SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2002

MATHEMATICS

EXTENSION 1

and

EXTENSION 2 (COMMON)

Time allowed - Two hours

(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions
- ALL questions are of equal value
- Start each question on a **new page**, clearly marked with the number of the question and your name and class.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Approved calculators may be used.
- A sheet of standard integrals is provided.

	Mark
Question 1	
Question 2	
Question 3	
Question 4	
Question 5	
Question 6	
Question 7	
TOTAL	

Question 1

a) $x + 2y + 2\sqrt{x - y} = 11 + \sqrt{20}$ find x and y. (3)

b) Solve $\dfrac{1-x}{1+x} \le 1$ (3)

c) Shade the region on the number plane where $y \le \log_e x$ *and* $y \le 1 - x^2$ hold simultaneously. (2)

d) If the line $y = mx + b$ is 1 unit from the origin, prove that $b^2 = m^2 + 1$. (2)

e) Given A (-7 ,2) and B (5, 8), find the coordinates of Q such that A Q : Q B = - 4 : 3. (2)

Question 2 (start a new page)

a) Find the gradients of the lines that make an angle of $45°$ with the line $2x - 3y + 6 = 0$. (3)

b) Find $\int \cos^2 \dfrac{x}{2} \, dx$ (3)

c) i) Express *tan 2A* in terms of *tan A* (1)

ii) Hence let $A = \tan^{-1} \dfrac{1}{3}$ to prove that $2\tan^{-1} \dfrac{1}{3} = \tan^{-1} \dfrac{3}{4}$ (2)

d) i) State the domain and range of $y = \sin^{-1}(\sin x)$ (2)

ii) Draw a neat sketch of $y = \sin^{-1}(\sin x)$ (1)

Question 3 (start a new page)

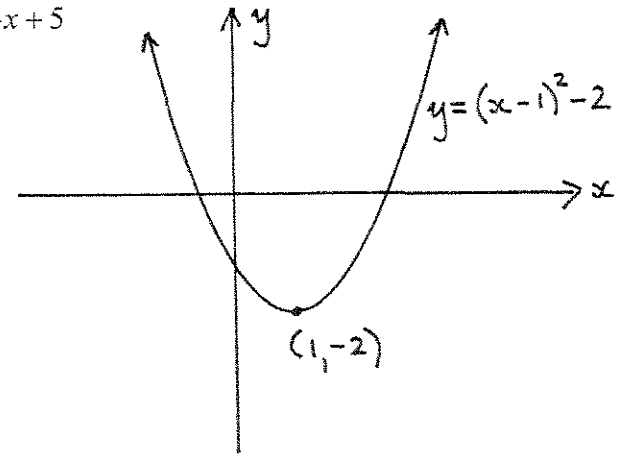
a) Differentiate i) $\cos^{-1}(x^2)$ (1)

ii) $e^{2x} \log_e 2x$ (2)

b) i) Find a and b such that $x^2+4x+5=(x+a)^2+b$ (1)

ii) Hence evaluate $\int \frac{dx}{x^2+4x+5}$ (2)

c)



The graph of $y=(x-1)^2-2$ is shown above.

For the part of the curve where $x \geq 1$, an inverse function $f^{-1}(x)$ exists.

i) Find $y=f^{-1}(x)$ and state its range and domain. (3)

ii) Sketch $y=f(x)$ and $y=f^{-1}(x)$ on the sames axes. Indicate any point(s) of intersection but do **not** find the coordinates. (1)

iii)Find $f^{-1}[f(p)]$ *if* $p<1$ (2)

Question 4 (start a new page)

a) i) Express $\cos x-\sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$, *if* $R>0$ *and* $0<\alpha<\frac{\pi}{2}$ (3)

