SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
TRIAL EXAMINATIONS 2007

FORM VI

MATHEMATICS EXTENSION 1

Examination date

Monday 6th August 2007

Time allowed

2 hours (plus 5 minutes reading time)

Instructions

All seven questions may be attempted.
All seven questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the seven questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 7 per boy. A total of 1000 booklets should be sufficient.
Candidature: 117 boys.

Examiner

MLS

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) If $(x-2)$ is a factor of the polynomial **1**

$$P(x) = 2x^3 + x + a,$$

find the value of a.

(b) Given that $\log_a b = 2{\cdot}8$ and $\log_a c = 4{\cdot}1$, find $\log_a \frac{b}{c}$. **1**

(c) Shade the region on the number plane satisfied by $y \geq |x+2|$. **2**

(d) Solve the inequality $\dfrac{5}{x-4} \geq 1$. **3**

(e) State the domain and range of $y = \cos^{-1} \dfrac{x}{4}$. **2**

(f) Evaluate $\displaystyle\int_0^3 \frac{dx}{9+x^2}$. **3**

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin \frac{x}{3}}{2x}$. **1**

(b) The point A has coordinates $(-2, 1)$ and the point B has coordinates $(b, -3)$. The point $P(13, -9)$ divides the interval AB externally in the ratio $5:3$. Find the value of b. **2**

(c) Using the substitution $u = e^x$, find **3**

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx.$$

(d) (i) Write down an expression for $\tan 2x$ in terms of $\tan x$. **1**

(ii) Hence show that if $f(x) = x \cot x$, then $f(2x) = (1 - \tan^2 x) f(x)$. **3**

(e) Find the coefficient of x^3 in the expansion of $(2-5x)^6$. **2**

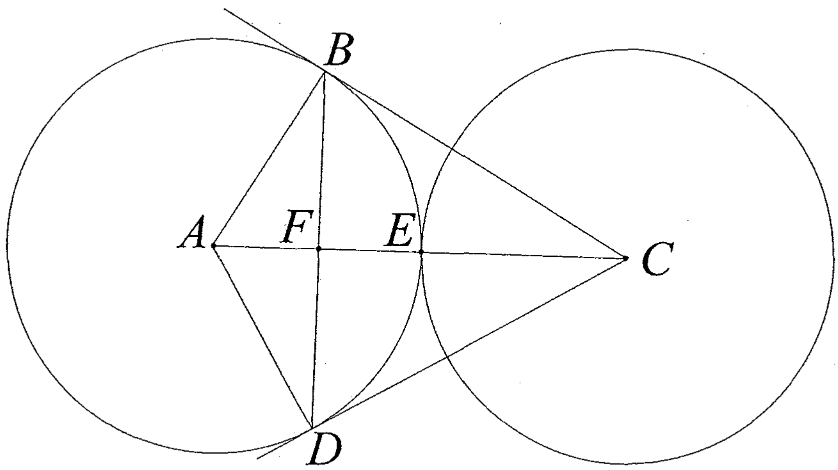
QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate $\cos^{-1} x^2$. **2**

(b) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{4}$. **2**

Exam continues next page ...

(c)



Two circles with equal radii and centres A and C touch externally at E as shown in the diagram. The lines BC and DC are tangents from C to the circle with centre A.

(i) Explain why $ABCD$ is a cyclic quadrilateral. **2**

(ii) Show that E is the centre of the circle that passes through A, B, C and D. **2**

(iii) Show that $\angle BCA = \angle DCA = 30°$. **2**

(iv) Deduce that $\triangle BCD$ is equilateral. **2**

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) The region between the curve $y = \dfrac{1}{\sqrt{1+4x^2}}$ and the x-axis is rotated about the x-axis. Find the volume of the solid enclosed between $x = \dfrac{2}{\sqrt{3}}$ and $x = 2\sqrt{3}$. **3**

(b) Use the substition $u = x - 3$ to evaluate $\displaystyle\int_3^4 x\sqrt{x-3}\,dx$. **4**

(c) A metal rod is taken from a freezer at $-8°$C into a room where the air temperature is $22°$C. The rate at which the rod warms follows Newton's law, that is

$$\frac{dT}{dt} = -k(T-22)$$

where k is a positive integer, time t is measured in minutes, and temperature T is measured in degrees Celsius.

