02

FORM VI MATHEMATICS EXTENSION 1

Time allowed: 2 hours (plus 5 minutes reading) **Exam date:** 15th August 2002

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

Checklist:

SGS Examination booklets required — seven 4-page books per boy.
120 candidates.

QUESTION ONE (Start a new answer booklet)

Marks

(a) Evaluate $\sum_{n=1}^{4} n^2$. (Show your working.) $\boxed{1}$

(b) Solve the inequation $\dfrac{x}{x-3} > 2$. $\boxed{3}$

(c) (i) Differentiate $y = e^{7x}$. $\boxed{1}$

(ii) Hence find the equation of the tangent to $y = e^{7x}$ at $x = 1$. $\boxed{3}$

(d) Simplify the expression $\dfrac{{}^nC_2}{{}^nC_1}$. $\boxed{2}$

(e) Write down the general solution of the equation $\cos 3x = \cos \frac{\pi}{5}$. $\boxed{2}$

QUESTION TWO (Start a new answer booklet)

(a) The radius of a circular oil slick is increasing at 0·1m/s.

Marks

(i) Show that the rate of increase of its area is given by $\dfrac{dA}{dt} = 0{\cdot}2\pi r$. $\boxed{2}$

(ii) What is the radius when the area is increasing at 2π m^2/s? $\boxed{1}$

(b) Find the volume generated when the region between $y = \sec 3x$ and the x-axis, from $x = -\frac{\pi}{12}$ to $x = \frac{\pi}{12}$, is rotated about the x-axis. $\boxed{3}$

(c) Prove the identity $\sin 2\theta(\tan\theta + \cot\theta) = 2$. $\boxed{3}$

(d) A particle is moving with acceleration $\ddot{x} = -9x$ and it is initially stationary at $x = 4$.

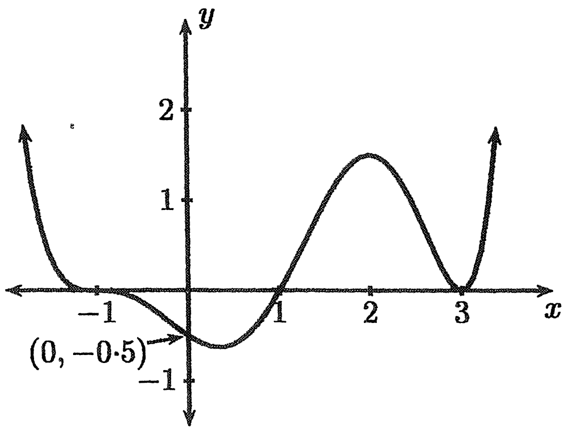
(i) Find v^2 as a function of x. $\boxed{2}$

(ii) What is the particle's maximum speed? $\boxed{1}$

QUESTION THREE (Start a new answer booklet)

Marks

(a) **3**



Write down a possible equation $y = P(x)$ for the polynomial function sketched above. (Do not use calculus.)

(b) Find the term independent of x in the expansion of $\left(x^2 + \dfrac{3}{x}\right)^{12}$. **3**

(c) A particle's displacement is $x = 5 - 3\sin(2t + \frac{\pi}{4})$, in units of centimetres and seconds.

(i) In what interval is the particle moving? **1**

(ii) Write down the period of the motion. **1**

(iii) Find the first two times after time zero when the particle is closest to the origin. **2**

(d) (i) Write down the expansion of $(1 + 2x)^n$. **1**

(ii) Hence prove the identity **1**

$$3^n = {}^nC_0 + 2 \times {}^nC_1 + 2^2 \times {}^nC_2 + \cdots + 2^n \times {}^nC_n.$$

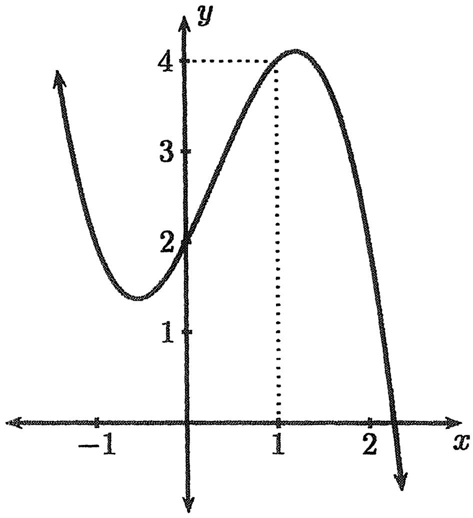
QUESTION FOUR (Start a new answer booklet)

Marks

(a) Use the substitution $u = 1 - x$ to evaluate **4**

$$\int_0^1 x(1-x)^7\,dx.$$

(b)



A pupil is using Newton's method to calculate an approximate value for the single zero of the polynomial $y = x^2 + 2x + 2 - x^3$. The polynomial is sketched above.