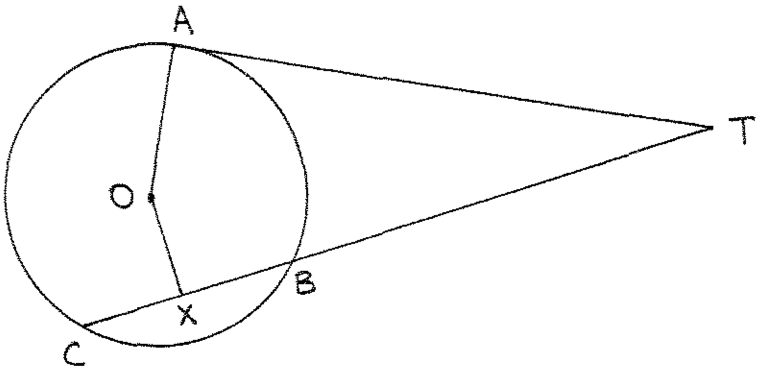
ii)Hence find the general solution to $\cos x-\sqrt{3}\sin x=2$ (2)

b) A particle moving in a straight line x centimetres from the origin 0, after t minutes has its displacement given by $x(t)=3-5\cos 2t$

i) Show that acceleration is given by $a=-4(x-3)$ (2)

ii) Find the period of its motion (1)

c)



A,B,C are 3 points on a circle centre O.

The tangent at A meets CB produced at T.

X is the midpoint of BC.

i) Copy the diagram onto your answer sheets

ii) Prove that AOXT is a cyclic quadrilateral without adding any constructions to the diagram. (3)

iii) Hence state why $\angle AOT = \angle AXT$ (1)

Question 5 (start a new page)

a) i) Prove $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$ (1)

ii) The acceleration of a particle x metres from the origin O, at time t seconds is given by $\ddot{x} = -\frac{1}{2}e^{-x}$. If the velocity v is 1 m/s when $x = 0$, find its velocity when $x = 4$. (3)

b) Use the substitution $u = \cos x$ or otherwise to evaluate (4)

$$\int_{\pi/3}^{\pi/2} \sin x \cos^2 x \, dx$$

c) The point P $(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The tangent at P meets the axis of the parabola at R and S is the focus.

Prove that RS = SP. (4)

Question 6 (start a new page)

a) Prove by mathematical induction that $\cos(x+n\pi)=(-1)^n\cos x$ for all integers $n\geq 1$ (4)

b) The rate of growth of the number of bacteria in a colony is proportional to the excess of the colonys population over 5000 and is given by $\frac{dN}{dt}=k(N-5000)$

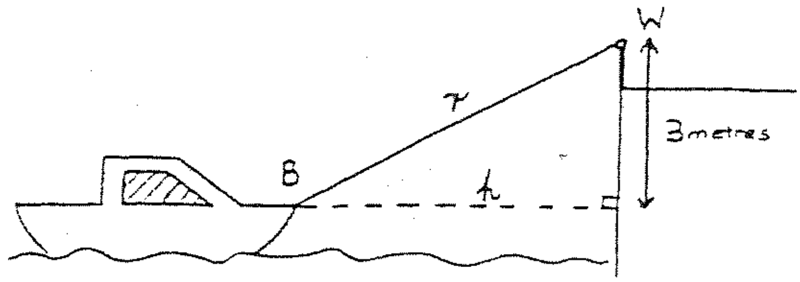
i) Show that $N=5000+Ae^{kt}$ is a solution to this differential equation (1)

ii) If the initial population is 15,000 and reaches 20,000 after 2 days, find the values of A and k. (2)

iii) Hence, calculate the expected population after 7 days. (1)

c) A boat is being winched towards a wharf by a rope attached to its bow, B. The winch, W is 3 metres above B. The rope (r) is wound at the rate of 12 metres per minute. Find the rate at which the boat is approaching the wharf when the horizontal distance (h) from the bow to the wharf is 5 metres.(assume the rope is stretched tightly in a straight line at all stages of the operation)

(4)



Question 7 (start a new page)

a) If α,β and γ are the roots of $x^3-4x^2+1=0$ find

i) $\alpha+\beta+\gamma$ (1)

ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ (2)

b) A projectile is fired horizontally with a speed of V m/s from a point h metres above the horizontal ground. Assume g is acceleration due to gravity.

i) Use calculus to show that the position of the projectile at time t is given by

$$x = Vt$$

and $$y = \frac{-gt^2}{2} + h$$ (4)

ii) Prove that the projectile will reach the ground after $\sqrt{\frac{2h}{g}}$ seconds. (1)

iii) If the angle at which the projectile reaches the ground is $60°$ to the horizontal, prove that $3V^2 = 2gh$ and that its speed on reaching the ground is 2V m/s. (4)

