File copy

139

SYDNEY GRAMMAR SCHOOL

1990 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

# MATHEMATICS (3 UNIT)

Time Allowed : 2 hours 8th August, 1990

* All questions may be attempted.
* All questions are of equal value.
* All necessary working should be shown.
* Marks may not be awarded for careless or badly arranged work.
* Approved calculators may be used.
* Hand in each question separately.

---

The following Standard Integrals may be used.

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1, \quad x \neq 0, \text{ if } n<0.$$
$$\int \frac{1}{x} \, dx = \log_e x, \quad x>0.$$
$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0.$$
$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0.$$
$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0.$$
$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0.$$
$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0.$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0.$$
$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a.$$
$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \log_e \left[x + \sqrt{x^2 - a^2}\right], \quad |x| > |a|.$$
$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \log_e \left[x + \sqrt{x^2 + a^2}\right].$$

## QUESTION 1

(a) Find primitives of:

(i) $\dfrac{5}{x^2 + 7}$ ;

(ii) $(3x - 4)^9$ ;

(iii) $\dfrac{\cos x}{\sin x + 1}$.

(b) Use the substitution u = x - 1 to evaluate:

$$\int_0^1 x(x - 1)^5 \, dx.$$

(c) What is the domain of log (x + 1)?

(d) Find (as a rational number) the term independent of x in the expansion of:

$$\left(\frac{x}{3} - \frac{2}{x^2}\right)^9.$$

## QUESTION 2

(a) The cubic equation $3x^3 + 4x^2 - 2x + 1 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

(i) Find $\alpha + \beta + \gamma$.

(ii) Find $\alpha\beta + \beta\gamma + \gamma\alpha$.

(iii) Show $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$, and hence find $\alpha^2 + \beta^2 + \gamma^2$.

(b) The quartic polynomial f(x) is given by: $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$.

(i) Show that x - 1 and x + 2 are factors of f(x).

(ii) Using sum and product of roots, or otherwise, factor f(x) into linear factors.

(c) Consider the equation: $x^3 - 6x^2 + 4 = 0$.

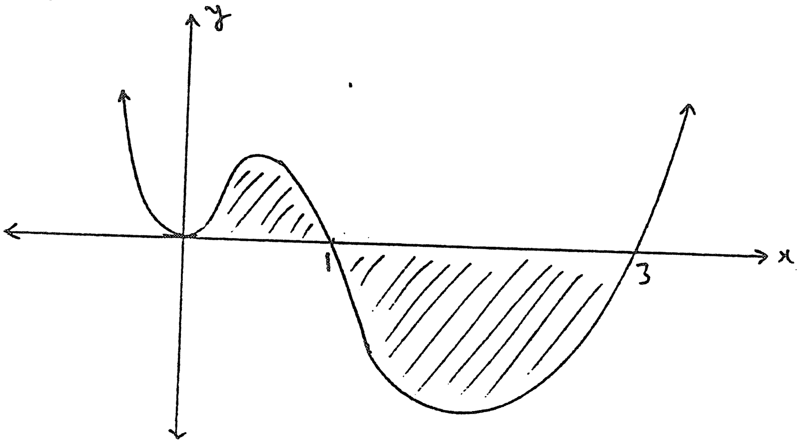
(i) Show that the equation has a solution between -1 and 0, a solution between 0 and 1, and a solution between 5 and 6.

(ii) Hence deduce how many solutions the equation has.

(iii) One of the solutions is near $x = 1$. Find a better approximation for this solution by using one application of Newton's method taking $x_0 = 1$ (leave your answer as a fraction).

**QUESTION 3**

(a)

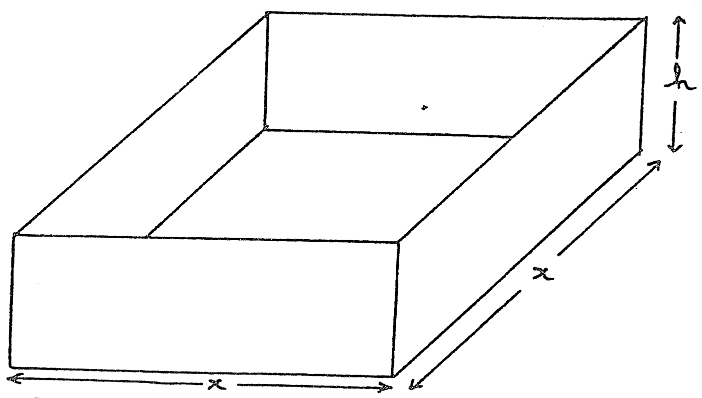



The diagram shows a sketch, not to scale, of the curve:

$y = x^2 (x - 1)(x - 3)$.

Find the shaded area.

(b)




The diagram shows a large steel box, open at the top. It has a square base, of side length x metres and walls of height h metres.
The volume of the box is $4m^3$.

(i) Show that the total surface area A of the base and four walls is given by:

$$A = x^2 + \frac{16}{x}$$

cubic metres.

(ii) Find the value of x which makes A a minimum, and show that for this value of x, the height is half the side length of the base.

