QUESTION ONE

(a) Evaluate:

(i) $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$,

(ii) $\displaystyle\int_{-2}^{2\sqrt{3}} \frac{1}{4+x^2}\,dx$.

(b) Find the gradient of the tangent to the curve $y = \tan^{-1}(\sin x)$ at $x = 0$.

(c) Solve $\dfrac{1}{x+1} < 3$.

(d) Give the general solution of the equation $\cos(\theta + \frac{\pi}{4}) = \dfrac{1}{\sqrt{2}}$.

(e) If $f(x) = 8x^3$ then find the inverse function $f^{-1}(x)$.

QUESTION TWO

(a) Prove the identity $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$.

(b) The equation $x^3 - 103x + 102{\cdot}5 = 0$ has a root near $x = 1$. Take $x = 1$ as a first approximation and use Newton's method once to obtain a closer approximation to this root.

(c) (i) Sketch the graph of $y = |2x - 4|$.

(ii) Using your graph or otherwise solve the inequation $|2x - 4| > x$.

(d) (i) Express $7\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° \le \alpha \le 90°$.

(ii) Hence solve $7\cos\theta - \sin\theta = 5$ for $0° \le \theta \le 360°$, giving your answer to the nearest degree.

QUESTION THREE

(a) Consider the function $f(x) = 3\sin^{-1}(\frac{x}{2} - 1)$.

(i) State the domain of $f(x)$.

(ii) State the range of $f(x)$.

(iii) Sketch the graph of $y = f(x)$.

(iv) Evaluate $f(1)$.

(Exam continues next page ...)

(b) Find $\int x\sqrt{1-x}\,dx$, using the substitution $u = 1 - x$.

(c) Find the values of the constants a and b if $x^2 - 2x - 3$ is a factor of the polynomial $P(x) = x^3 - 3x^2 + ax + b$.

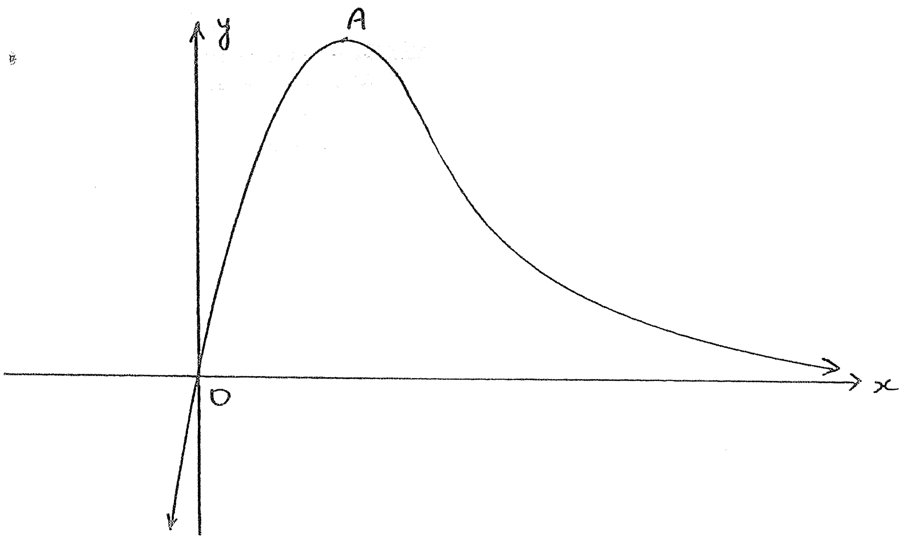
(d) (i) If $x > 0$, prove that $\frac{d}{dx}\left(\tan^{-1} x + \tan^{-1} \frac{1}{x}\right) = 0$.

(ii) Hence find the value of $\tan^{-1} x + \tan^{-1} \frac{1}{x}$ for $x > 0$.

QUESTION FOUR

(a) Find $\cos\theta$ if $\theta = \cos^{-1} \frac{24}{25} - \sin^{-1} \frac{15}{17}$.

(b)



The diagram shows the graph of the function $y = xe^{-x}$. A is a stationary point on the curve.