Question 1

a) $x+2y+2\sqrt{x-y} = 11+\sqrt{20}$

$= 11 + 2\sqrt{5}$

$\begin{cases} x+2y=11 \\ x-y=5 \end{cases}$

$3y = 6$

$\therefore y=2$ and $x=7$ ③

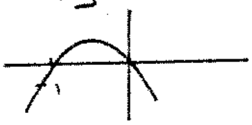
b) $\frac{1-x}{1+x} \leq 1$

$(1+x)(1-x) \leq (1+x)^2$

$(1+x)(1-x)-(1+x)^2 \leq 0$

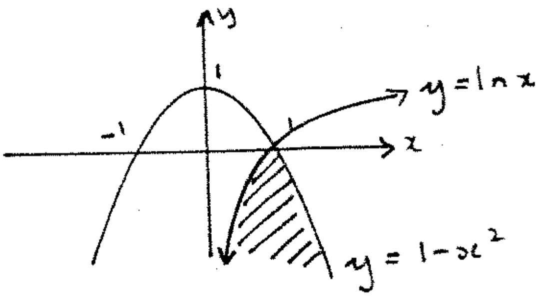
$(1+x)\left[(1-x)-(1+x)\right] \leq 0$

$(1+x)(-2x) \leq 0$



$\therefore x \geq 0$ and $x < -1$ ③

c)



②

d) rewrite line $mx - y + b = 0$

$1 = \left| \frac{m\times 0 - 1\times 0 + b}{\sqrt{m^2+1^2}} \right|$

$1 = \frac{b^2}{m^2+1}$

$\therefore m^2+1 = b^2$ ②

e) $A(-7,2)$ $B(5,8)$

$-4 : 3$

$Q\left(\frac{-21-20}{-1}, \frac{6-32}{-1}\right)$

$Q(41, 26)$ ②

Question 2

a) $\tan 45° = \left| \frac{m - \frac{2}{3}}{1+\frac{2}{3}m} \right|$

Let gradients be $m_1 = \frac{2}{3}$ & m

$1 = \left| \frac{3m-2}{3+2m} \right|$

$3m-2 = 3+2m$ & $-3-2m = 3m-2$

$m = 5$ & $-1 = 5m$

$m = -1/5$ ③

b) $\int \cos^2 \frac{x}{2}\, dx$

since $\cos^2 A = \frac{1}{2}(\cos 2A + 1)$

$\int \cos^2 \frac{x}{2}\, dx = \frac{1}{2}\int (\cos x + 1)\, dx$ ③

$= \frac{1}{2}\left[\sin x + x\right] + c$

c) i) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ ①

ii) $A = \tan^{-1}\frac{1}{3}$

$\tan A = \frac{1}{3}$

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$= \frac{2\times 1/3}{1-\frac{1}{9}}$

$\tan 2A = \frac{3}{4}$

$2A = \tan^{-1}\frac{3}{4}$

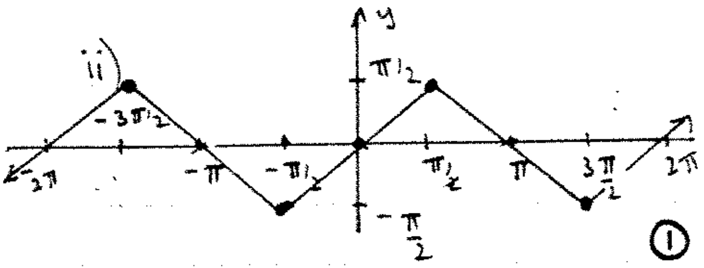
$\therefore 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4}$ ②

d) i) $y = \sin^{-1}(\sin x)$

D: all real x

R: $-\frac{\pi}{2} \leq y \leq \pi/2$ (2)

ii)



(1)

Question 3

a) i) $\frac{d}{dx} \cos^{-1}(x^2) = \frac{-2x}{\sqrt{1-x^4}}$ (1)

ii) $u = e^{2x}$ $\quad v = \ln 2x$

$u' = 2e^{2x}$ $\quad v' = \frac{1}{x}$

$\frac{d}{dx}(e^{2x} \cdot \ln 2x) = \ln 2x \cdot 2e^{2x} + \frac{e^{2x}}{x}$ (2)

b) i) $x^2 + 4x + 5 = (x^2 + 4x + 4) + 1$

$= (x+2)^2 + 1 \quad \therefore a = 2, \; b = 1$ (1)

ii) $\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{1+(x+2)^2}$

$= \tan^{-1}(x+2) + c$ (2)