## QUESTION 4

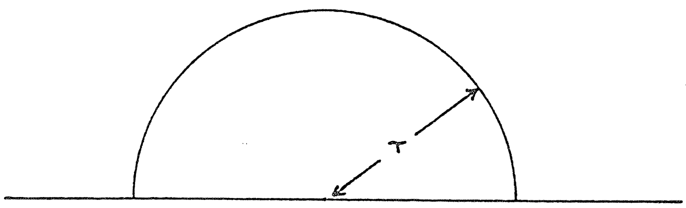
(a) (i) Draw a neat sketch of $y = \cos^{-1}x$, using the same scale on both axes. Mark clearly on your diagram the scale, and any intercepts and end-points.

(ii) Show that the equation of the tangent to $y = \cos^{-1}x$ at the point where $x = -\frac{1}{2}$ is:

$$2\sqrt{3}x + 3y = 2\pi - \sqrt{3}.$$

(iii) Find the x and y intercepts of this tangent correct to 3 significant figures, and hence draw this tangent on your sketch of $y = \cos^{-1}x$.

(b)




The diagram shows a hemispherical soap bubble which is expanding so that its volume increases at a steady rate of $4\text{cm}^3$ per second.

(i) Show that, at the time when the bubble's radius is r cm, its volume V $\text{cm}^3$ is given by:

$$V = \frac{2\pi r^3}{3},$$

and its total surface area A $\text{cm}^2$ (including the flat base) by :

$$A = 3\pi r^2.$$

(ii) Find the rate of change of :

(α) the radius, and

(β) the surface area,

when the radius is 2 cm.

**QUESTION 5**

(a) (i) Differentiate $x^2 \log x$.

(ii) Hence find a primitive of $x \log x$.

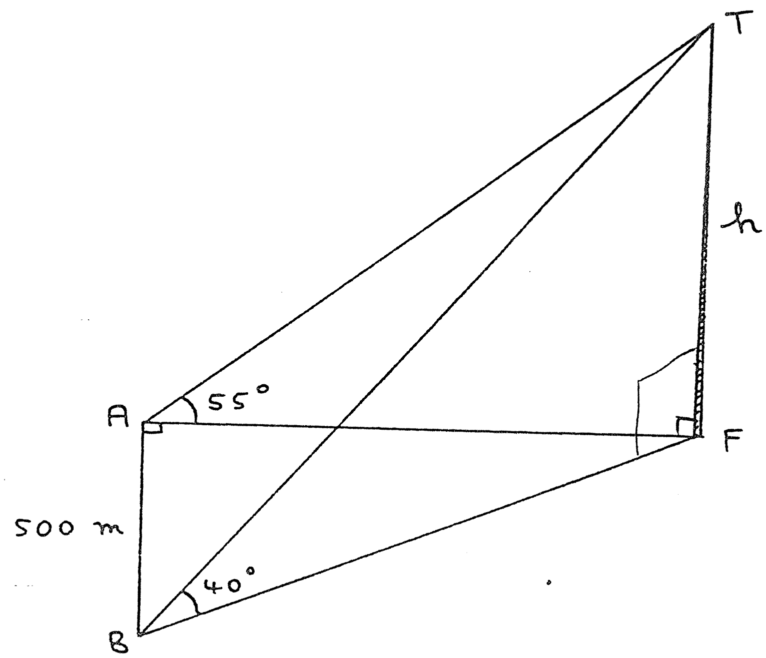
(b) A particle is moving in simple harmonic motion about the origin 0. Its displacement x cm from 0 at time t seconds satisfies :

$$\ddot{x} = -\pi x .$$

(i) If the velocity of the particle is v cm per second, derive an expression for $v^2$ as a function of x, given that the amplitude of the motion is 4 cm.

(ii) Where is the particle when its speed is 5 cm per second (answer to 4 significant figures) ?

(c)




The diagram shows a tower of height h metres standing on level ground. The angles of elevation of the top T of the tower from two points A and B on the ground nearby are 55° and 40° respectively. The distance AB is 500 metres, and the interval AB is perpendicular to the interval AF (where F is foot of the tower).

(i) Find AT and BT in terms of h.

(ii) Hence find h, first in exact form, then correct to the nearest centimetre.

## QUESTION 6

(a) A geometric series has first term 1 and ratio ¾ .

(i) Find the limiting sum .

(ii) Find an expression for the sum $s_n$ of the first n terms of the series.

(iii) Hence find the smallest value of n so that $s_n$ differs from the limiting sum by less than $10^{-15}$ .

(b) Under what conditions will the geometric series :

$$1 + (\sqrt{x} + \tfrac{1}{2}) + (\sqrt{x} + \tfrac{1}{2})^2 + \ldots$$

have a limiting sum?

(c) The points P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ lie on the parabola $\wp$ with equation $x^2 = 4ay$.

(i) Find the point A of intersection of the tangents to $\wp$ at P and at Q (you may use the fact that the tangent to $\wp$ at any point T $(2at, at^2)$ on $\wp$ has equation $y = tx - at^2$).

