta$.

Copy the diagram in your answer booklet and show that $\angle ABC = 2\theta$. **3**

(f) If α, β and γ are the roots of the equation $2x^3 - 5x^2 + 3x - 1 = 0$, find the value of :

(i) $\alpha\beta\gamma(\alpha + \beta + \gamma)$ **1**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

End of Question 12

Question 13 **(15 marks)** Begin a new page.

(a) Consider the quadratic polynomial, $P(x) = (x + h)^2 + k$, with constants h and k.

Find the values of h and k given that $(x + 2)$ is a factor of $P(x)$ and 16 is the remainder when $P(x)$ is divided by x. **3**

(b) Prove by mathematical induction that for any integer $n > 0$, **3**

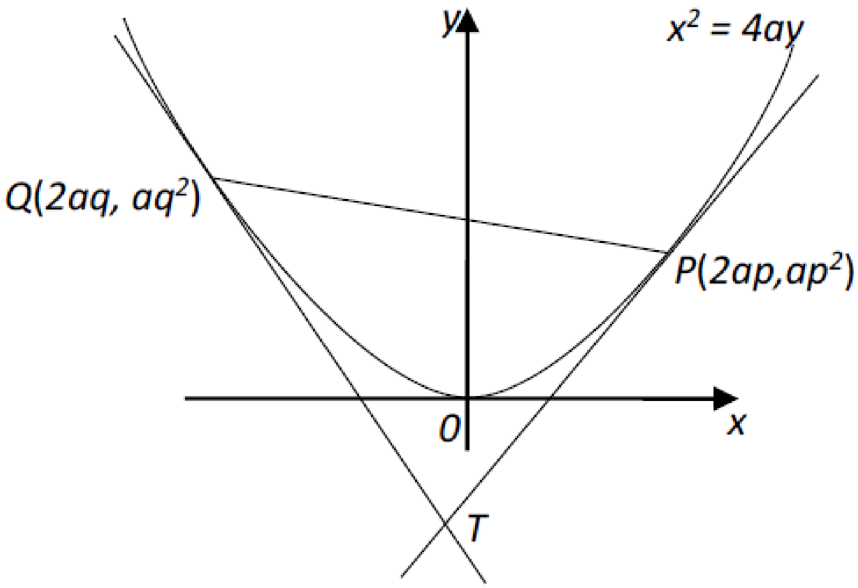
$$\frac{1}{3 \times 4 \times 5} + \frac{2}{4 \times 5 \times 6} + \cdots + \frac{n}{(n+2)(n+3)(n+4)}$$

$$= \frac{1}{6} - \frac{1}{n+3} + \frac{2}{(n+3)(n+4)}$$

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

The tangents to the parabola at P and Q intersect at the point T.

The coordinates of the point T is given by $x = a(p+q)$ and $y = apq$. (Do NOT prove this.)



(i) Show that $p = 1 + pq + q$ if the tangents at P and Q intersect at 45°. **2**

(ii) Find the Cartesian equation of the locus of T. **2**

(d) A particle is moving along the x-axis. Initially the particle is 1 metre to the right of the origin, travelling at a velocity of 3 metres per second and its acceleration is given by $\ddot{x} = 2x^3 + 4x$, where x is the displacement of the particle after t seconds.

(i) Show that $\dot{x} = x^2 + 2$. **2**

(ii) Hence or otherwise, show that $x = \sqrt{2}\tan\left(\sqrt{2}\,t + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$. **3**

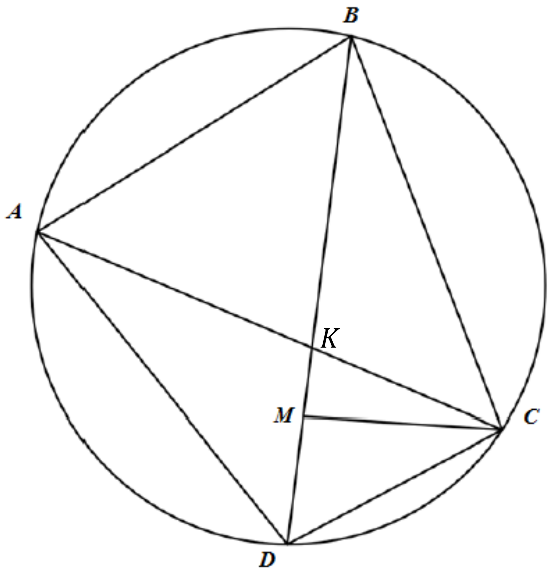
End of Question 13

Question 14 **(15 marks)** Begin a new page.