c) $f(x) = (x-1)^2 - 2$

$x = (y-1)^2 - 2$

$x + 2 = (y-1)^2$

$\sqrt{x+2} = y - 1$

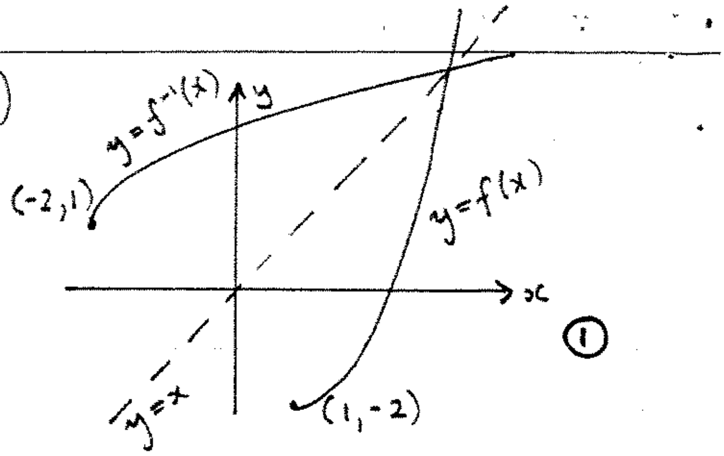
$\sqrt{x+2} + 1 = y$

$\therefore f^{-1}(x) = \sqrt{x+2} + 1$,

D: $x \geq -2$

R: $y \geq 1$ (3)

ii)



(1)

iii) $f(p) = (p-1)^2 - 2 \quad$ if $p < 1$

$f^{-1}(p) = -\sqrt{p+2} + 1$

$f^{-1}[f(p)] = -\sqrt{(p-1)^2 - 2 + 2} + 1$

$= -(p-1) + 1$ (2)

$f^{-1}(f(p)) = -p + 2$, if $p < 1$

Question 4

a) i) $\cos x - \sqrt{3}\sin x$

$R = \sqrt{4} = 2$

$\therefore \cos x - \sqrt{3}\sin x = 2\left[\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right]$

required form

$= 2[\cos x \cos\alpha - \sin x \sin\alpha$

$\therefore \cos\alpha = \frac{1}{2} \quad \sin\alpha = \frac{\sqrt{3}}{2}$

$\alpha = \pi/3$

$\therefore \cos x \; \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$ (2)

ii) $\therefore 2\cos\left(x + \frac{\pi}{3}\right) = 2$

$\cos\left(x + \frac{\pi}{3}\right) = 1$

$x + \frac{\pi}{3} = 2n\pi \pm \cos^{-1}(1)$ ~~$\cos^{-1}(1)$~~ 0

$\therefore$ general solution $\quad x = 2n\pi - \frac{\pi}{3}$

where n is an integer

(2)

b) i) $x(t) = 3 - 5\cos 2t$

$\dot{x} = 10\sin 2t$

$\ddot{x} = 20\cos 2t$

since $5\cos 2t = 3 - x$

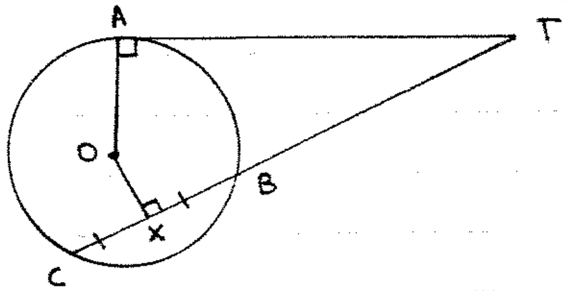
$\therefore \ddot{x} = 4(5\cos 2t)$

$\ddot{x} = 4(3 - x)$

$\therefore \ddot{x} = -4(x - 3)$ ②

ii) Period $= \frac{2\pi}{2} = \pi$ seconds ①

c) i)



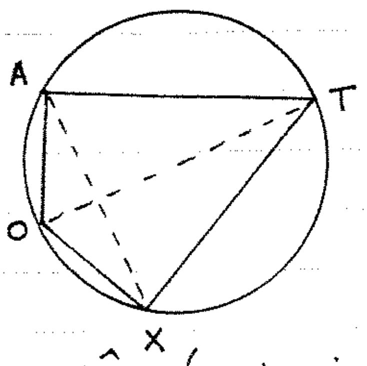
ii) $O\hat{A}T = 90°$ (tang ⊥ radius)

