Sydney Grammar School
Mathematics Department
Trial Examinations 2005

FORM VI

MATHEMATICS EXTENSION 1

Examination date

Wednesday 10th August 2005

Time allowed

2 hours

Instructions

All seven questions may be attempted.
All seven questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the seven questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 7 per boy. A total of 1000 booklets should be sufficient.
Candidature: 117 boys.

Examiner

KWM

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\sum_{n=1}^{4} n!$. $\boxed{1}$

(b) Differentiate the following with respect to x.

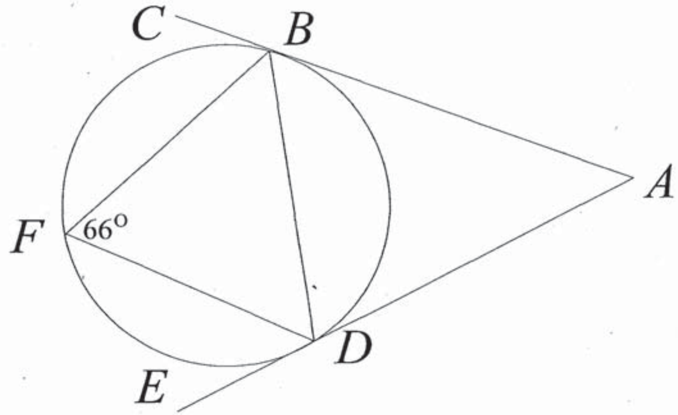
(i) $y = \log_e(\sin x)$ $\boxed{1}$

(ii) $y = \cos^{-1} 3x$ $\boxed{1}$

(c) State the domain and range of the function $f(x) = 2\cos^{-1}\frac{x}{3}$. $\boxed{2}$

(d) Given the points $A(5,1)$ and $B(-3,6)$, find the co-ordinates of the point P that divides the interval AB externally in the ratio $3:4$. $\boxed{2}$

(e) $\boxed{2}$



The diagram above shows the tangents AC and AE drawn to a circle. BF and DF are chords drawn from the points of contact at B and D respectively. Given that $\angle BFD = 66°$, find $\angle BAD$ giving reasons for your answer.

(f) Use the substitution $u = 1 - x^2$ to evaluate the definite integral $\boxed{3}$

$$\int_0^{\frac{\sqrt{3}}{2}} x\sqrt{1-x^2}\,dx.$$

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Simplify $\dfrac{{}^{n}C_{r+1}}{{}^{n}C_{r}}$. **2**

(b) Find the term independent of x in the expansion of $\left(3x^2 + \dfrac{2}{x}\right)^{12}$. **3**

(c) A couple, purchasing a house, negotiates a \$300 000 mortgage to be repaid in equal monthly instalments over a period of 25 years. The interest on the loan is 7·2% per annum, compounded monthly. Let $\$A_n$ be the amount owing on the loan after n months, and $\$M$ the monthly repayment.

(i) Write down an expression for A_1. **1**

(ii) Hence show that $A_2 = 300\,000(1{\cdot}006)^2 - 1{\cdot}006M - M$. **1**

(iii) Show that $A_n = 300\,000(1{\cdot}006)^n - \dfrac{M(1{\cdot}006^n - 1)}{0{\cdot}006}$. **1**

(iv) Find, to the nearest dollar, the monthly repayment M required to repay the loan over 25 years under the agreed terms. **1**

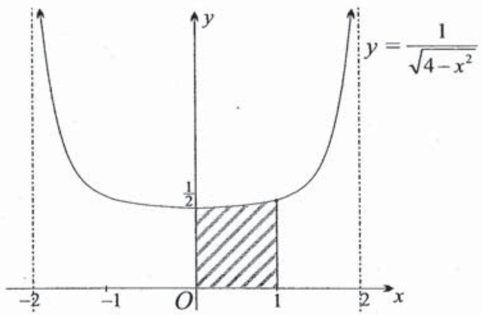
(d) Find $\displaystyle\int_0^{\frac{\pi}{3}} \tan^2 x \, dx$. **3**

Exam continues overleaf ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Prove that $\dfrac{1 - \cos 2A}{\sin 2A} = \tan A$. **2**