(i) Show that A is the point $(1, \frac{1}{e})$.

(ii) State the range of the function $y = xe^{-x}$.

(iii) How many real roots are there to the equation $xe^{-x} = k$ if:

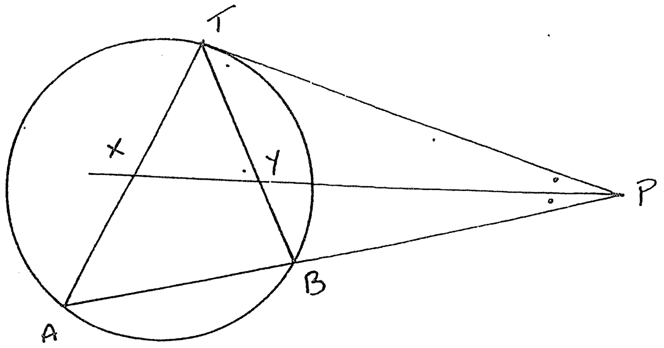
(α) $0 < k < \frac{1}{e}$,

(β) $k \leq 0$,

(γ) $k > \frac{1}{e}$?

(Exam continues overleaf ...)

(c)



The tangent at T on the circle meets a chord AB produced to P. The bisector of $\angle TPA$ meets TA and TB at X and Y respectively.

(i) Give the reason why $\angle PTB = \angle TAB$.

(ii) Prove $TX = TY$.

(iii) Prove $\dfrac{TX}{XA} = \dfrac{TP}{PA}$.

QUESTION FIVE

(a) A particle P moves along the x-axis so that at time t seconds it is x cm from the origin O and its velocity is v cm/s. Initially the particle is at rest at the origin.

(i) If the acceleration of P is given by $\ddot{x} = 4(40 - x)$ cm/s^2, use $\ddot{x} = \dfrac{d}{dx}(\frac{1}{2}v^2)$ to show $v^2 = 4(80x - x^2)$.

(ii) Prove that P moves in the interval $0 \le x \le 80$.

(iii) Find the maximum velocity of the particle and where the maximum occurs.

(b) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$.

(i) Show that the equation of the normal to the parabola at the point P is $x + py = 2ap + ap^3$.

(ii) If the normal at P cuts the y-axis at Q, show that the co-ordinates of Q are $(0, 2a + ap^2)$.

(iii) Show that the co-ordinates of R which divides the interval PQ externally in the ratio $2:1$ are $(-2ap, 4a + ap^2)$.

(iv) Find the Cartesian equation of the locus of R and describe this locus in geometric terms.

(v) Show that if the normal at P passes through a given point (h, k) then p must be a root to the equation $ap^3 + (2a - k)p - h = 0$.

(vi) What is the maximum number of normals of the parabola $x^2 = 4ay$ which can pass through any given point? Give reasons for your answer.

(Exam continues next page ...)

QUESTION SIX

(a) The rate at which a body cools in air is proportional to the difference between its temperature T and the constant temperature $20°C$ (in this case) of the surrounding air. This can be expressed by the differential equation:

$$\frac{dT}{dt} = -k(T - 20).$$

The original temperature of a heated metal bar was $100°C$. The bar cools to $70°C$ in 10 minutes.

(i) Show that $T = 20 + Ae^{-kt}$ is a solution to the differential equation.

(ii) Show $A = 80$.

(iii) Find the value of k.

(iv) Find the time taken for the temperature of the bar to reach $60°C$. (Give your answer to the nearest minute.)

(b) Suppose that $(5 + 2x)^{12} = \sum_{k=0}^{12} a_k x^k$.

(i) Use the binomial theorem to write an expression for a_k.

(ii) Show that $\frac{a_{k+1}}{a_k} = \frac{24 - 2k}{5k + 5}$.

(c) Consider the geometric series $S = 1 + (1 + x) + (1 + x)^2 + \ldots + (1 + x)^n$.

(i) Show that $S = \frac{(1 + x)^{n+1} - 1}{x}$.