(a) An ice sculpture in the form of a sphere melts in such a way that it maintains its spherical shape. The volume of the sphere is decreasing at a constant rate of 2π cubic metres per hour. At what rate, in square metres per hour, is the surface area of the sphere decreasing at the moment when the radius is 5 metres? **2**

(b) A team of 17 soccer players includes two Brown sisters and three Stefanovic sisters. How many different ways are there of choosing a group of 11 soccer players from the team, if the group can include no more than one of the Brown sisters and no more than two of the Stefanovic sisters? **2**

(c) In the diagram below, $ABCD$ is a cyclic quadrilateral and K is the intersection of the diagonals AC and BD. M is the point on BD such that $\angle ACB = \angle DCM$.

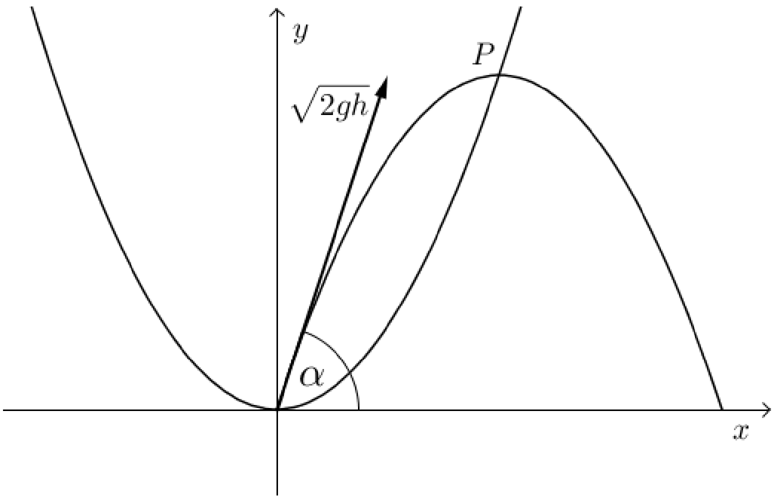


(i) Prove that $\dfrac{AC}{CD} = \dfrac{AB}{MD}$. **1**

(ii) Ptolemy's Theorem states that in a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the pairs of opposite sides, that is: $AC \times BD = AB \times CD + BC \times AD$. **2**

Prove Ptolemy's theorem.

(d) A cross-section of a valley is in the form of a parabola $x^2 = 4ay$ where a is a positive constant. A water cannon placed at the origin fires a jet of water with speed $\sqrt{2gh}$ at an angle α where $0 < \alpha < \frac{\pi}{2}$, h is a positive constant and g is the acceleration due to gravity.



The equations of motion of a projectile fired from the origin with initial velocity $V\ ms^{-1}$ at angle α to the horizontal are:

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$. (Do NOT prove these)

(i) If the water jet strikes the wall of the valley at the point $P(X, Y)$ show that: **3**

$$X = \frac{4ah}{(a+h)\cot\alpha + a\tan\alpha}$$

(ii) Let $f(\theta) = (a+h)\cot\theta + a\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. **3**

Show that the minimum value of $f(\theta)$ occurs when $\tan\theta = \sqrt{\frac{a+h}{a}}$.

(iii) Hence or otherwise, show that the greatest value of X is given by: **2**

$$X = 2h\sqrt{\frac{a}{a+h}}.$$

End of paper

Sydney Girls High School

Mathematics Faculty

Multiple Choice Answer Sheet
Trial HSC Mathematics Extension 1

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2+4=?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ☒ B ☒ C ○ D ○

Student Number: EXT 1

Completely fill the response oval representing the most correct answer.

1. A ○ B ○ C ● D ○
2. A ● B ○ C ○ D ○
3. A ● B ○ C ○ D ○
4. A ○ B ○ C ○ D ●
5. A ○ B ○ C ○ D ●
6. A ○ B ● C ○ D ○
7. A ○ B ○ C ● D ○
8. A ● B ○ C ○ D ○
9. A ○ B ○ C ○ D ●
10. A ● B ○ C ○ D ○

Q11

a) $y = \sin^{-1}(3x)$

$y' = \dfrac{3}{\sqrt{1-9x^2}}$ ✓✓

b) $2x - 3y - 4 = 0$, $y = mx - 3$

$y = \frac{2}{3}x + \frac{4}{3}$ $m_2 = m$

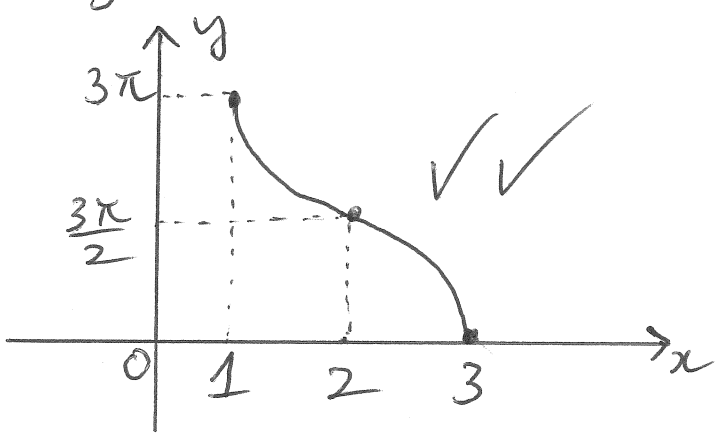
$m_1 = \frac{2}{3}$

$\tan\theta = \left| \dfrac{\frac{2}{3} - m}{1 + \frac{2m}{3}} \right| = \tan 45^\circ$ ✓