(i) Show that $T = 22 - Ae^{-kt}$ is a solution of the equation $\dfrac{dT}{dt} = -k(T-22)$, and find the value of A. **2**

(ii) The temperature of the rod reaches $4°$C in 90 minutes. Find the exact value of k. **2**

(iii) Find the temperature of the rod after another 90 minutes. **1**

Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) A particle is moving in a straight line so that its displacement x at time t seconds is given by $x = \sqrt{3}\cos 2t - \sin 2t$ metres.

(i) Write $x = \sqrt{3}\cos 2t - \sin 2t$ in the form $x = R\cos(2t + \alpha)$, where $R > 0$ and $0 \le \alpha < 2\pi$. **2**

(ii) When is the particle first at $x = 1$? **1**

(iii) What is the maximum velocity of the particle and when does it first occur? **2**

(b) (i) Show that $x^3 - x - 2 = 0$ has a root between $x = 1$ and $x = 2$. **1**

(ii) Given that $x = 1{\cdot}5$ is your first approximation to a root of $x^3 - x - 2 = 0$, use one application of Newton's method to find another approximation. Give your answer correct to one decimal place. **2**

(c) A particle is moving in simple harmonic motion on a straight line. Its velocity v is given by $v^2 = 4(2x - x^2)$, where x is its displacement from a fixed point O on the line.

(i) Show that its acceleration is given by $\ddot{x} = -4(x - 1)$. **1**

(ii) Find the centre of the motion. **1**

(iii) Find the displacement of the particle when its speed is half the maximum speed. **2**

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) The length of a rectangle is increasing at $6\,\text{cm s}^{-1}$, while the breadth is decreasing so that the area of the rectangle remains constant at $50\,\text{cm}^2$. Find the rate of change of the breadth when the length is 10 cm. **3**

(b) (i) Use the method of mathematical induction to show that if x is a positive integer, then $(1 + x)^n - 1$ is divisible by x, for all positive integers $n \ge 1$. **3**

(ii) Write $12^n - 4^n - 3^n + 1$ as a product of two factors. **1**

(iii) Use parts (i) and (ii) to deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all integers $n \ge 1$. **1**

(c) The quadratic equation $ax^2 + bx + c = 0$ has roots $x = \tan\alpha$ and $x = \tan\beta$.

(i) Show that $\tan(\alpha + \beta) = \dfrac{b}{c - a}$. **2**

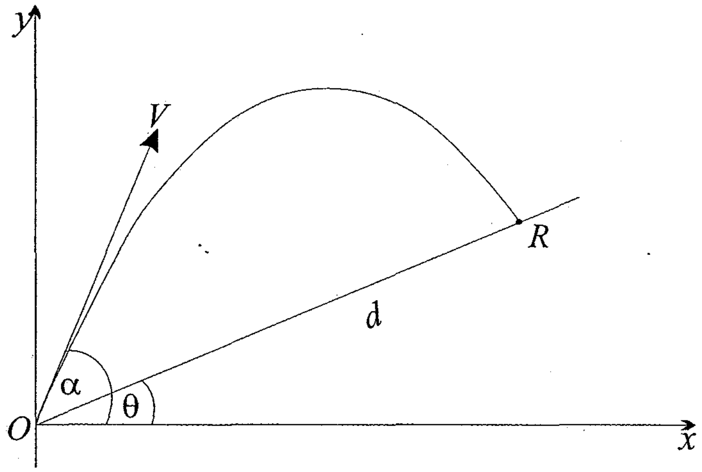
(ii) Show that $\tan^2(\alpha - \beta) = \dfrac{b^2 - 4ac}{(a + c)^2}$. **2**