(b)



In the diagram above the curve $y = \dfrac{1}{\sqrt{4-x^2}}$ is sketched showing vertical asymptotes at $x = -2$ and $x = 2$. Find the exact area of the shaded region bounded by the curve, the line $x = 1$ and the co-ordinate axes. **2**

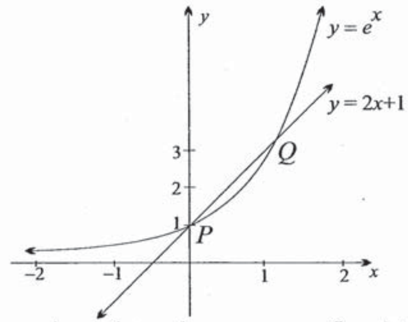
(c) Let the equation $x^3 - 3x^2 - 4x + 12 = 0$ have roots α, β and γ.

(i) Find the value of $\alpha + \beta + \gamma$. **1**

(ii) Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **2**

(iii) Given that two of its roots sum to zero, solve the equation. **2**

(d)



The diagram above shows the curve $y = e^x$ and the line $y = 2x + 1$ intersecting at point $P(0,1)$ and at another point Q. Use Newton's Method once, with initial approximation $x = 1$, to find a better approximation to the x co-ordinate of the point Q. Write your approximation correct to one decimal place. **3**

Exam continues next page ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Find $\cos^{-1}(-\frac{1}{2}) + \sin^{-1}(-\frac{\sqrt{3}}{2})$ in radians. $\boxed{1}$

(b) Find the values of a and b that make the polynomial $P(x) = 2x^3 + ax^2 - 13x + b$ exactly divisible by $x^2 - x - 6$. $\boxed{3}$

(c) (i) Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. $\boxed{2}$

(ii) Hence, or otherwise, find the general solution of the equation $\boxed{2}$

$$\cos x - \sqrt{3}\sin x = 1.$$

(d) If the surrounding air temperature is 20°C, it takes 15 minutes for a cup of tea at a temperature of 80°C to cool to a temperature of 40°C. Given that T is the temperature in degrees Celsius of the tea after t minutes, then Newton's Law of cooling states that T satisfies the differential equation $\dfrac{dT}{dt} = k(T - 20)$.

(i) Show that $T = 20 + Ae^{kt}$ is a solution of the differential equation. $\boxed{1}$

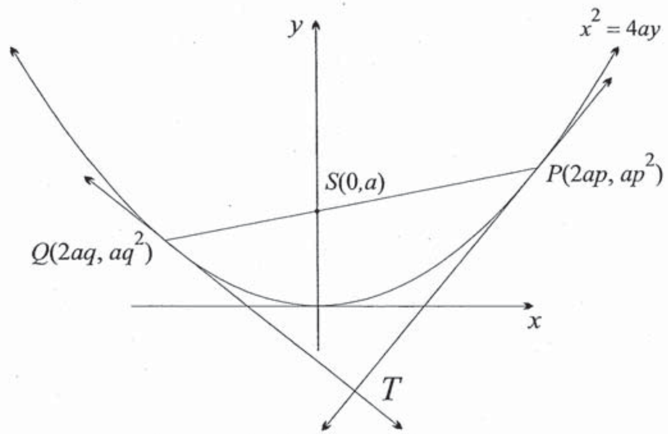
(ii) Find the value of A, and show that $k = -\dfrac{\ln 3}{15}$. $\boxed{2}$

(iii) Find the temperature of the tea after 30 minutes. $\boxed{1}$

Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above a focal chord PQ intersects the parabola $x^2 = 4ay$ at points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$. The tangents to the parabola at point P and point Q intersect at T.

(i) Show that the equation of the tangent to the parabola at the point P is given by $y = px - ap^2$. $\boxed{2}$

(ii) Show that $pq = -1$. $\boxed{2}$

(iii) Show that the acute angle between the focal chord QP and the tangent TP to the parabola at P is given by $\tan^{-1}|q|$. $\boxed{2}$

(b) A particle is moving in simple harmonic motion about the origin.