$\left| \dfrac{2-3m}{2m+3} \right| = 1 \quad \therefore \left[\begin{array}{l} 2 - 3m = 2m+3 \\ 2 - 3m = -2m-3 \end{array} \right.$

$\left[\begin{array}{l} m = -1/5 \\ m = 5 \end{array} \right.$ ✓✓

c) $y = 3\cos^{-1}(x-2)$



✓✓

$D: 1 \leq x \leq 3$

$R: 0 \leq y \leq 3\pi$

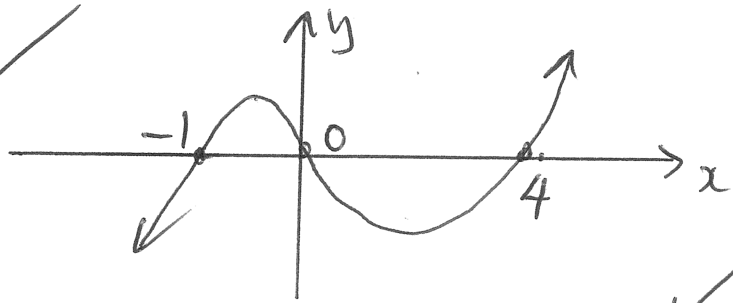
Q11

d) $\dfrac{x^2-4}{x} > 3$

$x(x^2-4) > 3x^2$

$x^3 - 3x^2 - 4x > 0$

$x(x-4)(x+1) > 0$



$-1 < x < 0$ OR

$x > 4$

e) $\displaystyle\int_0^2 \frac{5x}{(5x+2)^2}\,dx$

Let $u = 5x+2$

$\dfrac{du}{dx} = 5 \therefore du = 5dx$

$x = \dfrac{u-2}{5}$

$x = 2 \longrightarrow u = 12$

$x = 0 \longrightarrow u = 2$

$= \displaystyle\int_2^{12} \frac{\frac{u-2}{5}}{u^2}\,du$

$= \dfrac{1}{5}\displaystyle\int_2^{12} \frac{u-2}{u^2}\,du = \left[\frac{1}{5}\ln u + \frac{2}{5u}\right]_2^{12}$

$= \dfrac{1}{5}\ln 12 + \dfrac{1}{30} - \dfrac{1}{5}\ln 2 - \dfrac{1}{5}$

$= \dfrac{1}{5}\ln 6 - \dfrac{1}{6} = 0.192$

Q11

f) prove $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$

$$LHS = \frac{\sin 2\theta . \cos\theta - \cos 2\theta . \sin\theta}{\sin\theta . \cos\theta}$$

$$= \frac{\sin(2\theta - \theta)}{\sin\theta . \cos\theta}$$

$$= \frac{\sin\theta}{\sin\theta . \cos\theta}$$

$$= \frac{1}{\cos\theta} = \sec\theta = RHS$$

Ext 1 2019 Trial

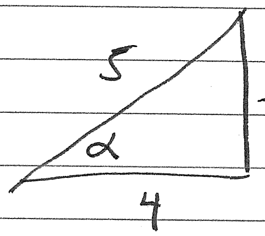
12a) i) $\frac{6!}{3!} = 120$ ✓

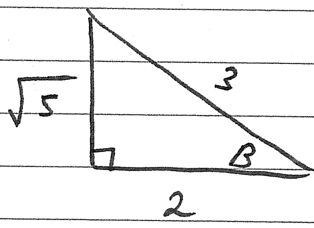
ii) $\frac{4!}{120} = \frac{1}{5}$ ✓

some students forgot to do the probability

b) $\alpha = \sin^{-1}\left(-\frac{3}{5}\right)$ $\quad B = \cos^{-1}\frac{2}{3}$

$\sin\alpha = -\frac{3}{5}$ $\quad \cos B = \frac{2}{3}$





$$\tan(\alpha+B) = \frac{\tan\alpha + \tan B}{1 - \tan\alpha\tan B}$$

$$= \frac{-\frac{3}{4} + \frac{\sqrt{5}}{2}}{1 - \left(-\frac{3}{4} \times \frac{\sqrt{5}}{2}\right)}$$

$$= \frac{\frac{-3 + 2\sqrt{5}}{4}}{1 + \frac{3\sqrt{5}}{8}}$$

$$= \frac{-6 + 4\sqrt{5}}{8 + 3\sqrt{5}}$$

some students didn't take ⊖ $\frac{3}{5}$ into their calcutions

c) $\int_0^2 \frac{dx}{4(4+x^2)}$

$= \frac{1}{4}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$

$= \frac{1}{8}\left[\tan^{-1}\left(\frac{2}{2}\right)\right] = \frac{1}{8} \times \frac{\pi}{4}$