(i) Use Newton's method with $x_0 = 2$ to approximate the zero correct to two decimal places. **3**

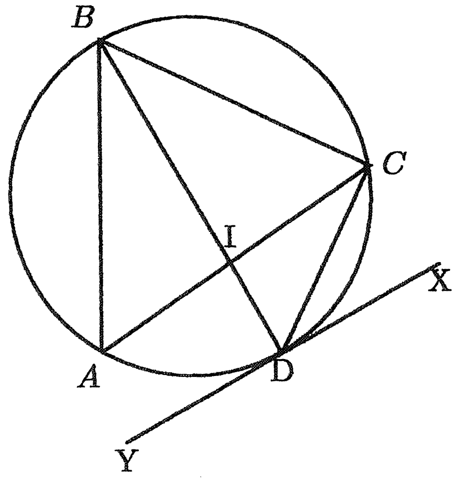
(ii) Quickly copy the graph above and use it to show why $x_0 = 1$ would be a bad initial approximation to the root (no calculations are required). **1**

(c) (i) Show that $y = \sqrt{3}\sin x + \cos x$ may be expressed in the form $y = 2\sin(x + \frac{\pi}{6})$. **2**

(ii) Hence sketch the curve $y = \sqrt{3}\sin x + \cos x$, for $0 \le x \le 2\pi$. Do not use calculus but do mark x- and y-intercepts. **2**

QUESTION FIVE (Start a new answer booklet)

(a)



In the circle above, $\angle ABD + \angle BCA = 90°$ and XY is a tangent to the circle at D. The chords AC and BD intersect at I.

Marks

(i) Prove that $\angle BCD = 90°$ and hence that BD is a diameter of the circle. **2**

(ii) Prove that if $\triangle ABC$ is isosceles with $AB = BC$, then $AC || XY$. **2**

(b) Harry is investing a certain amount each year for 20 years in a superannuation fund, which pays 6% per annum compound interest. Harry pays a yearly contribution $\$M$ at the start of each year.

(i) Show that after n years Harry's investment amounts to **2**

$$M \times 1{\cdot}06 + M \times 1{\cdot}06^2 + \cdots + M \times 1{\cdot}06^n.$$

(ii) By summing this geometric series, find M if the fund is to reach \$500 000 at the end of the 20 years. **2**

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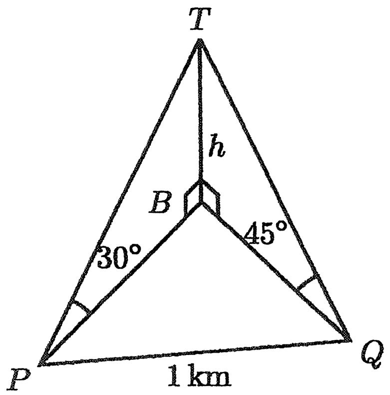
QUESTION FIVE (Continued)

(c)



The angle of elevation from a boat at P to a point T at the top of a vertical cliff is measured to be 30°. The boat sails 1 km to a second point Q, from which the angle of elevation to T is measured to be 45°. Let B be the point at the base of the cliff directly below T and let $h = BT$ be the height of the cliff in metres. The bearings of B from P and Q are 50° and 310° respectively.

(i) Show that $\angle PBQ = 100°$. **1**

(ii) Find expressions for PB and QB in terms of h. **1**

(iii) Hence show that **1**

$$h^2 = \frac{1000^2}{\cot^2 30° + \cot^2 45° - 2\cot 30° \cot 45° \cos 100°}$$

(iv) Use a calculator to find the height of the cliff, correct to the nearest metre. **1**

QUESTION SIX (Start a new answer booklet)

Marks

(a) (i) Show that the tangent to the parabola $x^2 = 8y$ at the point $P(4p, 2p^2)$ has equation $y = px - 2p^2$. **2**

(ii) By substituting the point $A(3, -2)$ into this equation of the tangent, or otherwise, find the points of contact of the tangent to the parabola from A. **2**

(b) (i) A particle moves along a path described by $y = \frac{1}{4}x^2$. Show that the perpendicular distance from any point on its path to the line $3x - 4y + 4 = 0$ is **1**

$$\tfrac{1}{5}\left|x^2 - 3x - 4\right|.$$

(ii) Sketch the curve $y = \frac{1}{5}\left|x^2 - 3x - 4\right|$, showing any intercepts and the vertex. **2**