(ii) Hence show that

$$S = {}^{n+1}C_1 + {}^{n+1}C_2 x + \ldots + {}^{n+1}C_{r+1} x^r + \ldots + {}^{n+1}C_{n+1} x^n.$$

(iii) Hence prove

$${}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \ldots + {}^{r}C_r = {}^{n+1}C_{r+1}.$$

(Exam continues overleaf ...)

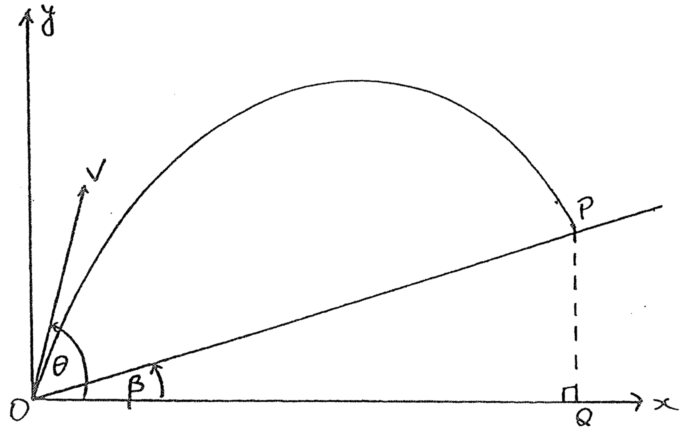
QUESTION SEVEN

(a) Consider the function $y = 2\sin(x-\beta)\cos x$, where $0 < \beta < \frac{\pi}{2}$.

(i) Show that $\dfrac{dy}{dx} = 2\cos(2x-\beta)$.

(ii) Hence or otherwise, show that $2\sin(x-\beta)\cos x = \sin(2x-\beta) - \sin\beta$.

(b)



A projectile is fired from the origin with a velocity V and an angle of elevation θ, where $\theta \neq 90°$. You may assume that:

$$x = Vt\cos\theta \qquad \text{and} \qquad y = -\tfrac{1}{2}gt^2 + Vt\sin\theta,$$

where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing and g is the acceleration due to gravity.

(i) Show that the cartesian equation of the flight of the projectile is

$$y = x\tan\theta - \frac{g}{2V^2\cos^2\theta}x^2.$$

(ii) Suppose the projectile is fired up a plane inclined at β to the horizontal so that $0° < \beta < \theta$. If the projectile strikes the plane at $P(h,k)$ show that

$$h = \frac{(\tan\theta - \tan\beta)2V^2\cos^2\theta}{g}.$$

(iii) Hence show that the range OP of the projectile can be given by:

$$OP = \frac{2V^2\sin(\theta-\beta)\cos\theta}{g\cos^2\beta}.$$

(iv) By referring to (ii) of part (a) or otherwise, show that the maximum value of the range OP is given by $\dfrac{V^2}{g(1+\sin\beta)}$.

(v) If the angle of inclination of the plane is 14°, at what angle to the horizontal should the projectile be fired in order to attain maximum range?

JCM

3 unit Trial 1994
Recommended Marking Scheme.

15 marks / question.

1 (a) (i) $\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\left[\ln(x^2+1)\right]_0^1$ ✓
$= \frac{1}{2}(\ln 2 - \ln 1)$
$= \frac{1}{2}\ln 2$ ✓

(ii) $\int_{-2}^{2\sqrt{3}} \frac{1}{4+x^2}\,dx = \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_{-2}^{2\sqrt{3}}$ ✓
$= \frac{1}{2}\left(\tan^{-1}\sqrt{3} - \tan^{-1}-1\right)$ ✓
$= \frac{1}{2}\left(\pi/3 + \pi/4\right)$ ✓
$= \frac{7\pi}{24}$ ✓ 5m

(b) $y = \tan^{-1}(\sin x)$
$\frac{dy}{dx} = \frac{\cos x}{1+\sin^2 x}$ ✓
$f'(0) = 1$ ✓ 2m