$= \frac{\pi}{32}$

Some students couldn't find the correct inverse tan

d) $2\sin\theta\cos\theta = \sin^2\theta$

$2\sin\theta\cos\theta - \sin^2\theta = 0$

$\sin\theta(2\cos\theta - \sin\theta) = 0$

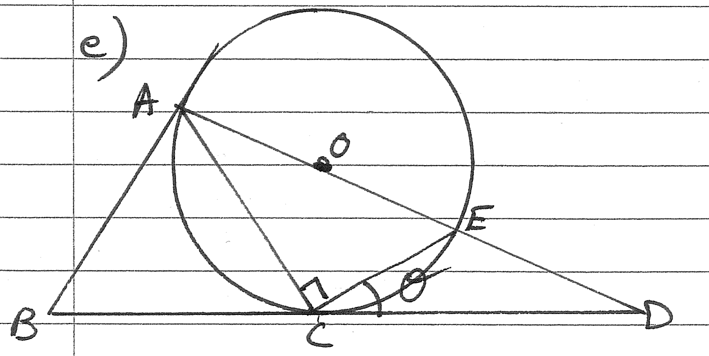
$\sin\theta = 0 \quad \therefore \theta = n\pi$

$2\cos\theta = \sin\theta$

$\tan\theta = 2 \quad \therefore \theta = n\pi + \tan^{-1}2$

* many students were careless with factorising or loosing $\sin\theta = 0$.

e)



$\angle ACE = 90$ ($\angle$ in a semicircle)

$\angle ACB + 90 + \theta = 180$ (st $\angle$)

$\angle ACB = 90 - \theta$

$AB = BC$ (tangents from external pt are equal)

$\therefore \angle BAC = \angle ACB = 90 - \theta$ (base $\angle$s of isosceles Δ)

$\angle ABC + 90 - \theta + 90 - \theta = 180$ ($\angle$ sum of Δ)

$\therefore \angle ABC = 2\theta$

* many students had poor setting out and the handwriting was very messy.

f) i) $\frac{1}{2} \times \frac{5}{2} = \frac{5}{4}$

ii) $(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= \left(\frac{5}{2}\right)^2 - 2\left(\frac{3}{2}\right) = \frac{13}{4}$

Question 13

(a) $P(0) = h^2 + k = 16$

$$P(-2) = (h-2)^2 + k = 0$$

$$h^2 + k - 4h + 4 = 0$$

$$16 - 4h + 4 = 0$$

$$20 = 4h$$

$$\underline{h = 5}$$

$$25 + k = 16$$

$$\underline{k = -9}$$

This question was poorly done because if P(x) is divided by x the remainder is P(0).

(b) Prove for $n = 1$

$$LHS = \frac{1}{3 \times 4 \times 5} = \frac{1}{60} \qquad RHS = \frac{1}{6} - \frac{1}{4} + \frac{2}{20} = \frac{1}{60} = LHS$$

Assume for $n = k$

$$\frac{1}{3 \times 4 \times 5} + \frac{1}{4 \times 5 \times 6} + \cdots + \frac{k}{(k+2)(k+3)(k+4)} = \frac{1}{6} - \frac{1}{k+3} + \frac{2}{(k+3)(k+4)}$$

Prove for $n = k + 1$

Required to prove

$$\frac{1}{3 \times 4 \times 5} + \frac{1}{4 \times 5 \times 6} + \cdots + \frac{k+1}{(k+3)(k+4)(k+5)} = \frac{1}{6} - \frac{1}{k+4} + \frac{2}{(k+4)(k+5)}$$

$$RHS = \frac{1}{6} - \frac{k+5-2}{(k+4)(k+5)}$$
$$= \frac{1}{6} - \frac{k+3}{(k+4)(k+5)}$$

$$LHS = \frac{1}{6} - \frac{1}{k+3} + \frac{2}{(k+3)(k+4)} + \frac{k+1}{(k+3)(k+4)(k+5)} \quad \text{by assumption}$$

$$= \frac{1}{6} - \frac{(k+4)(k+5) - 2(k+5) - (k+1)}{(k+3)(k+4)(k+5)}$$

$$= \frac{1}{6} - \frac{k^2 + 9k + 20 - 2k - 10 - k - 1}{(k+3)(k+4)(k+5)}$$

$$= \frac{1}{6} - \frac{k^2 + 6k + 9}{(k+3)(k+4)(k+5)}$$

Simplifying both the left and right sides of the identity is easier than trying to make the left look like the right.

$$= \frac{1}{6} - \frac{(k+3)^2}{(k+3)(k+4)(k+5)}$$

$$= \frac{1}{6} - \frac{k+3}{(k+4)(k+5)}$$

$$= RHS$$

Therefore by the principles of mathematical induction the statement is true for any integer $n > 0$.