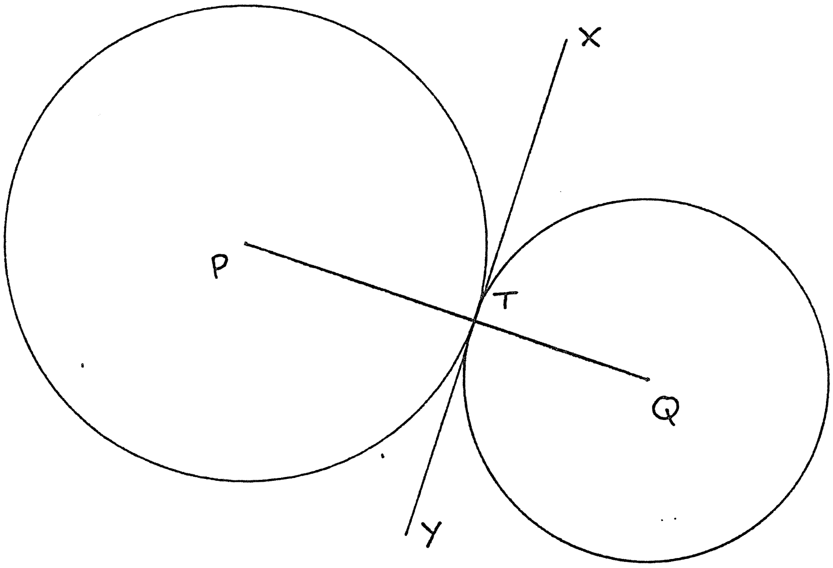
(ii) Suppose further that A lies on the line containing the latus rectum of $\wp$ .

(α) Show that $pq = 1$.

(β) Show that the chord PQ intersects the axis of symmetry of $\wp$ on the directrix.

## QUESTION 7

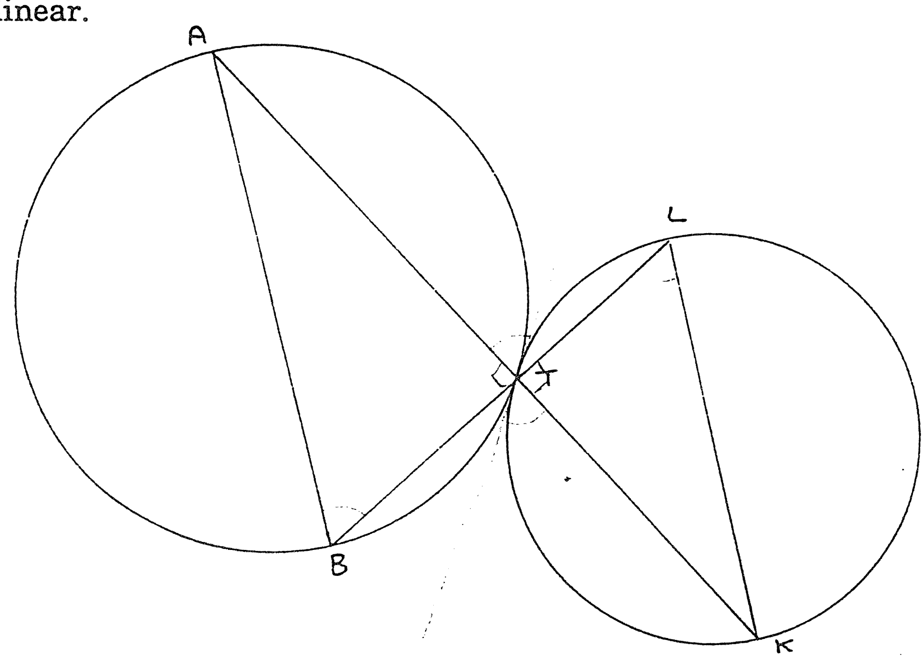
(a) (i)




The diagram shows two circles with centres P and Q respectively, and touching externally at T. XTY is the common tangent at T.

Copy the diagram into your workbook, and then prove that P, T and Q are collinear.

(ii)




The diagram shows two circles touching externally at T. AB is any diameter of the first circle, and AT and BT are produced to meet the second circle again at K and L respectively.

Copy the diagram into your workbook. Then prove that:

(α) KL is a diameter of the second circle, and
(β) LK is parallel to AB.

Question 7 (continued)

(b) A new species of ant was once introduced onto the uninhabited island of Zildon. Earlier studies of these ants suggested that once they had begun to spread, the number N of nests on the island at any time t will tend towards a stable value A according to the differential equation :

$$\frac{dN}{dt} = k(N-A),$$

where k is a negative constant.

(i) Prove that $N = A + (N_0 - A)e^{kt}$, where $N_0$ is a constant, is a solution of the differential equation.

(ii) A few years later, a scientific survey ship visited the island of Zildon in the first week of May on three successive years to count the number of nests. The values they obtained were successively 1100, 1500 and 1800.

Use these values with the equations above to find estimates for the stable population A and the constant k (give k correct to 3 significant figures).

W.M.P.