(i) Assuming that $\ddot{x} = -n^2 x$, show that $v^2 = n^2(a^2 - x^2)$, where a is the amplitude. $\boxed{2}$

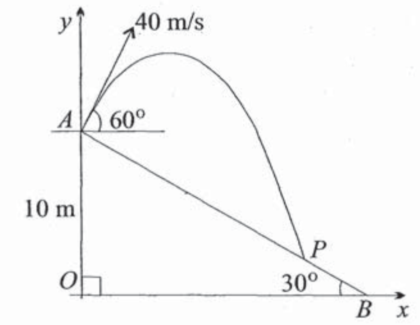
(ii) When the particle is 3 metres from the origin, its speed is 8 m/s, and when it is 4 metres from the origin its speed is 6 m/s. Find the period and amplitude of the motion. $\boxed{3}$

(iii) Find the greatest acceleration of the particle. $\boxed{1}$

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows a plane inclined at 30° to the horizontal, meeting level ground at B. A ball is projected from a point A on the plane, 10 metres above the horizontal. The angle of projection is 60° to the horizontal and the initial speed of the ball is 40 m/s.

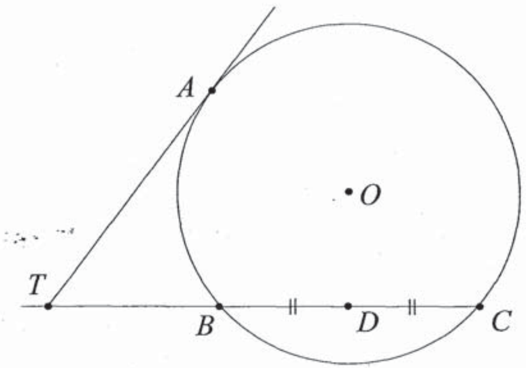
(i) Take g $=10\,\text{m}/s^2$, and show that the displacement equations of motion of the ball are given by **3**

$$y = 20\sqrt{3}t - 5t^2 + 10 \quad \text{and}$$

$$x = 20t.$$

(ii) Show that the ball hits the inclined plane at the point P after $t = \dfrac{16\sqrt{3}}{3}$ seconds. **3**

(b)



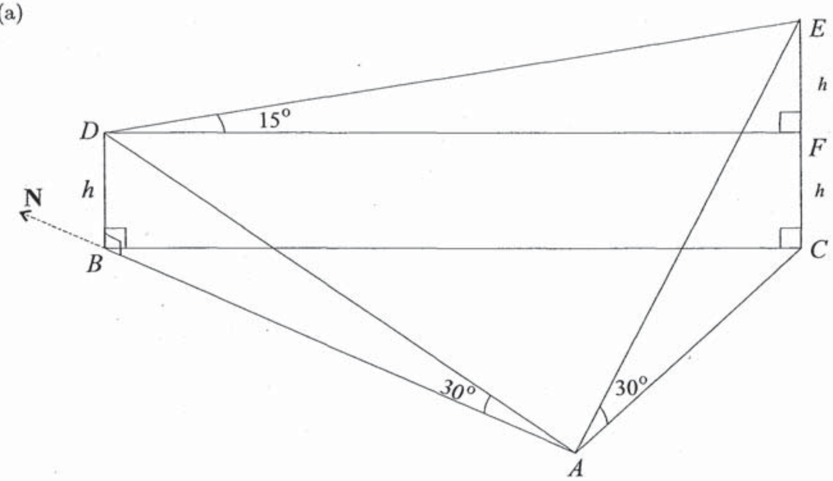
In the diagram above, TA is a tangent and TBC is a secant drawn to a circle of centre O. Let the midpoint of the chord BC be D. Prove that $\angle AOT = \angle ADT$. **3**

Exam continues overleaf ...