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Find the value of n if **2**

$$^{n}C_2 + {}^{n}C_1 + {}^{n}C_0 = 37.$$

(b)



In the diagram above, a particle is projected at an angle of elevation α with velocity V from a point O which is at the bottom of an inclined plane. The plane is inclined to the horizontal at an angle θ, where $\theta < \alpha$. The particle meets the inclined plane again at R. The acceleration due to gravity is g, and $0^\circ < \alpha < 90^\circ$. Let $OR = d$.

(i) Given that $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$, where t is the time elapsed, show that the Cartesian equation of the path of the particle is **1**

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2}.$$

(ii) Find expressions for the coordinates of R in terms of θ and d. **1**

(iii) Show that the range of the particle up the inclined plane is given by **3**

$$d = \frac{2V^2\cos\alpha\sin(\alpha - \theta)}{g\cos^2\theta}.$$

(c) Consider the identity $(1+x)^n = 1 + {}^{n}C_1\,x + {}^{n}C_2\,x^2 + {}^{n}C_3\,x^3 + \cdots + {}^{n}C_n\,x^n$, where n is a positive integer.

(i) Use the formula for the sum of a GP to simplify **1**

$$1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^{n-1}.$$

(ii) Use part(i) to show that **1**

$$1+(1+x)+(1+x)^2+(1+x)^3+\cdots+(1+x)^{n-1} = {}^{n}C_1 + {}^{n}C_2\,x + {}^{n}C_3\,x^2 + \cdots + {}^{n}C_n\,x^{n-1}.$$

(iii) Find $\displaystyle\int_{-1}^{0} {}^{n}C_1 + {}^{n}C_2\,x + {}^{n}C_3\,x^2 + \cdots + {}^{n}C_n\,x^{n-1}\,dx.$ **1**

(iv) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{(-1)^{r+1}}{r}\,{}^{n}C_r = \sum_{r=1}^{n} \frac{1}{r}.$ **2**

END OF EXAMINATION

Solutions Trial Ext 1 Maths

2007.

Q.1.

(a) $P(x) = 2x^3 + x + a$

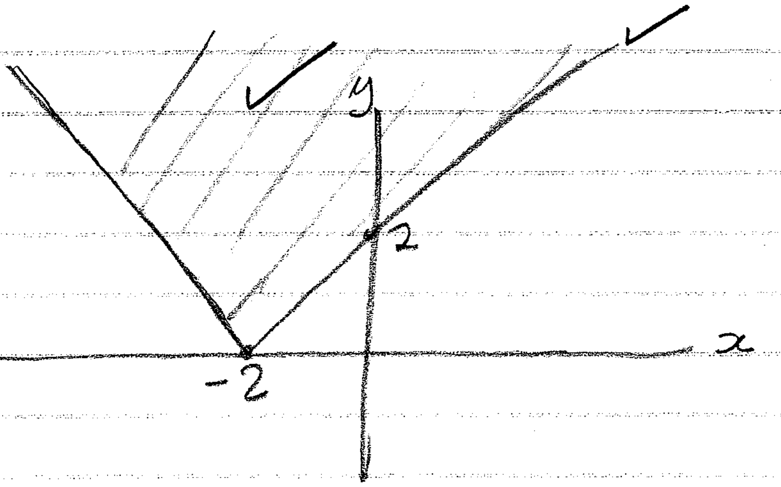
$P(2) = 16 + 2 + a = 0$

$a = -18$ ✓

(b) $\log \frac{b}{c} = \log b - \log c$

$= 2.8 - 4.1$

$= -1.3$

(c) 