(iii) Use your graph in part (ii) to show that the distance from the path to the line is $1\frac{1}{4}$ units on exactly three occasions. **1**

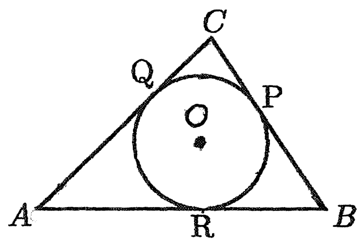
(c) Use mathematical induction to prove that for all integers $n > 0$, **4**

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + n(n+1)^2 = \tfrac{1}{12}n(n+1)(n+2)(3n+5).$$

QUESTION SEVEN (Start a new answer booklet)

Marks

(a) **3**



The incircle of a triangle is the circle inscribed to touch the triangle at exactly three points, as in the diagram above. In the diagram, O is the centre of the incircle, and P, Q and R are the points where the incircle touches the triangle. Let r be the radius of the incircle and p be the perimeter of the triangle. Prove that

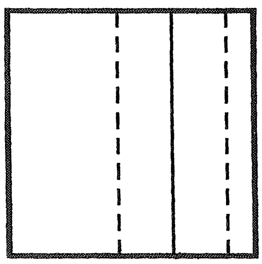
$$\text{area } \triangle ABC = \frac{1}{2}rp.$$

(b) (i) By differentiating, or otherwise, prove that for $x > -1$, **3**

$$\tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} = \frac{\pi}{4}.$$

(ii) Hence, or otherwise, calculate $\tan^{-1}(\sqrt{2}-1)$. **1**

(c)



An artist has been commissioned to draw a piece of modern art, using his pack of 30 long sharp coloured pencils.
First he draws a square. Then he draws a vertical line that divides the area of the square in the ratio 2 : 1. Then he draws further vertical lines that divide the areas of each region in the ratio 2 : 1 (see the diagram). The artist repeats this process eleven times, resulting in $2^{11} = 2048$ thin rectangles.
The artist then begins to colour in the rectangles, using the same colour for rectangles of the same size, but different colours for rectangles of different sizes.

(i) Let the number of rectangles of size $\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k}$ be written $r(n,k)$. Show that **2**

$$r(n,k) = r(n-1,k-1) + r(n-1,k).$$

(ii) Show that he has enough colours to colour the entire square, and find what area is coloured by the pencil that he uses the most of. **3**

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02

The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$\text{NOTE}: \ \ln x = \log_e x, \quad x > 0$$

QUESTION ONE

(a) $\displaystyle\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2$

$= 30$

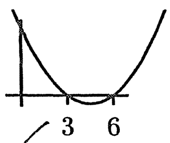
(b) Since $\dfrac{x}{x-3} > 2$

then $x(x-3) > 2(x-3)^2$

Thus $2(x-3)^2 - x(x-3) < 0$

$(x-3)(2x-6-x) < 0$

$(x-3)(x-6) < 0$



Hence from the graph $3 < x < 6$.

(c) (i) $y' = 7e^{7x}$.

(ii) At $x = 1$, $y = e^7$ and $y' = 7e^7$. Thus using the point-gradient form the required tangent is

$$y - e^7 = 7e^7(x-1)$$
$$y = 7e^7 x - 6e^7$$

(d) $$\frac{{}^{n}\mathrm{C}_2}{{}^{n}\mathrm{C}_1} = \frac{n!}{(n-2)!2!} \div \frac{n!}{(n-1)!1!}$$
$$= \frac{(n-1)!}{(n-2)!2}$$
$$= \tfrac{1}{2}(n-1)$$

(e) For any integer $n \in \mathbf{Z}$,

$$3x = \tfrac{\pi}{5} + 2n\pi \quad \text{or} \quad 3x = -\tfrac{\pi}{5} + 2n\pi$$
$$x = \tfrac{\pi}{15} + \tfrac{2}{3}n\pi \quad \text{or} \quad x = -\tfrac{\pi}{15} + \tfrac{2}{3}n\pi$$

QUESTION TWO

(a) (i) Now $A = \pi r^2$ and $\dfrac{dr}{dt} = 0{\cdot}1$ (given). Hence by the chain rule

$$\frac{dA}{dt} = \frac{dA}{dr}\,\frac{dr}{dt}$$
$$= 2\pi r \times 0{\cdot}1.$$

(ii) If $\dfrac{dA}{dt} = 2\pi$, then using the result from part (i),

$$0.2\pi r = 2\pi$$

Thus $r = \frac{2}{0.2} = 10$ metres.