(c) $\frac{1}{x+1} < 3$
$x+1 < 3(x+1)^2$ ✓
$3x^2+6x+3-x-1>0$
$3x^2+5x+2>0$ ✓
$(3x+2)(x+1)>0$ 3m
$x<-1$ or $x>-2/3$ ✓

(d) $\cos\left(\theta+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$\theta + \pi/4 = 2n\pi \pm \pi/4$ ✓ 3m
$\theta = 2n\pi$ or $2n\pi - \pi/2$ ✓✓

(e) f: let $y = 8x^3$
f^{-1}: then $x = 8y^3$ ✓
$y^3 = \frac{x}{8}$
$y = \frac{1}{2}\sqrt[3]{x}$ 2m
or $f^{-1}(x) = \frac{1}{2}\sqrt[3]{x}$ ✓

(a) R.T.P. $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

$$\text{L.H.S.} = \frac{2\sin x \cos x}{1+2\cos^2 x - 1} \quad ✓✓$$

$$\left.\begin{aligned} &= \frac{2\sin x\cos x}{2\cos^2 x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}\right\} \quad ✓ \qquad 3m$$

(b) let $f(x) = x^3 - 103x + 102{\cdot}5$

$f'(x) = 3x - 103$

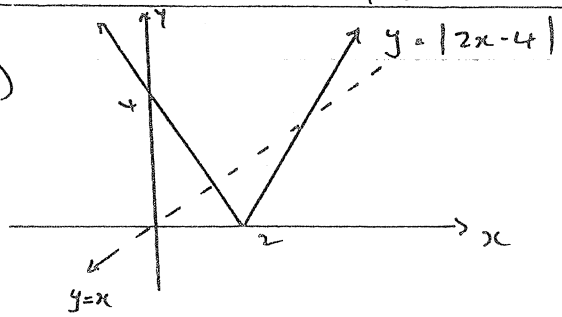
$f(1) = {\cdot}5$

$f'(1) = -100$

$$\left.\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{{\cdot}5}{-100} \end{aligned}\right\} \quad ✓ ✓ \qquad 3m$$

$$= 1{\cdot}005 \quad ✓$$

(c)



✓

4m

for points of intersection

$$\left.\begin{aligned} x &= 2x - 4 \quad \text{or} \quad x = -(2x-4) \\ x &= 4 \qquad\qquad\quad x = 4/3 \end{aligned}\right\} \quad ✓✓$$

So, for $|2x - 4| > x$

$x < 4/3$ or $x > 4$ ✓

(d)(i) $7\cos\theta - \sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$ ✓

$R\cos\alpha = 7$

$R\sin\alpha = 1$

$R = \sqrt{7^2 + 1^2} = 5\sqrt{2}$ ✓

5m

$$\left.\begin{aligned} \alpha &= \tan^{-1}\tfrac{1}{7} \\ &\doteqdot 8^\circ \end{aligned}\right\} \quad ✓$$

(ii) $7\cos\theta - \sin\theta = 5$

$5\sqrt{2}\cos(\theta + \alpha) = 5$

$\cos(\theta + \alpha) = 1/\sqrt{2}$ ✓

$\theta = 37^\circ, 308^\circ$ ✓

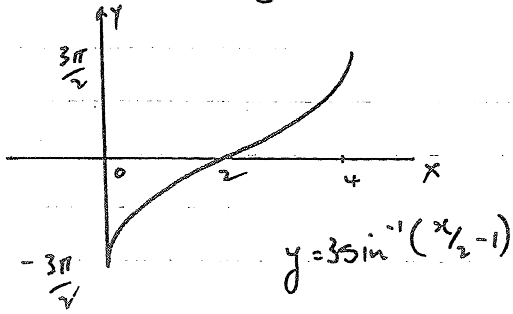
Q3

(a) (i) $D: \ -1 \le \frac{x}{2} - 1 \le 1$

$0 \le x \le 4$ ✓

(ii) $R: \ -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ ✓

(iii)