(c) A rectangle is expanding in such a way that at all times it is twice as long as it is wide. If its area is increasing at a rate of $18\,\text{cm}^2/\text{s}$, find the rate at which its perimeter is increasing at the instant its width is 1 metre. **3**

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows two vertical towers BD and CE of heights h and $2h$ respectively, on a horizontal plane ABC. Point A is due south of point B, and the angles of elevation of the tops of the towers from A are both 30°. Given that the angle of elevation from D to E is 15°, find the bearing of the taller tower from point A correct to the nearest degree. **4**

(b) By considering the expansion of $(1+x)^{n-1}$, prove that: **4**

$$\frac{7}{1}\binom{n-1}{0} + \frac{7^2}{2}\binom{n-1}{1} + \frac{7^3}{3}\binom{n-1}{2} + \cdots + \frac{7^n}{n}\binom{n-1}{n-1} = \frac{1}{n}(2^{3n}-1).$$

(c) Use induction, or otherwise, to prove that the sum of the products of all the pairs of different integers that can be formed from the first n positive integers is **4**

$$\frac{n}{24}(n-1)(n+1)(3n+2).$$

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ \ n \neq -1;\ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\text{NOTE : } \ln x = \log_e x,\ \ x > 0$$

FORM $\overline{\text{VI}}$ TRIAL 2005

MATHEMATICS (~~2 UNIT~~) (EXT. 1)

QUESTION 1

(a)
$$\sum_{n=1}^{4} n! = 1! + 2! + 3! + 4!$$
$$= 1 + 2 + 6 + 24$$
$$= 33 \checkmark$$

(b) (i)
$$y = \ln(\sin x)$$
$$y' = \frac{1}{\sin x} \times \cos x$$
$$y' = \cot x \checkmark$$

(ii)
$$y = \cos^{-1} 3x$$
$$y' = \frac{1}{\sqrt{1-9x^2}} \times 3$$
$$y' = \frac{3}{\sqrt{1-9x^2}} \checkmark$$

(c) $f(x) = 2\cos^{-1}\frac{x}{3}$

Domain: $-1 \le \frac{x}{3} \le 1$

so $-3 \le x \le 3 \checkmark$

Range: $0 \le y \le 2\pi \checkmark$

(d) $A(5,1)$, $B(-3,6)$

$k : l = -3 : 4$

$$P\left(\frac{lx_1 + kx_2}{l+k}, \frac{ly_1 + ky_2}{l+k}\right) \checkmark$$

$$P\left(\frac{20+9}{1}, \frac{4-18}{1}\right)$$

$$P(29, -14). \checkmark$$

(e)

$\angle DBA = 66°$ (The angle between a chord and tangent at the point of contact equals the angle drawn in the alternate segment.) $\checkmark$

Tangents drawn from an external point are equal, so $AB = AD$ and $\triangle ABD$ is isosceles.

$\angle BAD = \angle DBA = 66°$

(base angles of an isosceles triangle are equal.)

So $\angle BAD = 180° - 2 \times 66°$

$= 48° \checkmark$

(angle sum of a triangle.)

(f)
$$\int_0^{\frac{\sqrt{3}}{2}} x\sqrt{1-x^2}\,dx \qquad u = 1-x^2$$
$$du = -2x\,dx$$
$$-\tfrac{1}{2}\,du = x\,dx$$

when $x = 0$, $u = 1$ $\checkmark$

when $x = \frac{\sqrt{3}}{2}$, $u = \frac{1}{4}$:

$$-\frac{1}{2}\int_1^{\frac{1}{4}} u^{\frac{1}{2}}\,du \checkmark = \frac{1}{3}\left[u^{\frac{3}{2}}\right]_{\frac{1}{4}}^{1}$$
$$= \frac{1}{3}\left(1 - \frac{1}{8}\right)$$
$$= \frac{1}{3} \times \frac{7}{8}$$
$$= \frac{7}{24} \checkmark$$

(12)

QUESTION 2

(a)
$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n!}{(n-r-1)!\,(r+1)!} \times \frac{(n-r)!\,r!}{n!} \checkmark$$
$$= \frac{n-r}{r+1} \checkmark$$

(b) $\left(3x^2 + \frac{2}{x}\right)^{12}$

General term $T_r = {}^{12}C_r\,(3x^2)^{12-r}\left(\frac{2}{x}\right)^r \checkmark$

using rules of indices:

$24 - 3r = 0$

$3r = 24$

$r = 8 \checkmark$

$${}^{12}C_8 \times 3^4 \times 2^8 \checkmark$$

(c)