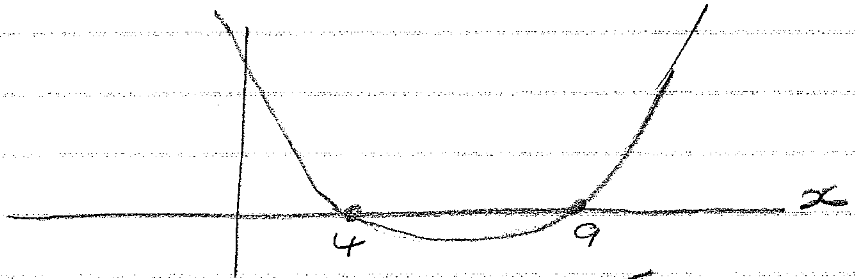
(d) $\frac{5}{x-4} \geq 1, \quad x \neq 4$ ✓

$5(x-4) \geq (x-4)^2$

$(x-4)^2 - 5(x-4) \leq 0$

$(x-4)(x-4-5) \leq 0$

$(x-4)(x-9) \leq 0$ ✓



$4 < x \leq 9$ ✓

(e)

Range: $0 \leq y \leq \pi$ ✓

Domain $-1 \leq \frac{x}{4} \leq 1$

$-4 \leq x \leq 4$ ✓

f. $\int_0^3 \frac{dx}{9+x^2} = \frac{1}{3} \tan^{-1} \frac{x}{3} \Big]_0^3$ ✓ ✓

$= \frac{1}{3}(\tan^{-1} 1 - \tan^{-1} 0)$

$= \frac{1}{3}(\frac{\pi}{4} - 0)$

$= \frac{\pi}{12}$ ✓

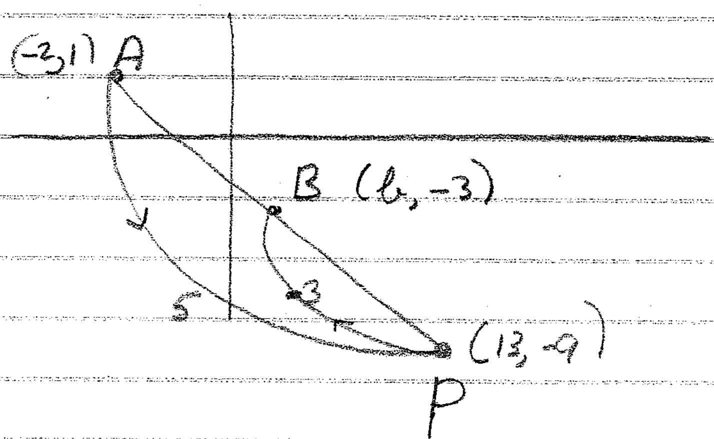
Q2.

(a) $$\lim_{x \to 0} \frac{\sin \frac{x}{3}}{2x} = \frac{1}{6} \lim_{x \to 0} \frac{\sin \frac{x}{3}}{\frac{x}{3}}$$

$$= \frac{1}{6}$$

✓ correct intermediate step is necessary here.

(b)



$$\frac{(-3) \times (-2) + 5b}{5-3} = 13$$ ✓

$$\frac{6+5b}{2} = 13$$

$$6+5b = 26$$

$$5b = 20$$

$$b = 4$$ ✓

(c) $u = e^x$

$du = e^x\, dx$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}}$$ ✓

$$= \sin^{-1} u + c$$ ✓

$$= \sin^{-1} e^x + c$$ ✓

(ignore $+c$).

(d)

(i) $$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$ ✓

(ii) $f(x) = x \cot x$

$$f(2x) = 2x \cot 2x$$ ✓

$$= 2x \times \frac{1-\tan^2 x}{2\tan x}$$ ✓

$$= \frac{x(1-\tan^2 x)}{\tan x}$$

$$= x(1-\tan^2 x)\cot x$$ ✓

$$= (1-\tan^2 x)\, f(x)$$

(e) $(2-5x)^6$

general term is ${}^6C_r\, 2^r (-5x)^{6-r}$

need $6-r=3$ so $r=3$ ✓✓

so coefficient is ${}^6C_3\, 2^3 (-5)^3 = -20000$

✓ for ${}^6C_3$ ($=20$), ✓ for $2^3 \times (-5)^3$, ✓ for $r=3$,

Q3.