✓ ✓

5m

(iv) $f(1) = 3\sin^{-1}(-\frac{1}{2})$ ✓

$= -\frac{\pi}{2}$

(b) $I = \int x\sqrt{1-x}\, dx$

$u = 1-x$

$x = 1-u$

$\frac{du}{dx} = -1$ ✓

$I = \int (1-u)u^{1/2} \cdot -du$

$= \int u^{3/2} - u^{1/2}\, du$ ✓

3m

$= \frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} + c$

$= \frac{2}{5}\sqrt{(1-x)^5} - \frac{2}{3}\sqrt{(1-x)^3} + c$ ✓

(c) since $x^2 - 2x - 3 = (x-3)(x+1)$ and (x^2-2x-3) is a factor of $P(x)$ then $(x-3)$ & $(x+1)$ are factors of $P(x)$ ✓

$\therefore \ P(3) = 0$ i.e. $27 - 27 + 3a + b = 0$

$3a + b = 0$ — ① ✓

$P(-1) = 0$ i.e. $-1 - 3 - a + b = 0$

$a - b = -4$ — ② ✓

4m

solve ① & ② $a = -1$

$b = 3$ ✓

(d) $\frac{d}{dx}\left(\tan^{-1}x + \tan^{-1}\frac{1}{x}\right) = \frac{1}{1+x^2} + \frac{-1/x^2}{1+\frac{1}{x^2}}$

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$ ✓

$= 0$

(ii) since $\frac{d}{dx}\left(\tan^{-1}x + \tan^{-1}\frac{1}{x}\right) = 0$

3m

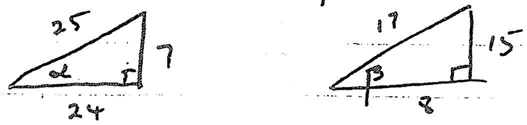
$\tan^{-1}x + \tan^{-1}\frac{1}{x} = c$ ✓

let $x = 1$: $\tan^{-1}1 + \tan^{-1}1 = c$

$c = \frac{\pi}{2}$ ✓

Q4

(a) let $\alpha = \cos^{-1}\frac{24}{25}$ & $\beta = \sin^{-1}\frac{15}{17}$



✓✓

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha.\sin\beta$$
$$= \frac{24}{25}\,\frac{8}{17} + \frac{7}{25}\,\frac{15}{17}$$ ✓

4 m

$$= \frac{295}{425}$$ ✓

(b)(i) $y = xe^{-x}$

$$\frac{dy}{dx} = e^{-x} - xe^{-x}$$
$$= e^{-x}(1-x)$$

for stationary points $\frac{dy}{dx} = 0$

$$e^{-x}(1-x) = 0$$
$$x = 1$$

when $x = 1$, $y = e^{-1}$

$\therefore$ A is $(1, 1/e)$ ✓✓

(ii) R : $y \le 1/e$ ✓

(iii) (α) 2 ✓

(β) 1 ✓

(γ) 0 ✓

6 m

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c (i) $\angle PTB = \angle TAB$ (alternate segment theorem) ✓

(ii) $\angle TYX = \angle PTB + \angle TPY$ (exterior $\angle$ ΔTYP)

$\angle TXY = \angle TAB + \angle XPA$ (ext. $\angle$ ΔPAX)

Since $\angle PTB = \angle TAB$ & $\angle TPY = \angle APX$

$\angle TYX = \angle TXY$

So $TX = TY$ (isosceles Δ) ✓✓

(iii) $\Delta TPY \,|||\, \Delta APX$ (A.A.) ✓

So $\frac{TY}{AX} = \frac{TP}{AP}$ (corresp. sides of sim. Δ's in proportion)

$\therefore \frac{TX}{XA} = \frac{TP}{PA}$ (fr. (ii)) ✓

5 m

Q5

(a) (i) $\ddot{x} = 4(40-x)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4(40-x)$

$\frac{1}{2}v^2 = 4(40x - \frac{1}{2}x^2) + c$

when $x=0$, $v=0$ $\therefore c=0$

$\frac{1}{2}v^2 = 4(40x - \frac{1}{2}x^2)$

$v^2 = 4(80x - x^2)$

(ii)

$v^2 = 4(80x - x^2)$

since $v^2 \geq 0$ for all x

then v^2 is defined for $0 \leq x \leq 80$

hence the particle moves in the interval $0 \leq x \leq 80$.