(i) $A_1 = 300\,000 \times 1.006 - M$ $\checkmark$

(ii) $A_2 = A_1 \times 1.006 - M$

$= 300\,000\,(1.006)^2 - M(1.006) - M$ $\checkmark$

(iii) $A_n = 300\,000\,(1.006)^n - M(1 + 1.006 + 1.006^2 + \dots 1.006^{n-1})$

a geometric progression

$a = 1$ and $r = 1.006$.

$$S_n = \frac{a(r^n - 1)}{r-1} \checkmark$$

$$S_n = \frac{(1.066)^n - 1}{0.006}$$

$$A_n = 300000(1.006)^n - \frac{M(1.006^n - 1)}{0.006}.$$

(iv) after the loan is repaid

$A_n = 0$, and $n = 300$

$$M = \frac{300000(1.006)^{300}(0.006)}{(1.006)^{300} - 1}$$

$M \doteqdot \$2159 \checkmark$

(d)
$$\int_0^{\frac{\pi}{3}} \tan^2 x\,dx$$
$$= \int_0^{\frac{\pi}{3}} \sec^2 x - 1\,dx \checkmark$$
$$= \left[\tan x - x\right]_0^{\frac{\pi}{3}} \checkmark$$
$$= \left(\sqrt{3} - \frac{\pi}{3}\right) - (0)$$
$$= \sqrt{3} - \frac{\pi}{3}. \checkmark$$

(12)

QUESTION 3

(a) $LHS = \dfrac{1-\cos 2A}{\sin 2A}$

$= \dfrac{1-(1-2\sin^2 A)}{2\sin A\cos A}$ ✓

$= \dfrac{2\sin^2 A}{2\sin A\cos A}$

$= \dfrac{\sin A}{\cos A}$ ✓

$= \tan A$

$= RHS$

(b) $\text{Area} = \displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

$= \left[\sin^{-1}\dfrac{x}{2}\right]_0^1$ ✓

$= \dfrac{\pi}{6} - 0$

$= \dfrac{\pi}{6}$ sq. units. ✓

(c) $x^3 - 3x^2 - 4x + 12 = 0$

(i) $\alpha + \beta + \gamma = -\dfrac{b}{a}$

$= 3$ ✓

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ ✓

$= \dfrac{\frac{c}{a}}{-\frac{d}{a}}$

$= -\dfrac{c}{d}$

$= \dfrac{1}{3}$ ✓

(iii) Let the roots be $\alpha, -\alpha, \beta$.

then $\alpha - \alpha + \beta = 3$

and $\beta = 3$. ✓

now $\alpha(-\alpha)\beta = -12$

$-\alpha^2 = -4$

$\alpha = \pm 2$

the roots are $-2, 2$ and 3. ✓

(d) $y = e^x$

$y = 2x+1$

The points of intersection correspond to the roots of the equation

$e^x - 2x - 1 = 0$

$f(x) = e^x - 2x - 1$ ✓

$f'(x) = e^x - 2$

put $x_0 = 1$: $f(1) = -0.282$

$f'(1) = 0.718$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ ✓

$= 1 + \dfrac{0.282}{0.718}$

$\doteqdot 1.4$ (1 dec. place.) ✓

(12)

QUESTION 4.

(a) $\cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \dfrac{2\pi}{3} - -\dfrac{\pi}{3}$

$= \pi$ ✓

(b)

$P(x) = 2x^3 + ax^2 - 13x + b$

$x^2 - x - 6 = (x-3)(x+2)$

$(x+2)$ is a factor: $P(-2) = 0$

$(x-3)$ is a factor: $P(3) = 0$.

$P(-2)$: $-16 + 4a + 26 + b = 0$ ①

$P(3)$: ✓ $54 + 9a - 39 + b = 0$ ②

$4a + b = -10$ — ①

$9a + b = -15$ — ②

② − ①: $5a = -5$

$a = -1$ ✓

$b = -6$ ✓

(c) (i)

Let $\cos x - \sqrt{3}\sin x \equiv R\cos(x+\alpha)$

$\cos x - \sqrt{3}\sin x \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$

equating co-efficients:

$\cos x$: $R\cos\alpha = 1$ — ①

$\sin x$: $R\sin\alpha = \sqrt{3}$ — ②

$\dfrac{②}{①}$: $\tan\alpha = \sqrt{3}$

$\alpha = \dfrac{\pi}{3}$ ✓

$①^2 + ②^2$: $R^2 = 4$

$R = 2$ ✓

$\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$.