(b) The volume of revolution is

$$\begin{aligned} V &= \pi \int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} y^2\, dx \\ &= \pi \int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \sec^2 3x\, dx \\ &= \left[\tfrac{1}{3}\pi \tan 3x\right]_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \\ &= \tfrac{1}{3}\pi\left(\tan\tfrac{\pi}{4} - \tan(-\tfrac{\pi}{4})\right) \\ &= \tfrac{2}{3}\pi \text{ units}^3 \end{aligned}$$

(c) $\text{LHS} = \sin 2\theta(\tan\theta + \cot\theta)$

$$\begin{aligned} &= 2\sin\theta\cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right) \\ &= 2\sin\theta\cos\theta\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right) \\ &= 2(\sin^2\theta + \cos^2\theta) \\ &= 2 \\ &= \text{RHS} \end{aligned}$$

(d) Since the acceleration is given by $\ddot{x} = \dfrac{d\frac{1}{2}v^2}{dx}$, then

$$\begin{aligned} \frac{d\frac{1}{2}v^2}{dx} &= -9x \\ \tfrac{1}{2}v^2 &= -\tfrac{9}{2}x^2 + C \end{aligned}$$

for some constant C. When $x = 4$, $v = 0$ so

$$\begin{aligned} C &= \tfrac{1}{2}v^2 + \tfrac{9}{2}x^2 \\ &= 0 + \tfrac{9}{2}(4)^2 \\ &= 72. \end{aligned}$$

Thus

$$\begin{aligned} v^2 &= -9x^2 + 144. \\ &= -9(x^2 - 16) \end{aligned}$$

(e) The function v^2 found in part (i) is a concave down parabola with a maximum of $v^2 = 9 \times 16$ at $x = 0$. Thus the maximum speed is $|v| = 12\,\text{m/s}$. (or 12 units)

QUESTION THREE

(a) By examining the zeros a possible polynomial is

$$y = a(x+1)^3(x-1)(x-3)^2,$$

for some constant $a \neq 0$. Now the polynomial passes through $(0, -0{\cdot}5)$, so

$$\begin{aligned} -\tfrac{1}{2} &= a \times 1^3 \times -1 \times (-3)^2 \\ -\tfrac{1}{2} &= -9a \\ a &= \tfrac{1}{18} \end{aligned}$$

Thus a possible polynomial is

$$y = \tfrac{1}{18}(x+1)^3(x-1)(x-3)^2,$$

(b) We have the expansion

$$(x^2 + \tfrac{3}{x})^{12} = \sum_{k=0}^{12} {}^{12}C_k (x^2)^{12-k}(\tfrac{3}{x})^k$$

Thus the general term is

$$\begin{aligned}{}^{12}\mathrm{C}_k(x^2)^{12-k}(\tfrac{3}{x})^k &= {}^{12}\mathrm{C}_k x^{24-2k}3^k x^{-k}\\ &= {}^{12}\mathrm{C}_k x^{24-3k}3^k\end{aligned}$$

The term independent of x occurs when $24-3k=0$, that is when $k=8$. Hence the term independent of x is

$$\begin{aligned}{}^{12}\mathrm{C}_8 3^8 &= \frac{12!}{8!4!}3^8\\ &= 495\times 3^8\\ &= 3\,247\,695\end{aligned}$$

(c) (i) The interval $5-3\le x\le 5+3$, i.e. $2\le x\le 8$.

(ii) The period is

$$T=\frac{2\pi}{n}=\frac{2\pi}{2}=\pi.$$

(iii) That is, find the two smallest positive solutions of

$$\begin{aligned}\sin(2t+\tfrac{\pi}{4}) &= 1\\ 2t+\tfrac{\pi}{4} &= \tfrac{\pi}{2} \text{ or } \tfrac{5\pi}{2}\\ 2t &= \tfrac{\pi}{4} \text{ or } \tfrac{9\pi}{4}\\ t &= \tfrac{\pi}{8} \text{ or } \tfrac{9\pi}{8}\end{aligned}$$

Thus the first two times are after $\frac{\pi}{8}$ and $\frac{9\pi}{8}$ seconds.