(iii) fr. graph max velocity when $x = 40$

i.e. $v^2 = 4(80 \times 40 - 40^2)$

$= 80$.

max vel. 80 cm/s

when $x = 40$ cm.

5m

(b) (i) $x^2 = 4ay$

grad. of tangent at $P = p$

grad. of normal at $P = -1/p$

equn. of normal at P:

$y - ap^2 = -1/p\,(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$

(ii) for Q let $x = 0$

$py = 2ap + ap^3$

$y = 2a + ap^2$

hence Q is $(0, 2a + ap^2)$

(iii) $x = \dfrac{-mx_2 + nx_1}{-m+n}$

$= \dfrac{0 + 1 \times 2ap}{-2+1}$

$= -2ap$

$y = \dfrac{-2(2a + ap^2) + ap^2}{-1}$

$= 4a + ap^2$

$\therefore$ R is $(-2ap, 4a + ap^2)$

(iv) $x = -2ap$

$p = \dfrac{-x}{2a}$

$y = 4a + ap^2$

$= 4a + a\left(\dfrac{x}{-2a}\right)^2$

$= 4a + x^2/4a$

$x^2 = 4a(y - 4a)$

parabola

vertex $(0, 4a)$

focal length $= a$

(v) since $x + py = 2ap + ap^3$ is through (h, k)

$h + pk = 2ap + ap^3$

$ap^3 + (2a - k)p - h = 0$

(vi) Since $ap^3 + (2a-h)p - h = 0$ is a cubic equation in p the maximum number of solutions for p is 3. Hence the maximum number of points for P is 3.

10m

Q6

(a) (i) $T = 20 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T-20)$$

(ii) when $t=0$, $T=100$

So, $100 = 20 + Ae^{0}$

$$A = 80$$

(iii) when $t = 10$, $T = 70$

$$70 = 20 + 80e^{-10k}$$

$$e^{-10k} = 5/8$$

$$k = -\frac{1}{10} \ln 5/8$$

(iv) when $T = 60$

$$60 = 20 + 80e^{-kt}$$

$$e^{-kt} = \cdot 5$$

$$t = \frac{10 \ln \cdot 5}{\ln 5/8}$$

$$\doteqdot 15 \text{ min}.$$

6m

(b) (i) $U_{k+1} = {}^{12}C_k 5^{12-k} (2x)^k$

$$\therefore a_k = {}^{12}C_k 5^{12-k} 2^k$$

(ii)
$$\frac{a_{k+1}}{a_k} = \frac{{}^{12}C_{k+1} 5^{11-k} 2^{k+1}}{{}^{12}C_k 5^{12-k} 2^k}$$

$$= \frac{\dfrac{12!}{[12-(k+1)]!\,(k+1)!}}{\dfrac{12!}{(12-k)!\,k!}} \quad \frac{2}{5}$$

$$= \frac{(12-k)}{(k+1)} \frac{2}{5}$$

$$= \frac{24-2k}{5k+5}$$

3m

(c) (i) $S = 1 + (1+x) + (1+x)^2 + \cdots + (1+x)^n$ — ①

$a = 1$, $r = (1+x)$, no. terms $= n+1$

$$\therefore S = \frac{a(r^{n+1} - 1)}{r - 1}$$

$$= \frac{1\{(1+x)^{n+1} - 1\}}{(1+x) - 1}$$

$$= \frac{(1+x)^{n+1} - 1}{x}$$

✓

(ii) $$S = \frac{1 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + \cdots + {}^{n+1}C_{n+1} x^{n+1} - 1}{x}$$ ✓

$$= {}^{n+1}C_1 + {}^{n+1}C_2 x \cdots + {}^{n+1}C_{r+1} x^r + \cdots + {}^{n+1}C_{n+1} x^n$$ — ② ✓