(a) $y = \cos^{-1} x^2$

$$\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^4}}$$

$\left(\frac{2x}{\sqrt{1-x^4}} \text{ is worth one mark.} \right)$

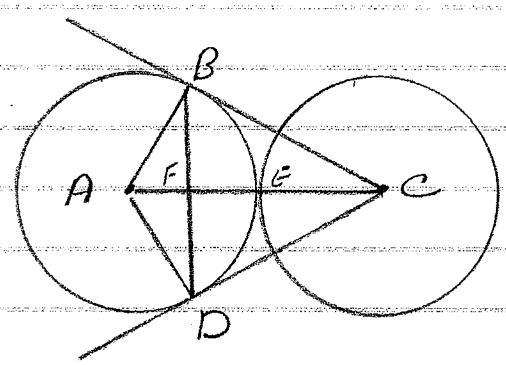
(b)
$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \tfrac{1}{2}(1+\cos 2x) \, dx$$

$$= \tfrac{1}{2}\left[x + \tfrac{1}{2}\sin 2x \right]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{2} + \tfrac{1}{2}\sin\pi\right) - \left(0 + \tfrac{1}{2}\sin 0\right) \right]$$

$$= \tfrac{\pi}{4}$$

(c)



(i) The angle between a tangent and chord is a right angle
so $\angle ABC$ and $\angle ADC$ are right angles
so the opposite angles of ABCD are supplementary
so ABCD is cyclic

(ii) Right angles are subtended at the circumference by diameters
So AC is a diameter of ABCD
But AE = EC, since these are radii of equal circles
So E is the centre of the circle

(iii) $\sin \angle BCA = \frac{AB}{AC} = \frac{1}{2}$

so $\angle BCA = 30°$

Similarly $\sin \angle ACD = \frac{AD}{AC} = \frac{1}{2}$

so $\angle ACD = 30°$

(iv) Tangents from an external point are equal
So $BC = DC$ and $\triangle BCD$ is isosceles
So $\angle CBD = \angle BDC$
$\angle CBD + \angle BDC = 180° - 60°$, angle sum of triangle
$= 120°$
So $\angle CBD = \angle BDC = 60°$
and $\triangle BCD$ is equilateral

(There are many other correct ways to do this question)

Q4.

(a) $y = \frac{1}{\sqrt{1+4x^2}} = \frac{1}{\sqrt{4(\frac{1}{4}+x^2)}} = \frac{\frac{1}{2}}{\sqrt{\frac{1}{4}+x^2}}$

$$V = \frac{\pi}{2}\int_{\frac{2}{\sqrt{3}}}^{2\sqrt{3}} \frac{1}{\frac{1}{4}+x^2}\,dx$$ $a=\frac{1}{2}$

$$= \frac{\pi}{2}\left[\tan^{-1}2x\right]_{\frac{2}{\sqrt{3}}}^{2\sqrt{3}}$$

$$= \frac{\pi}{2}\left(\tan^{-1}4\sqrt{3} - \tan^{-1}\frac{4}{\sqrt{3}}\right)$$

(b) $u = x-3$

$du = dx$

limits: when $x=4$, $u=1$

$x=3$ $u=0$

$$\int_3^4 x\sqrt{x-3}\,dx = \int_0^1 (u+3)\sqrt{u}\,du$$

$$= \int_0^1 (u^{3/2} + 3u^{\frac{1}{2}})\,du$$

$$= \left[\frac{2u^{\frac{5}{2}}}{5} + 3u^{\frac{3}{2}} \times \frac{2}{3}\right]_0^1$$

$$= \left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right]_0^1$$

$$= 2\tfrac{2}{5}$$

c.