(d) (i) $$\begin{aligned}(1+2x)^n &= \sum_{k=0}^{n} {}^n\mathrm{C}_k(2x)^k\\ &= {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1(2x) + {}^n\mathrm{C}_2(2x)^2 + \cdots + {}^n\mathrm{C}_n(2x)^n\end{aligned}$$

(ii) Now if we let $x=1$ in the identity in part (i);

$$\begin{aligned}(1+2)^n &= {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1(2) + {}^n\mathrm{C}_2(2)^2 + \cdots + {}^n\mathrm{C}_n(2)^n\\ 3^n &= {}^n\mathrm{C}_0 + 2\times{}^n\mathrm{C}_1 + 2^2\times{}^n\mathrm{C}_2 + \cdots + 2^n\times{}^n\mathrm{C}_n.\end{aligned}$$

QUESTION FOUR

(a) (i) Let $u=1-x$. Then $du=-dx$. If $x=0$, $u=1$. If $x=1$, $u=0$. Thus

$$\begin{aligned}\int_0^1 x(1-x)^7\,dx &= -\int_1^0 (1-u)u^7\,du\\ &= \int_0^1 u^7-u^8\,du\\ &= \Big[\tfrac{1}{8}u^8-\tfrac{1}{9}u^9\Big]_0^1\\ &= \tfrac{1}{8}-\tfrac{1}{9}\\ &= \tfrac{1}{72}\end{aligned}$$

(b) (i) Let $f(x)=x^2+2x+2-x^3$ and let $x_0=2$ be the initial approximation to the root of $y=f(x)$. Then Newton's method says that if x_n is an approximation to the root then a better approximation is

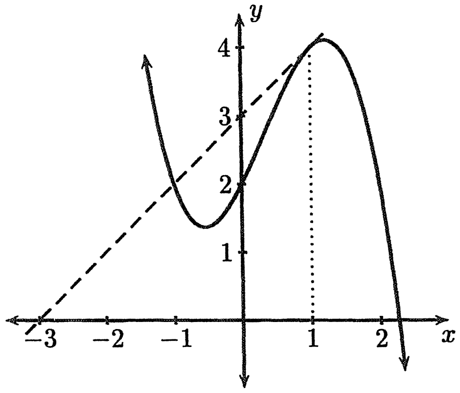
$$\begin{aligned}x_{n+1} &= x_n-\frac{f(x_n)}{f'(x_n)}\\ &= x_n-\frac{x_n^2+2x_n+2-x_n^3}{2x_n+2-3x_n^2}.\end{aligned}$$

From a calculator,

$$x_1=2{\cdot}\dot{3}\qquad x_2 \doteqdot 2{\cdot}2720\qquad x_3 \doteqdot 2{\cdot}2695$$

Since x_2 and x_3 agree to 2 decimal places, to 2 decimal places the root is 2·27.

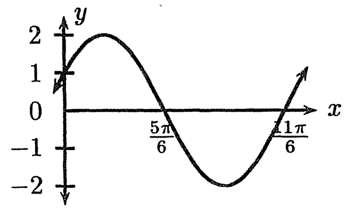
(ii) A tangent to the graph at $x = 1$ does not intersect the x-axis anywhere near the root. This occurs since $(1, 4)$ is on the other side of the turning point from the root. (See the graph below, the tangent is the dashed line).



(c) (i) The easy method is to expand $y = 2\sin(x + \frac{\pi}{6})$. We have

$$\begin{aligned} y &= 2\sin(x + \tfrac{\pi}{6}) \\ &= 2(\sin x \cos \tfrac{\pi}{6} + \cos x \sin \tfrac{\pi}{6}) \\ &= 2(\sin x \times \tfrac{\sqrt{3}}{2} + \cos x \times \tfrac{1}{2}) \\ &= \sqrt{3}\sin x + \cos x \end{aligned}$$

(ii)



✓ shape
✓ intercepts.

QUESTION FIVE

(a) (i) Since $\angle ACD = \angle ABD$ (angles standing on the arc AD), then

$$\begin{aligned} \angle BCD &= \angle ACD + \angle BCA \\ &= \angle ABD + \angle BCA \\ &= 90^\circ \text{(given)} \end{aligned}$$

Hence BD must be a diameter (since $\angle BCD = 90^\circ$).

(ii) Notice $\angle BAC = \angle BCA$ (equal angles opposite equal sides)

Thus $\angle ABD + \angle BAC = \angle ABD + \angle BCA$

$= 90^\circ$ (given)

But $\angle ABD + \angle BAC + \angle BIA = 180^\circ$ (angle sum of triangle)

Hence $\angle BIA = 90^\circ$

But $\angle IDY = 90^\circ$ (radius and tangent)

It follows that $AC \parallel YX$, since the corresponding angles $\angle BIA$ and $\angle IDY$ are equal.