(ii) $\cos x - \sqrt{3}\sin x = 1$

$2\cos\left(x + \frac{\pi}{3}\right) = 1$

$\cos\left(x + \frac{\pi}{3}\right) = \dfrac{1}{2}$

$x + \dfrac{\pi}{3} = 2\pi n + \dfrac{\pi}{3}$ or $2\pi n - \dfrac{\pi}{3}$

$x = 2\pi n$ ✓ or $x = 2\pi n - \dfrac{2\pi}{3}$ ✓

(d)(i) $T = 20 + Ae^{kt}$

$\dfrac{dT}{dt} = kAe^{kt}$

$= k(T-20)$

It satisfies the DE. ✓

(ii) when $t = 0$, $T = 80°c$

$80 = 20 + A$

$A = 60$ ✓

$T = 20 + 60e^{kt}$

when $t = 15$, $T = 40°c$

$40 = 20 + 60e^{15k}$

$e^{15k} = \dfrac{1}{3}$

$15k = \ln\dfrac{1}{3}$ ✓

$15k = -\ln 3$

$k = -\dfrac{1}{15}\ln 3$ as required.

(iii) $T = 20 + 60e^{-30\frac{\ln 3}{15}}$

$= 20 + 60e^{-2\ln 3}$

$= 20 + 60e^{\ln\frac{1}{9}}$

$= 20 + 60 \times \dfrac{1}{9}$

$= 26\tfrac{2}{3}\,°c$. ✓

(12)

QUESTION 5

(i) $x = 2ap$ $\quad y = ap^2$

$\frac{dx}{dp} = 2a$ $\quad \frac{dy}{dp} = 2ap$

$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$

$= 2ap \times \frac{1}{2a}$

$= p$ ✓

gradient at $P = p$., $(2ap, ap^2)$

$y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$\therefore y = px - ap^2$ is ✓

the equation of the tangent at P.

(ii)

Gradient of PS = Gradient of QP

$\frac{ap^2 - a}{2ap} = \frac{ap^2 - aq^2}{2ap - 2aq}$ ✓

$\frac{p^2-1}{2p} = \frac{a(p-q)(p+q)}{2a(p-q)}$

$p^2 - 1 = p(p+q)$

$p^2 - 1 = p^2 + pq$ ✓

$\therefore pq = -1$ as required.

(iii)

Let m_1 be the gradient of PQ and m_2 be the gradient of PT.

$m_1 = \frac{p+q}{2}$ $\quad m_2 = p$

$\tan \angle TPQ = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$= \left| \frac{\frac{p+q}{2} - p}{1 + p\left(\frac{p+q}{2}\right)} \right|$ ✓

$= \left| \frac{p + q - 2p}{2 + p^2 + pq} \right|$ $\quad (pq = -1)$

$= \left| \frac{q - p}{p^2 + 1} \right|$ $\quad \left(p = -\frac{1}{q}\right)$

$= \left| \frac{q + \frac{1}{q}}{\frac{1}{q^2} + 1} \right|$ ✓

$= \left| \frac{q^3 + q}{1 + q^2} \right|$

$= \left| \frac{q(q^2+1)}{(q^2+1)} \right|$

$= |q|$.

(b) (i)

$\ddot{x} = -n^2 x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2 x$ ✓

$\frac{1}{2}v^2 = -\frac{n^2 x^2}{2} + c$

when $x = a$, $v = 0$ and $c = \frac{n^2 a^2}{2}$

so $v^2 = -n^2 x^2 + n^2 a^2$ ✓

$v^2 = n^2(a^2 - x^2)$

(ii) $v^2 = n^2(a^2 - x^2)$

when $x = 3$, $v = 8$

$64 = n^2(a^2 - 9)$ — ①

when $x = 4$, $v = 6$ ✓

$36 = n^2(a^2 - 16)$ — ②

$\frac{①}{②}$: $\frac{a^2 - 9}{a^2 - 16} = \frac{64}{36}$

Q5 continued.