(i) $T = 22 - Ae^{-kt}$

$\frac{dT}{dt} = kAe^{-kt}$

$= k(22 - T)$

$= -k(T - 22)$

when $t = 0$, $T = -8$

$-8 = 22 - A$

$A = 30$

ii) $T = 4$ when $t = 90$

$4 = 22 - 30e^{-90k}$

$30e^{-90k} = 18$

$e^{-90k} = \frac{18}{30}$

$-90k = \ln \frac{3}{5}$

$k = -\frac{1}{90} \ln \frac{3}{5}$

iii) Find T when $t = 180$

$T = 22 - 30e^{-180k}$

$= 22 - 30e^{-180 \times (-\frac{1}{90} \ln \frac{3}{5})}$

$= 22 - 30e^{2 \ln \frac{3}{5}}$

$= 22 - \cancel{30}^{6} \times \frac{9}{\cancel{25}_{5}}$

$= 22 - \frac{54}{5}$

$= 11\frac{1}{5}$

Q5

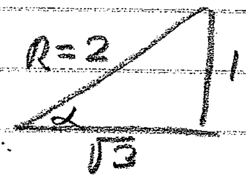
(a)

(i) $x = \sqrt{3}\cos 2t - \sin 2t = R\cos(2t + \alpha)$

$= R\cos 2t \cos\alpha - R\sin 2t \sin\alpha$

$R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$

$\cos\alpha = \frac{\sqrt{3}}{R}$ $\quad$ $\sin\alpha = \frac{1}{R}$



so $R = \sqrt{3+1} = 2$ ✓

and $\alpha = \frac{\pi}{6}$ ✓

So $x = 2\cos(2t + \frac{\pi}{6})$

(ii) Find t when $x = 1$

$2\cos(2t + \frac{\pi}{6}) = 1$

$\cos(2t + \frac{\pi}{6}) = \frac{1}{2}$

$2t + \frac{\pi}{6} = \frac{\pi}{3}$

$2t = \frac{\pi}{6}$

$t = \frac{\pi}{12}$ seconds ✓

(iii) $x = 2\cos(2t + \frac{\pi}{6})$

$\dot{x} = -4\sin(2t + \frac{\pi}{6})$ ✓

maximum value of $\dot{x}$ is $4\ ms^{-1}$, it occurs when

$\sin(2t + \frac{\pi}{6}) = -1$

$2t + \frac{\pi}{6} = \frac{3\pi}{2}$

$2t = \frac{3\pi}{2} - \frac{\pi}{6}$

$= \frac{4\pi}{3}$

$t = \frac{2}{3}\pi$ ✓

it reaches maximum velocity after $\frac{2\pi}{3}$ s.

(b) (i)

$f(x) = x^3 - x - 2$

$f(1) = 1 - 1 - 2 = -2$

$f(2) = 8 - 2 - 2 = 4$

$f(x)$ changes sign between $x = 1$ and 2, it is continuous, so there is a root between $x = 1$ and 2.

(ii) $f'(x) = 3x^2 - 1$

$f(1.5) = 1.5^3 - 1.5 - 2$

$f'(1.5) = 3(1.5)^2 - 1$

$$x_1 = 1.5 - \frac{1.5^3 - 1.5 - 2}{3(1.5)^2 - 1}$$

$\doteqdot 1.5$

($\frac{35}{23}$ is exact also accept)

(c) i) $\frac{1}{2}v^2 = 2(2x - x^2) = 4x - 2x^2$

$\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx}$

$= 4 - 4x$

$= -4(x - 1)$

ii) Centre is when $\ddot{x} = 0$, $x = 1$

iii) Maximum speed is when $\ddot{x} = 0$, $x = 1$.