(b) At 6% per annum, the investment grows by a factor of 1·06 each year.
Harry's first contribution is invested for n-years and grows to $M \times 1{\cdot}06^n$.
Harry's second contribution is invested for $(n-1)$-years and grows to $M \times 1{\cdot}06^{n-1}$.... and so on.
His last contribution is invested for 1 year and grows to $M \times 1{\cdot}06$.
Following this pattern, the total value of the investment after n-years is

$$M \times 1{\cdot}06 + M \times 1{\cdot}06^2 + \cdots + M \times 1{\cdot}06^n.$$

(c) This is a geometric progression with first term $a = M \times 1{\cdot}06$, common ratio $r = 1{\cdot}06$ and we are required to sum n terms. The sum is

$$\frac{a(r^n - 1)}{r-1} = \frac{M \times 1{\cdot}06 \times (1{\cdot}06^n - 1)}{1{\cdot}06 - 1}$$

If this sum is \$500 000 then

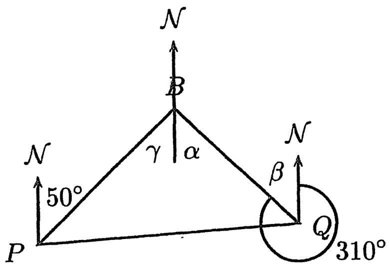
$$\frac{M \times 1{\cdot}06 \times (1{\cdot}06^n - 1)}{1{\cdot}06 - 1} = 500\,000.$$

Thus $M = 500\,000 \times 0{\cdot}06 \div 1{\cdot}06 \div (1{\cdot}06^n - 1)$
and when $n = 20$ years,

$$\begin{aligned} M &= 500\,000 \times 0{\cdot}06 \div 1{\cdot}06 \div (1{\cdot}06^{20} - 1) \\ &\doteqdot \$12823 \end{aligned}$$

(to the nearest dollar).

(d)



(i) In the diagram,

$$\begin{aligned} \beta &= 50^\circ \quad \text{(revolution)} \\ \alpha &= 50^\circ \quad \text{(alternate angles, parallel lines)} \\ \gamma &= 50^\circ \quad \text{(alternate angles, parallel lines)} \end{aligned}$$

Hence $\angle PBQ = 100^\circ$.

(ii) In $\triangle TBP$, $\dfrac{h}{PB} = \tan 30^\circ$, thus $PB = h \cot 30^\circ$.

In $\triangle TBQ$, $\dfrac{h}{BQ} = \tan 45^\circ$, thus $BQ = h \cot 45^\circ$.

(iii) By the cosine rule,

$$\begin{aligned} PQ^2 &= PB^2 + BQ^2 - 2PB \times BQ \cos \angle PBQ. \\ 1000^2 &= h^2 \cot^2 30^\circ + h^2 \cot^2 45^\circ - 2h^2 \cot 30^\circ \cot 45^\circ \cos 100^\circ \\ 1000^2 &= \left(\cot^2 30^\circ + \cot^2 45^\circ - 2 \cot 30^\circ \cot 45^\circ \cos 100^\circ\right) h^2. \end{aligned}$$

Now dividing by the coefficient of h^2 gives the required equation.

(iv) The height of the cliff is about 466 metres.

QUESTION SIX

(a) (i) The gradient is

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tfrac{1}{8}x^2\right) = \tfrac{1}{4}x$$

Thus $\dfrac{dy}{dx} = p$ at $P(4p, 2p^2)$.

Using the point-gradient form, the tangent has equation

$$\begin{aligned} y - 2p^2 &= p(x - 4p) \\ y &= px - 2p^2 \end{aligned}$$

(ii) If $A(3, -2)$ lies on the tangent then

$$\begin{aligned} -2 &= 3p - 2p^2 \\ 2p^2 - 3p - 2 &= 0 \\ (2p+1)(p-2) &= 0 \end{aligned}$$

Hence the points of contact are defined by $p = -0{\cdot}5$ or $p = 2$. The points are $(-2, 0{\cdot}5)$ and $P(8, 8)$.