(c)(i) Let the angles tangents from P and Q subtend with the horizontal be θ and ϕ respectively.

Thus $p = \tan\theta$ and $q = \tan\phi$.

$$\tan(\theta - \phi) = \frac{p - q}{1 + pq}$$

$$\tan(45°) = \frac{p - q}{1 + pq}$$

$$1 = \frac{p - q}{1 + pq}$$

$$1 + pq = p - q$$

Therefore $p = 1 + pq + q$

(ii) Note that: $p^2 + q^2 = (p - q)^2 + 2pq$

$$\frac{x^2}{a^2} = (p + q)^2$$

$$\frac{x^2}{a^2} = p^2 + q^2 + 2pq$$

$$\frac{x^2}{a^2} = (p - q)^2 + 4pq$$

$$\frac{x^2}{a^2} = (1 + pq)^2 + 4pq \quad \text{from part (i)}$$

$$\frac{x^2}{a^2} = (pq)^2 + 6pq + 1$$

$$\frac{x^2}{a^2} = \frac{y^2}{a^2} + \frac{6y}{a} + 1$$

$$x^2 = y^2 + 6ay + a^2$$

(d)(i) $$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = 2x^3 + 4x$$

$$\frac{1}{2}\dot{x}^2 = \frac{x^4}{2} + 2x^2 + \frac{C_1}{2}$$

$$\dot{x}^2 = x^4 + 4x^2 + C_1$$

When $t = 0$

$x = 1$ and $\dot{x} = 3$

$9 = 1 + 4 + C_1$

$C_1 = 4$

$$\dot{x}^2 = x^4 + 4x^2 + 4$$

$$\dot{x}^2 = (x^2 + 2)^2$$

$$\dot{x} = \pm(x^2 + 2)$$

$\dot{x} = x^2 + 2$ because of initial conditions

(ii) $$\frac{dx}{dt} = x^2 + 2$$

$$\frac{dt}{dx} = \frac{1}{x^2 + 2}$$

$$t = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C_2$$

$$C_2 = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$t = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\sqrt{2}t = \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) = \sqrt{2}t - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$x = \sqrt{2}\tan\left(\sqrt{2}t - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$$

Question 14 **(15 marks)** Begin a new page.

(a) An ice sculpture in the form of a sphere melts in such a way that it maintains its spherical shape. The volume of the sphere is decreasing at a constant rate of 2π cubic metres per hour. At what rate, in square metres per hour, is the surface area of the sphere decreasing at the moment when the radius is 5 metres? **2**

$V = \frac{4}{3}\pi r^3$ $\qquad \frac{dV}{dt} = -2\pi \ m^3/hr$

$S = 4\pi r^2$ $\qquad \frac{dS}{dt} = ? \ \ldots$ when $r = 5m$.

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$-2\pi = 4\pi r^2 \cdot \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-2\pi}{4\pi r^2}$

$\frac{dr}{dt} = \frac{-1}{2r^2}$ ✓

$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$

$\frac{dS}{dt} = 8\pi r \cdot \frac{-1}{2r^2}$

$\therefore \frac{dS}{dt} = -\frac{4\pi}{r}$

when $r = 5$:

$\frac{dS}{dt} = -\frac{4\pi}{5} \ m^2/hour$ ✓

(2)

(b) A team of 17 soccer players includes two Brown sisters and three Stefanovic sisters. How many different ways are there of choosing a group of 11 soccer players from the team, if the group can include no more than one of the Brown sisters and no more than two of the Stefanovic sisters? **2**

- Team = 17 ⇒ 2 x Brown and 3 x Stefanovic
- Choose a group of 11 players, such that:
  - → more than 1x Brown sister ⇒ (0 or 1)
  - → no more than 2x Stefanovic sisters ⇒ (0 or 1 or 2).
- Number of different combinations:

$\bar{B}\bar{S}$. $\quad {}^{12}C_{11} \times {}^{2}C_0 \times {}^{3}C_0 = 12 \times 1 \times 1 = 12$

$\bar{B}S$. $\quad {}^{12}C_{10} \times {}^{2}C_0 \times {}^{3}C_1 = 66 \times 1 \times 3 = 198$

$\bar{B}SS$. $\quad {}^{12}C_{9} \times {}^{2}C_0 \times {}^{3}C_2 = 220 \times 1 \times 3 = 660$ ✓

$B\bar{S}$. $\quad {}^{12}C_{10} \times {}^{2}C_1 \times {}^{3}C_0 = 66 \times 2 \times 1 = 132$

BS $\quad {}^{12}C_{9} \times {}^{2}C_1 \times {}^{3}C_1 = 220 \times 2 \times 3 = 1320$

BSS. $\quad {}^{12}C_{8} \times {}^{2}C_1 \times {}^{3}C_2 = 495 \times 2 \times 3 = 2970$

∴ Total no. of ways = 5292. ✓

(2)

Alternate Solution.