$9(a^2 - 9) = 16(a^2 - 16)$

$9a^2 - 81 = 16a^2 - 256$

$7a^2 = 175$

$a^2 = 25$ ✓

$a = 5$ (amplitude > 0)

② $6 = n\sqrt{25 - 16}$

$3n = 6$

$n = 2$.

$T = \frac{2\pi}{n}$

$= \frac{2\pi}{2}$

$T = \pi$ s ✓

(ii) The maximum acceleration occurs when $x = a$.

$\ddot{x} = -n^2 x$

$\ddot{x} = -4 \times 5$

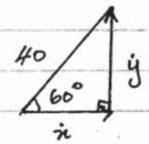
12

maximum acceleration is 20 m/s^2. ✓

(negative implies direction.)

QUESTION 6.

(a)(i) when $t=0$.



$\dot{x} = 40\cos 60°$ $\quad$ $\dot{y} = 40\sin 60°$

$\dot{x} = 20\,m/s$ ✓ $\quad$ $\dot{y} = 20\sqrt{3}\ m/s$

$\ddot{y} = -10$ $\quad$ $\ddot{x} = 0$

integrate w.r.t t.

$\dot{y} = -10t + c_1$ $\quad$ $\dot{x} = c_2$

when $t=0$, $\dot{y} = 20\sqrt{3}$ and $\dot{x} = 20$

thus $c_1 = 20\sqrt{3}$ and $c_2 = 20$.

$\dot{y} = 20\sqrt{3} - 10t$ $\quad$ $\dot{x} = 20$ ✓

integrate w.r.t t.

$y = 20t\sqrt{3} - 5t^2 + c_3$ $\quad$ $x = 20t + c_4$

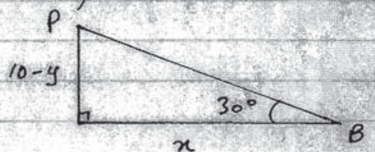
when $t=0$, $y=10$, $x=0$.

thus $c_3 = 10$ and $c_4 = 0$

The equations of motion ✓ are $y = 20t\sqrt{3} - 5t^2 + 10$

$x = 20t$.

(ii) When the ball hits the plane at P:



$$\frac{10-y}{x} = \tan 30° \checkmark$$

$$\frac{5t^2 - 20t\sqrt{3}}{20t} = \frac{1}{\sqrt{3}} \checkmark$$

$$\frac{t - 4\sqrt{3}}{4} = \frac{1}{\sqrt{3}}$$

$$t - 4\sqrt{3} = \frac{4}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$t = 4\sqrt{3} + \frac{4\sqrt{3}}{3}$$

$$t = \frac{16\sqrt{3}}{3}\ s. \checkmark$$

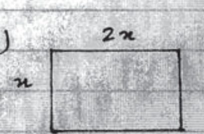
(b) $\angle TAO = 90°$ (TA is a tangent)

$\angle TDO = 90°$ (a line drawn from the centre to the mid-point of the ✓ chord is perpendicular to the chord.)

TAOD is a cyclic quadrilateral since opposite angles are supplementary

Hence $\angle AOT = \angle ADT$ (angles standing on the same arc - drawn to the circumference are equal.) ✓

(c)



let P = perimeter

let A = area

$$A = 2x^2$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} \quad \text{by chain rule}$$

$$\frac{dA}{dt} = 4x \times \frac{dx}{dt}$$

at $x = 100$ cm. $\quad 18 = 400 \times \frac{dx}{dt}$

$$\frac{dx}{dt} = \frac{9}{200} \checkmark$$

Now $P = 6x$ ✓

$$\frac{dP}{dt} = 6 \times \frac{dx}{dt}$$

$$= 6 \times \frac{9}{200}$$

$$= \frac{27}{100}\ cm/s. \checkmark$$

12

QUESTION 7

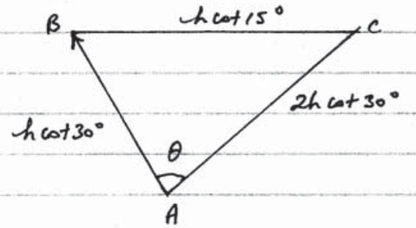
(a) ΔDEF: $\tan 15° = \frac{h}{DF}$

$DF = h\cot 15°$

and $BC = h\cot 15°$. ✓

ΔABD: $AB = h\cot 30°$

ΔAEC: $AC = 2h\cot 30°$ ✓



ΔABC, using the Cosine rule:

$$\cos\theta = \frac{h^2\cot^2 30° + 4h^2\cot^2 30° - h^2\cot^2 15°}{4h^2\cot^2 30°} \checkmark$$

$$= \frac{5\cot^2 30° - \cot^2 15°}{4\cot^2 30°}$$

$$= \frac{15 - \cot^2 15°}{12}$$

$\cos\theta = 0.089316$

$\theta = 84°53'$

The bearing of the taller ✓ tower from A is N 84°53' E.

(b) $(1+x)^{n-1} = \binom{n-1}{0} + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \dots + \binom{n-1}{n-1}x^{n-1}$ ✓

integrating both sides w.r.t x,

$$\frac{(1+x)^n}{n} = \binom{n-1}{0}x + \frac{1}{2}\binom{n-1}{1}x^2 + \frac{1}{3}\binom{n-1}{2}x^3 + \dots + \frac{1}{n}\binom{n-1}{n-1}x^n + c_1$$

put $x = 0$.

$$\frac{1}{n} = c_1$$

$$\frac{(1+x)^n}{n} = \binom{n-1}{0}x + \frac{1}{2}\binom{n-1}{1}x^2 + \frac{1}{3}\binom{n-1}{2}x^3 + \dots \frac{1}{n}\binom{n-1}{n-1}x^n + \frac{1}{n} \checkmark$$

put $x = 7$.

$$\frac{8^n}{n} = 7\binom{n-1}{0} + \frac{7^2}{2}\binom{n-1}{1} + \frac{7^3}{3}\binom{n-1}{2} + \dots + \frac{7^n}{n}\binom{n-1}{n-1} + \frac{1}{n} \checkmark$$

$$\frac{1}{n}(8^n - 1) = 7\binom{n-1}{0} + \frac{7^2}{2}\binom{n-1}{1} + \frac{7^3}{3}\binom{n-1}{2} + \dots + \frac{7^n}{n}\binom{n-1}{n-1}$$

$$\therefore\ 7\binom{n-1}{0} + \frac{7^2}{2}\binom{n-1}{1} + \frac{7^3}{3}\binom{n-1}{2} + \dots + \frac{7^n}{n}\binom{n-1}{n-1} = \frac{1}{n}(2^{3n} - 1) \checkmark$$

(c) Show the statement is true for $n=2$.

A 1, 2. Sum of the products $= 1 \times 2 = 2$.

$$\frac{n}{24}(n-1)(n+1)(3n+2) = \frac{2}{24}(2-1)(2+1)(6+2)$$
$$= \frac{1}{24} \times 1 \times 3 \times 8$$
$$= 2 \checkmark$$

It's true for $n = 2$.

B Assume that the result is true for $n = K$.

ie/ The sum of the products of all the pairs of integers that can be formed from the first K positive integers is $\frac{K}{24}(K-1)(K+1)(3K+2)$.

Prove that it is true for $n = K+1$.

The addition of the integer $(K+1)$ will add another $(1+2+3+\dots+K)(K+1)$ products in pairs.

When $n = K+1$ the sum is :

$$\frac{K}{24}(K-1)(K+1)(3K+2) + (K+1)(1+2+3+\dots+K) \checkmark$$

($1+2+3+\dots+K$ is an arithmetic series. Sum $= \frac{K}{2}(1+K)$

$$= \frac{K}{24}(K-1)(K+1)(3K+2) + (K+1)\frac{K}{2}(K+1) \checkmark$$

$$= \frac{K}{24}(K+1)\{(K-1)(3K+2) + 12(K+1)\}$$

$$= \frac{K}{24}(K+1)\{3K^2 + 11K + 10\}$$

$$= \frac{K}{24}(K+1)(K+2)(3K+5)$$ as required. ✓

C It follows from parts A and B by mathematical induction that the statement is true for all positive integers $n \geq 2$.

(12)