(b) (i) Let $P(x, y)$ be any point on the parabola $y = \frac{1}{4}x^2$. Then the distance d from P to the line $3x - 4y + 4 = 0$ is

$$\begin{aligned} d &= \frac{|3x - 4y + 4|}{\sqrt{3^2 + 4^2}} \\ &= \tfrac{1}{5}|3x - 4(\tfrac{1}{4}x^2) + 4| \\ &= \tfrac{1}{5}|x^2 - 3x - 4| \end{aligned}$$

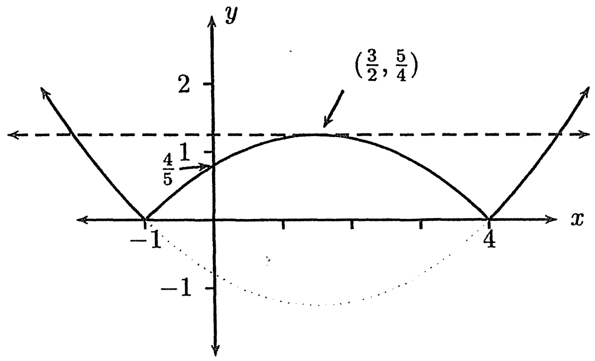
(ii) The quadratic $\frac{1}{5}|x^2 - 3x - 4| = \frac{1}{5}|(x-4)(x+1)$ has zeros at $x = -1$ and $x = 4$. The vertex is half-way between the zeros, at $x = 1.5$ and

$$\begin{aligned} y &= \tfrac{1}{5}|(1{\cdot}5 - 4)(1{\cdot}5 + 1)| \\ &= -1.25 \end{aligned}$$

The graph is below (the solid dark line).

√ shape

√ x-intercepts.



(iii) This is clear if we add the horizontal line $y = 1\frac{1}{4}$ onto our graph (the dashed line)- it touches or cuts the graph in part (ii) three times.

(c) **Part A:** If $n = 1$, then

$$\begin{aligned} \text{RHS} &= \tfrac{1}{12} \times 1 \times 2 \times 3 \times 8 \\ &= 4 \\ &= 1 \times 2^2 \\ &= \text{LHS} \end{aligned}$$

Hence the statement is true for $n = 1$.

Part B: Suppose k is a positive integer for which the result is true.

That is, suppose that

$$1 \times 2^2 \ + \ \cdots \ + \ k(k+1)^2 = \tfrac{1}{12}k(k+1)(k+2)(3k+5) \qquad ---(*)$$

We shal prove the statement is then true for $n = k + 1$. That is, we shall prove that

$$1 \times 2^2 \ + \ \cdots \ + \ (k+1)(k+2)^2 = \tfrac{1}{12}(k+1)(k+2)(k+3)(3k+8).$$

Here is the proof;

$$\begin{aligned} \text{LHS} &= 1 \times 2^2 \ + \ \cdots \ + \ k(k+1)^2 \ + \ (k+1)(k+2)^2 \\ &= \tfrac{1}{12}k(k+1)(k+2)(3k+5) \ + \ (k+1)(k+2)^2, \end{aligned}$$

(by the inductive hypothesis (*).)

Thus LHS$= \frac{1}{12}k(k+1)(k+2)(3k+5) + \frac{1}{12} \times 12(k+1)(k+2)^2$,

$= \frac{1}{12}(k+1)(k+2)\big(k(3k+5)+12(k+2)\big)$

$= \frac{1}{12}(k+1)(k+2)\big(3k^2+17k+24)\big)$

$= \frac{1}{12}(k+1)(k+2)\big((3k+8)(k+3)\big)$

$=$ RHS

Part C: It follows from Parts A and B, and the principle of mathematical induction, that the statement is true for all integers $n \geq 1$.

QUESTION SEVEN

(a) Join OA, OB, OC, OP, OQ, OR. Then the radii OP, OQ and OR are perpendicular to the tangents CB, AC and AB respectively.
The total area of the triangle is

$$\begin{aligned}\text{Area } \triangle ABC &= \text{Area } \triangle ABO + \text{Area } \triangle ACO + \text{Area } \triangle BCO \\ &= \tfrac{1}{2} \times AB \times OR + \tfrac{1}{2} \times AC \times OQ + \tfrac{1}{2} \times BC \times OP \\ &= \tfrac{1}{2}r(AB + AC + BC) \\ &= \tfrac{1}{2}rp\end{aligned}$$

(b) (i) Let $u = \dfrac{1-x}{1+x}$ and $y = \tan^{-1} x + \tan^{-1} u$. By the quotient rule,

$$\begin{aligned}\frac{du}{dx} &= \frac{-1(1+x) - 1(1-x)}{(1+x)^2}. \\ &= \frac{-2}{(1+x)^2}\end{aligned}$$