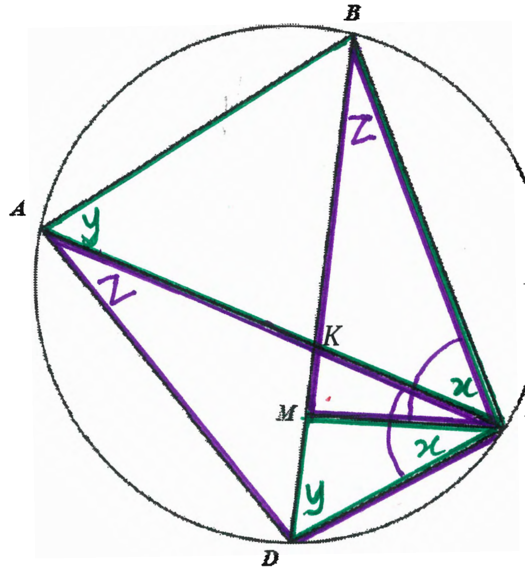
$${}^{17}C_{11} - [2B + 3S - (2B \text{ and } 3S)]$$

$$= {}^{17}C_{11} - [{}^{15}C_9 + {}^{14}C_8 - {}^{12}C_6]$$

$$= 12376 - 5005 - 3003 + 924$$

$$= 5292.$$

(c) In the diagram below, $ABCD$ is a cyclic quadrilateral and K is the intersection of the diagonals AC and BD. M is the point on BD such that $\angle ACB = \angle DCM$.



(i) Prove that $\dfrac{AC}{CD} = \dfrac{AB}{MD}$. **1**

In ΔABC and ΔDMC

$\angle ACB = \angle DCM = x$ (given)

$\angle BAC = \angle CDM = y$ (angles in same segment, standing on chord BC).

$\therefore \Delta ABC \,|||\, \Delta DMC$ (equiangular)

$\therefore \quad \dfrac{AC}{DC} = \dfrac{AB}{MD}$ (sides in same ratio)

✓ (1)

$\Rightarrow \quad AC \cdot MD = \boxed{AB \cdot CD}$ ..... (1)

(ii) Ptolemy's Theorem states that in a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the pairs of opposite sides, that is: $AC \times BD = \boxed{AB \times CD} + \boxed{BC \times AD}$. **2**

Prove Ptolemy's theorem.

In ΔACD and ΔBCM

$\angle DAC = \angle CBM = z$ (angles in same segment standing on chord DC).

$\angle ACD = \angle ACM + \angle D\overset{x}{C}M$ (adj. $\angle$s)

$= \angle ACM + \angle A\overset{x}{C}B$ ($\angle ACB = \angle DCM$ given)

$\therefore \angle ACD = \angle BCM$ (adj. $\angle$s)

$\therefore \Delta ACD \mathbin{|||} \Delta BCM$ (equiangular) ✓

$$\frac{AD}{BM} = \frac{AC}{BC}$$

$\boxed{AD.BC} = AC.BM$ .... ②

From ① and ②:

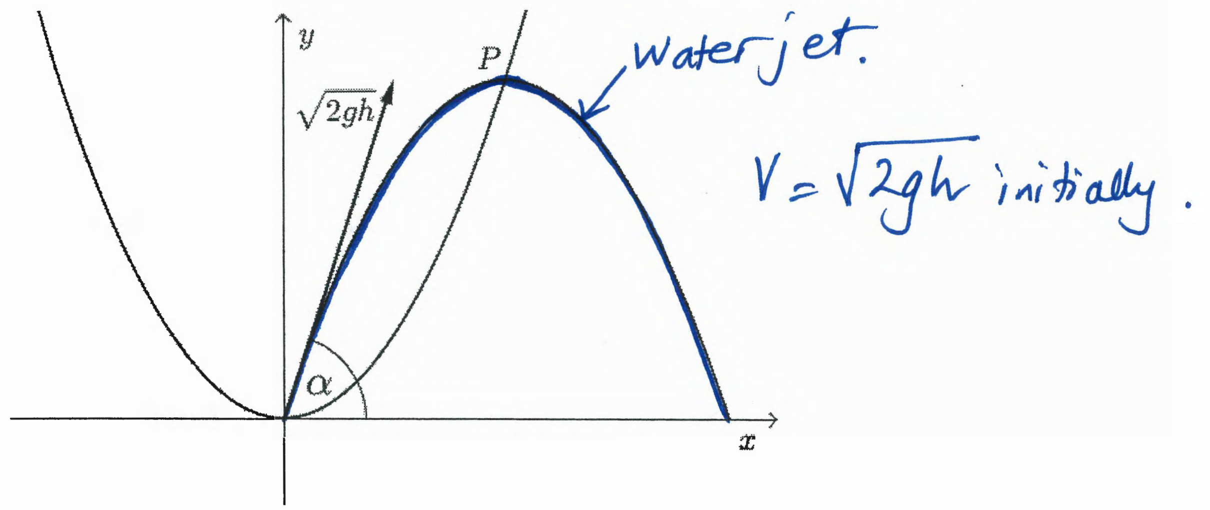
$\boxed{AB \times CD} + \boxed{BC \times AD} = AC \times MD + AC \times BM$