$x = 1$, $v^2 = 4(2 - 1) = 4$.

so maximum speed is 2, and half is 1.

so $2\sqrt{2x - x^2} = 1$

$2x - x^2 = \frac{1}{4}$

$4x^2 - 8x + 1 = 0$

$$x = \frac{8 \pm \sqrt{64-16}}{8}$$

$$= \frac{8 \pm \sqrt{48}}{8} \qquad 48 = 16 \times 3$$

$$= \frac{8 \pm 4\sqrt{3}}{8}$$

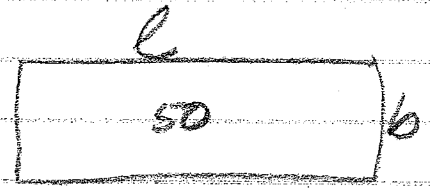
$$= \frac{2 \pm \sqrt{3}}{2}$$

$$= 1 \pm \frac{\sqrt{3}}{2} \text{ m}$$

So when the particle is $1+\frac{\sqrt{3}}{2}$ or $1-\frac{\sqrt{3}}{2}$ m from O it has half the maximum speed

6.

(a)



$$lb = 50 \quad \text{so} \quad b = \frac{50}{l} = 50l^{-1}$$

$$\frac{db}{dl} = -50l^{-2} = \frac{-50}{l^2}$$ ✓

$$\frac{db}{dt} = \frac{db}{dl}\,\frac{dl}{dt}$$

$$= \frac{-50}{l^2} \times 6$$ ✓

$$= \frac{-50}{100} \times 6 \quad \text{when} \quad l = 10$$

$$= -3 \text{ cm s}^{-1}$$ ✓

The breadth is decreasing at 3 cm s^{-1}

(b)

A. For $n=1$, we have $(1+x)^1 - 1 = x$ which is divisible by x. ✓

B. Assume the expression is divisible by x for some integer k

then $(1+x)^k - 1 = xM$, M an integer ✓

Now prove that $(1+x)^{k+1} - 1$ is divisible by x.

Now, $(1+x)^{k+1} - 1 = (1+x)(1+x)^k - 1$

$= (1+x)(xM+1) - 1$, using the induction hypothesis ✓

$= xM + 1 + x^2M + x - 1$

$= xM + x^2M + x$

$= xM + x^2M + x$
$= x(M + xM + 1)$ which is divisible by x

So, from parts A & B using the principle of mathematical induction the expression is divisible by x for all $n \geq 1$.

(ii) $12^n - 4^n - 3^n + 1$
$= 4^n(3^n - 1) - 1(3^n - 1)$
$= (3^n - 1)(4^n - 1)$

(iii) From (i), $3^n - 1 = (1+2)^n - 1$ is divisible by 2
$4^n - 1 = (1+3)^n - 1$ is divisible by 3
so the product is divisible by 6 for $n \geq 1$.

c)
(i) Sum of roots $= \tan\alpha + \tan\beta = -\frac{b}{a}$
product of roots $= \tan\alpha \tan\beta = \frac{c}{a}$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{-\frac{b}{a}}{1 - \frac{c}{a}}$$

$$= \frac{-b}{a-c}$$

$$= \frac{b}{c-a}$$

11)
$$\tan^2(\alpha-\beta) = \left(\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right)^2$$

$$= \frac{\tan^2\alpha - 2\tan\alpha\tan\beta + \tan^2\beta}{(1+\tan\alpha\tan\beta)^2}$$

$$= \frac{(\tan\alpha+\tan\beta)^2 - 4\tan\alpha\tan\beta}{(1+\tan\alpha\tan\beta)^2} ✓$$

$$= \frac{\frac{b^2}{a^2} - \frac{4c}{a}}{\left(1+\frac{c}{a}\right)^2} ✓$$

$$= \frac{b^2-4ac}{(a+c)^2}$$

Q7.