(iii) equate terms in x^r from ① & ② ✓

from ① take co-efft of x^r from $(1+x)^r + (1+x)^{r+1} + \cdots + (1+x)^n$

i.e. ${}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^nC_r$ ✓

counting back this is equivalent to

${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \cdots + {}^rC_r$

from ② co-efft of x^r is ${}^{n+1}C_{r+1}$ ✓

So $${}^{n+1}C_{r+1} = {}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \cdots + {}^rC_r$$

6m

Q7

(a) $y = 2\sin(x-\beta)\cos x$

(i) $\dfrac{dy}{dx} = 2[\cos(x-\beta)\cos x - \sin x \sin(x-\beta)]$

$= 2\cos(x-\beta+x)$

$= 2\cos(2x-\beta)$ ✓

(ii) $\displaystyle\int 2\cos(2x-\beta)\,dx = 2.\left[\tfrac{1}{2}\sin(2x-\beta)+c\right]$

$= \sin(2x-\beta) + K$ ✓

$\therefore\ 2\sin(x-\beta)\cos x = \sin(2x-\beta) + K$

let $x=0$: $\quad 2\sin(-\beta) = \sin(-\beta) + K$

$K = \sin(-\beta)$

$= -\sin\beta$

So, $2\sin(x-\beta)\cos x = \sin(2x-\beta) - \sin\beta$ ✓

(b) (i) $t = \dfrac{x}{v\cos\theta}$

$\therefore\ y = -\tfrac{1}{2}g\left(\dfrac{x}{v\cos\theta}\right)^2 + v\left(\dfrac{x}{v\cos\theta}\right)\sin\theta$

$= x\tan\theta - \dfrac{g}{2v^2\cos^2\theta}x^2 \quad$ — (A) ✓

(ii) for P: solve (A) & $y = x\tan\beta$

i.e. $x\tan\beta = x\tan\theta - \dfrac{g}{2v^2\cos^2\theta}x^2$ ✓

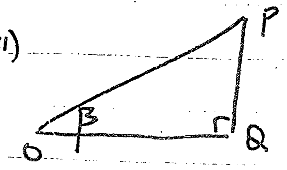
$\dfrac{g}{2v^2\cos^2\theta}x^2 + x(\tan\beta - \tan\theta) = 0$

$x\left[\dfrac{gx}{2v^2\cos^2\theta} - (\tan\theta - \tan\beta)\right] = 0$ ✓

$x = 0$ or $\dfrac{(\tan\theta - \tan\beta)\,2v^2\cos^2\theta}{g}$

So, $h = \dfrac{(\tan\theta - \tan\beta)\,2v^2\cos^2\theta}{g}$ ✓

(iii)



$$OP = \frac{OQ}{\cos\beta} = \frac{h}{\cos\beta}$$

So,
$$OP = \frac{2v^2}{g}\left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\beta}{\cos\beta}\right)\frac{\cos^2\theta}{\cos\beta}$$

$$= \frac{2v^2}{g}\left(\frac{\sin\theta.\cos\beta - \cos\theta\sin\beta}{\cos\theta.\cos^2\beta}\right)\cos^2\theta$$

$$= \frac{2v^2}{g}\,\frac{\sin(\theta-\beta)\cos\theta}{\cos^2\beta}$$

(iv) from (a) (ii)

$$OP = \frac{2v^2\sin(\theta-\beta)\cos\theta}{g\cos^2\beta}$$

$$= \frac{v^2[\sin(2\theta-\beta) - \sin\beta]}{g\cos^2\beta}$$

for maximum $\sin(2\theta-\beta) = 1$

So
$$OP = \frac{v^2[1-\sin\beta]}{g\cos^2\beta}$$

$$= \frac{v^2(1-\sin\beta)}{g(1-\sin^2\beta)}$$

$$= \frac{v^2}{g(1+\sin\beta)}$$

(v) for max. range $2\theta - \beta = 90°$

$$2\theta = (90°+14)°$$

$$\theta = 52°$$

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