$= AC(MD + BM)$

$= AC \times BD$. ✓ ②.

* This was very challenging for most students. Two marks were given for the correct solution only.

(d) A cross-section of a valley is in the form of a parabola $x^2 = 4ay$ where a is a positive constant. A water cannon placed at the origin fires a jet of water with speed $\sqrt{2gh}$ at an angle α where $0 < \alpha < \frac{\pi}{2}$, h is a positive constant and g is the acceleration due to gravity.



The equations of motion of a projectile fired from the origin with initial velocity $V\ ms^{-1}$ at angle α to the horizontal are:

$$x = Vt\cos\alpha \text{ and } y = Vt\sin\alpha - \frac{1}{2}gt^2. \text{ (Do NOT prove these)}$$

(i) If the water jet strikes the wall of the valley at the point $P(X,Y)$ show that: **3**

$$X = \frac{4ah}{(a+h)\cot\alpha + a\tan\alpha}$$

(ii) Let $f(\theta) = (a+h)\cot\theta + a\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. **3**

Show that the minimum value of $f(\theta)$ occurs when $\tan\theta = \sqrt{\frac{a+h}{a}}$.

(iii) Hence or otherwise, show that the greatest value of X is given by: **2**

$$X = 2h\sqrt{\frac{a}{a+h}}.$$

End of paper

d) i) Need to eliminate t, V, g and y from equation of trajectory of water jet:

$$y = Vt\sin\alpha - \frac{gt^2}{2}$$

$$= V\sin\alpha\left(\frac{x}{V\cos\alpha}\right) - \frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2$$ ✓

$$= x\tan\alpha - \frac{g}{2}\cdot\frac{x^2}{2gh\cos^2\alpha}$$

$$\therefore y = x\tan\alpha - \frac{x^2}{4h}\cdot\sec^2\alpha$$

Students should substitute $V=\sqrt{2gh}$ at this step.

At $P(X,Y)$, $y = \frac{X^2}{4a}$:

$$\frac{X^2}{4a} = X\tan\alpha - \frac{X^2}{4h}\sec^2\alpha$$ ✓

$$X^2h = 4ahX\tan\alpha - aX^2\sec^2\alpha$$

$$\therefore X^2(h + a\sec^2\alpha) - 4ahX\tan\alpha = 0$$

$$X\left[X(h+a\sec^2\alpha) - 4ah\tan\alpha\right] = 0.$$

$X \neq 0 \quad \therefore \quad X = \frac{4ah\tan\alpha}{h + a\sec^2\alpha}$

$$= \frac{4ah\tan\alpha}{h + a(1+\tan^2\alpha)}$$

$$= \frac{4ah\tan\alpha}{(a+h) + a\tan^2\alpha}$$

(dividing by $\tan\alpha$)

$$\therefore X = \frac{4ah}{(a+h)\cot\alpha + a\tan\alpha}$$ ✓

Students had to show clear steps to reach final result for last mark.

(3)

d ii) $f(\theta) = (a+h)\cot\theta + a\tan\theta$

$= (a+h)\cos\theta(\sin\theta)^{-1} + a\tan\theta$

$f'(\theta) = -(a+h)\cos^2\theta(\sin\theta)^{-2} - (a+h)\sin\theta(\sin\theta)^{-1} + a\sec^2\theta.$

$= -(a+h)\cot^2\theta - (a+h) + a\sec^2\theta.$

$= -(a+h)[\cot^2\theta + 1] + a\sec^2\theta.$

$\therefore f'(\theta) = -(a+h)\operatorname{cosec}^2\theta + a\sec^2\theta$ ✓

For minimum $f'(\theta) = 0 \Rightarrow$

$(a+h)\operatorname{cosec}^2\theta = a\sec^2\theta$

$$\frac{(a+h)}{\sin^2\theta} = \frac{a}{\cos^2\theta}$$

$$\frac{a+h}{a} = \frac{\sin^2\theta}{\cos^2\theta}$$

$\therefore \tan^2\theta = \frac{a+h}{a} \Rightarrow \tan\theta = \sqrt{\frac{a+h}{a}}$ ✓ $, \ 0 < \theta < \frac{\pi}{2}.$

Students had difficulty proving a minimum occurs.