a) $^nC_2 + {}^nC_1 + {}^nC_0 = 37$

$\frac{n(n-1)}{2} + n + 1 = 37$ ✓

$n^2 - n + 2n + 2 = 74$

$n^2 + n - 72 = 0$

$(n+8)(n+9) = 0, \quad n \geq 2$

so $n = 8$ ✓

(b) (i) $t = \frac{x}{V\cos\alpha}$

$y = V\frac{x}{V\cos\alpha}\sin\alpha - \frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha}$ ✓

$= x\tan\alpha - \frac{1}{2}\frac{g}{V^2}x^2\sec^2\alpha$

ii) $R = (d\cos\theta, d\sin\theta)$ ✓

iii) from (i) and (ii) ✓

$d\sin\theta = d\cos\theta\tan\alpha - \frac{1}{2}\frac{g}{V^2}d^2\cos^2\theta\sec^2\alpha$

$\sin\theta = \cos\theta\tan\alpha - \frac{gd\cos^2\theta\sec^2\alpha}{2V^2}$

$d\frac{g\cos^2\theta}{2V^2\cos^2\alpha} = \cos\theta\tan\alpha - \sin\theta$

$d = \left(\cos\theta\frac{\sin\alpha}{\cos\alpha} - \sin\theta\right)\frac{2V^2\cos^2\alpha}{g\cos^2\theta}$ ✓

$= \frac{2V^2(\cos\theta\sin\alpha\cos\alpha - \sin\theta\cos^2\alpha)}{g\cos^2\theta}$

$= \frac{2V^2\cos\alpha(\cos\theta\sin\alpha - \sin\theta\cos\alpha)}{g\cos^2\theta}$ ✓

recognising $\sin(\alpha-\theta)$

$$= \frac{2V^2 \cos\alpha \sin(\alpha - \theta)}{g \cos^2\theta}$$

c) next page

C

i) $1+(1+x)+(1+x)^2+\dots+(1+x)^{n-1} = \dfrac{1\left((1+x)^n-1\right)}{(1+x)-1}$

$= \dfrac{(1+x)^n-1}{x}$

ii)

$1+(1+x)+(1+x)^2+\dots+(1+x)^{n-1} = \dfrac{(1+x)^n-1}{x}$

$= \dfrac{\left(1+{}^nC_1x+{}^nC_2x^2+\dots{}^nC_nx^n\right)-1}{x}$

$= \dfrac{{}^nC_1x+{}^nC_2x^2+\dots{}^nC_nx^n}{x}$

$= {}^nC_1+{}^nC_2x+\dots{}^nC_nx^{n-1}$

iii) $\displaystyle\int_{-1}^{0} {}^nC_1+{}^nC_2x+\dots{}^nC_nx^{n-1}\,dx$

$= \left[{}^nC_1x+\tfrac{1}{2}{}^nC_2x^2+\tfrac{1}{3}{}^nC_3x^3+\dots\tfrac{1}{n}{}^nC_nx^n\right]_{-1}^{0}$

$= {}^nC_1-\tfrac{1}{2}{}^nC_2+\tfrac{1}{3}{}^nC_3-\dots+\dfrac{(-1)^{n+1}}{n}{}^nC_n$

iv)

Now $\displaystyle\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r}{}^nC_r = {}^nC_1-\tfrac{1}{2}{}^nC_2+\tfrac{1}{3}{}^nC_3-\dots\frac{(-1)^{n+1}}{n}{}^nC_n$

$\displaystyle= \int_{-1}^{0}{}^nC_1+{}^nC_2x+\dots{}^nC_nx^{n-1}\,dx$

(over)

$$= \int_{-1}^{0} 1 + (1+x) + (1+x)^2 + \dots (1+x)^{n-1} \, dx$$

$$= x + \frac{(1+x)^2}{2} + \frac{(1+x)^3}{3} + \dots \frac{(1+x)^n}{n} \Big]_{-1}^{0}$$

$$= \left(\frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}\right) - (-1)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}$$

$$= \sum_{r=1}^{n} \frac{1}{r}$$