Now using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+x^2} + \frac{1}{1+u^2} \times \frac{du}{dx} \\ &= \frac{1}{1+x^2} + \frac{1}{1+u^2} \times \frac{-2}{(1+x)^2}\end{aligned}$$

Since $u \times (1+x) = 1-x$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+x^2} + \frac{-2}{(1+x)^2 + (1-x)^2} \\ &= \frac{1}{1+x^2} + \frac{-2}{2+2x^2} \\ &= 0\end{aligned}$$

Now the function y is differentiable on the restricted domain $x > 1$, and since it has derivative zero there it must be a constant. Thus $\tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} = C$, for some constant C. To find the value of this constant, substitute any value of $x > -1$, say $x = 0$. Thus

$$\begin{aligned}\tan^{-1} 0 + \tan^{-1} 1 &= C \\ 0 + \frac{\pi}{4} &= C\end{aligned}$$

So $\tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} = \frac{\pi}{4}$, for $x > -1$

(ii) Let $x = \sqrt{2} - 1$. Then $u = \dfrac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$.
Thus using part (i),

$$\begin{aligned}\tan^{-1}(\sqrt{2}-1) + \tan^{-1}(\sqrt{2}-1) &= \frac{\pi}{4} \\ \tan^{-1}(\sqrt{2}-1) &= \frac{\pi}{8}\end{aligned}$$

(c) (i) The cardinal n represents the number of times this dissection process has been carried out. Initially $n = 0$ and we have the whole square of 1×1 units. Thus $r(0,0) = 1$. At each stage of the division process, rectangles of size $\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$ may arise in only one of two ways;

- if one takes $\frac{1}{3}$ rectangle of size $\left(\frac{1}{3}\right)^{k-1} \left(\frac{2}{3}\right)^{n-k}$;
- or if one takes $\frac{2}{3}$ of a rectangle of size $\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k-1}$.

Thus $r(n,k) =$ number of rectangles of size $\left(\frac{1}{3}\right)^{k-1} \left(\frac{2}{3}\right)^{n-k}$

$+$ number of rectangles of size $\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k-1}$

$= r(n-1, k-1) + r(n-1, k)$

(ii) It is clear also that $r(n,0) = 1$, hence the numbers $r(n,k)$ form the entries of Pascal's triangle, that is $r(n,k) = {}^n\mathrm{C}_k$.

The number of rectangles of size $\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$ is the term in $\left(\frac{1}{3}\right)^k$ in the expansion of $\left(\frac{1}{3} + \frac{2}{3}\right)^n$. Since this has 12 terms when $n = 11$, the artist will use 12 different colours, and he has plenty of colours in his pack of pencils. The greatest term will corresponding to the colour he uses the most of. Thus to complete the question we need to find the greatest term in this expansion.

To simplify the algebra, let $x = \frac{1}{3}$ and $y = \frac{2}{3}$. Then

$$(x+y)^n = \sum_{k=0}^{11} T_k$$

where $T_k = {}^{11}\mathrm{C}_k x^k y^{11-k}$. Hence

$$\begin{aligned}
\frac{T_{k+1}}{T_k} &= \frac{11!}{(k+1)!(10-k)!} \times \frac{k!(11-k)!}{11!} \times \frac{x^{k+1}y^{10-k}}{x^k y^{11-k}} \\
&= \frac{(11-k)x}{(k+1)y} \\
&= \frac{(11-k) \times \frac{1}{3}}{(k+1) \times \frac{2}{3}} \\
&= \frac{11-k}{2k+2}
\end{aligned}$$

The terms will be increasing if $\dfrac{T_{k+1}}{T_k} > 1$, that is if

$$\begin{aligned}
\frac{11-k}{2k+2} &> 1 \\
11 - k &> 2k + 2 \qquad (\text{since } (2k+2) > 0) \\
9 &> 3k
\end{aligned}$$

That is, if $k < 3$. So $T_0 < T_1 < T_2 < T_3 = T_4 > T_5 > \cdots > T_{11}$.
Hence the greatest area coloured by one colour is of size

$$\begin{aligned}
T_3 &= {}^{11}\mathrm{C}_3 \times \left(\tfrac{1}{3}\right)^3 \times \left(\tfrac{2}{3}\right)^8 \\
&= \frac{11 \times 10 \times 9}{1 \times 2 \times 3} \times \frac{2^8}{3^{11}} \\
&= \frac{2^8 \times 5 \times 11}{3^{10}}
\end{aligned}$$

(Since $T_3 = T_4$ the same answer is obtained for the fourth term).

BDD