*To prove minimum show $f''(\theta) > 0$:

$f'(\theta) = -(a+h)(\sin\theta)^{-2} + a(\cos\theta)^{-2}$

$f''(\theta) = 2(a+h)(\sin\theta)^{-3}\cos\theta + 2a(\cos\theta)^{-3}\sin\theta.$

$$= \frac{2(a+h)}{\sin^2\theta}.\cot\theta + \frac{2a}{\cos^2\theta}.\tan\theta$$

$$f''\left(\sqrt{\frac{a+h}{a}}\right) = \frac{2(a+h)}{\sin^2\theta}.\sqrt{\frac{a}{a+h}} + \frac{2a}{\cos^2\theta}.\sqrt{\frac{a+h}{a}}$$ ✓

> 0 (since a, h are positive constants)

$\therefore$ minimum occurs when $\tan\theta = \sqrt{\frac{a+h}{a}}.$

(<u>OR</u>: $f''(\theta) = 2(a+h)\cot\theta\operatorname{cosec}^2\theta + 2a\sec^2\theta\tan\theta$

(Since $0 < \theta < \frac{\pi}{2}, \quad a, h > 0$)

$\operatorname{cosec}^2\theta > 0, \cot\theta > 0, \sec^2\theta > 0, \tan\theta > 0$

$\therefore f''(\theta) > 0$ at $\tan\theta = \sqrt{\frac{a+h}{a}}.$).

d ii) Given $f(\theta) = (a+h)\cot\theta + a\tan\theta$

→ rewrite $f(\theta)$ in terms of $t = \tan\theta$:-

Alternative Solution

$$f(\theta) = \frac{a+h}{\tan\theta} + a\tan\theta$$

$$= \frac{a+h}{t} + at$$

$$\therefore f(\theta) = \frac{a+h+at^2}{t}.$$

→ for minimum value of $f(\theta)$ find $\frac{d[f(\theta)]}{dt} = 0$:-

$$\frac{d[f(\theta)]}{dt} = \frac{(t)(2at) - (a+h+at^2)(1)}{t^2} \quad \text{(quotient rule)}$$

$$= \frac{2at^2 - a - h - at^2}{t^2}$$

$$0 = \frac{at^2 - a - h}{t^2}$$

$$0 = at^2 - (a+h)$$

$$t^2 = \frac{a+h}{a} \Rightarrow t = \sqrt{\frac{a+h}{a}} \quad \text{i.e. } \tan\theta = \sqrt{\frac{a+h}{a}}.$$

→ Justify this is a minimum show $\frac{d^2[f(\theta)]}{dt^2} > 0$:-

$$\frac{d^2[f(\theta)]}{dt^2} = \frac{(t^2)(2at) - (at^2 - a - h)(2t)}{t^4}$$

$$= \frac{2at + 2ht}{t^4} \Rightarrow \frac{2(a+h)}{t^3}$$

3

When $t = \sqrt{\frac{a+h}{a}}$, $\frac{d^2[f(\theta)]}{dt^2} = \frac{2(a+h)}{\left(\frac{a+h}{a}\right)^{3/2}} > 0$ (since $a, h > 0$).

∴ Minimum value of $f(\theta)$ occurs when $\tan\theta = \sqrt{\frac{a+h}{a}}$.

d) iii) Since $X = \dfrac{4ah}{(a+h)\cot\alpha + a\tan\alpha}$

then the greatest value of X occurs when $(a+h)\cot\alpha + a\tan\alpha$ is a minimum

That is, when $\tan\alpha = \sqrt{\dfrac{a+h}{a}}$ from ii).

Hence: $$X = \frac{4ah}{(a+h)\sqrt{\dfrac{a}{a+h}} + a\sqrt{\dfrac{a+h}{a}}}$$ ✓

$$= \frac{4ah}{\dfrac{\sqrt{a}\,(a+h)}{\sqrt{a+h}} + \dfrac{a\sqrt{a+h}}{\sqrt{a}}}$$

$$= \frac{4ah}{\sqrt{a}.\sqrt{a+h} + \sqrt{a}.\sqrt{a+h}}$$

$$= \frac{4ah}{2\sqrt{a}.\sqrt{a+h}} \qquad \left(\frac{a}{\sqrt{a}} = \frac{a^1}{a^{1/2}} = a^{1/2} = \sqrt{a}\right).$$

$$= 2.\frac{a}{\sqrt{a}}.\frac{h}{\sqrt{a+h}}$$

$$= \frac{2h\sqrt{a}}{\sqrt{a+h}}$$ ✓ ②.

$\therefore\ X = 2h\sqrt{\dfrac{a}{a+h}}$, as required.

Students had to show clear steps to reach final result for last mark