$O\hat{X}T = 90°$ (line from centre perp to bisected chord)

$\therefore$ opposite angles supplementry

AOXT is a cyclic quad ③

iii)



①

$A\hat{O}T = A\hat{X}T$ (angles in same segment)

Question 5

a) i) RHS $= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \cdot \frac{dv}{dx}$

$= v \cdot \frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$ $\therefore$ LHS = RHS

$= \frac{dv}{dt}$

$= \ddot{x}$ ①

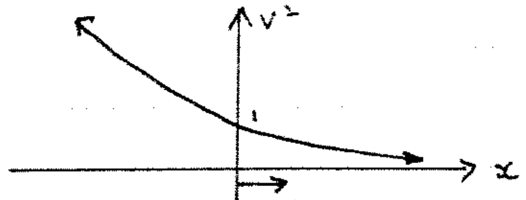
ii) $\ddot{x} = -\frac{1}{2}e^{-x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{2}e^{-x}$

$\frac{1}{2}v^2 = \frac{1}{2}e^{-x} + c$

sub $v = 1$ $x = 0$ $\therefore c = 0$

$v^2 = e^{-x}$



initially $x = 0$ $\therefore v$ +ve

$\therefore$ when $x = 4$ ③

$v = \sqrt{e^{-4}}$ m/s OR (·14 m/s)

b) $u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

$x = \pi/2$ $\therefore u = \cos \pi/2$

$u = 0$

$x = \pi/3$ $u = \cos \pi/3$

$u = 1/2$

$$\int_{\pi/3}^{\pi/2} \sin x \cdot \cos^2 x \, dx$$

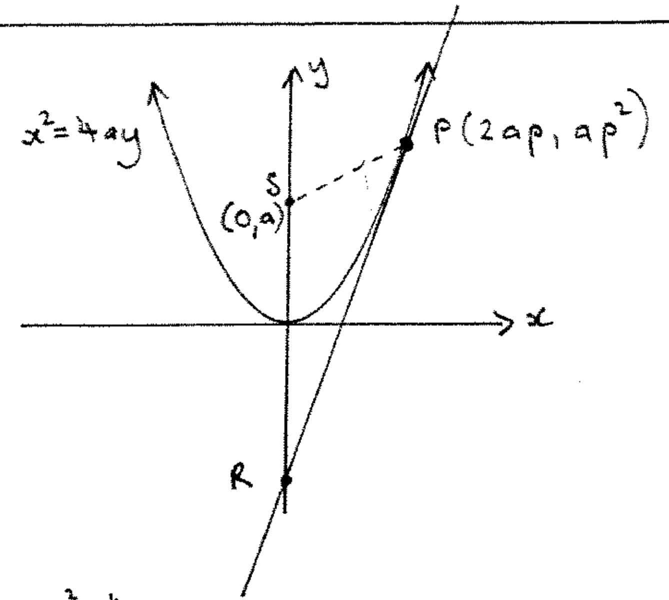
$$\Rightarrow \int_{1/2}^{0} \sin x \cdot u^2 \cdot \frac{du}{-\sin x}$$

$$= \int_{0}^{1/2} u^2 \, du$$

$$= \left[\frac{u^3}{3}\right]_0^{1/2}$$

$$= \frac{1}{24}$$ ④

c)



$x^2=4ay$

$\therefore y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$ $\therefore$ grad tang at P

$m_T = p$

eqn tang: $y - ap^2 = p(x - 2ap)$

$y = px - ap^2$

$\therefore$ $R(0, -ap^2)$

$\therefore$ $SR = a + ap^2 = a(p^2+1)$

$SP = \sqrt{(2ap-0)^2 + (ap^2-a)^2}$

$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$

$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$

$= a\sqrt{p^4 + 2p^2 + 1}$

$= a\sqrt{(p^2+1)^2}$

$SP = a(p^2+1)$

$\therefore$ $SP = SR$ (4)

Question 6

a) Prove $\cos(x + n\pi) = (-1)^n . \cos x$

Show true for $n = 1$

ie $\cos(x + \pi) = (-1)^1 \cos x$

LHS $= \cos x \cos\pi - \sin x \sin\pi$
$= (-1)\cos x$
$=$ RHS

Assume true $n = k$

$\cos(x + k\pi) = (-1)^k . \cos x$ — *

Prove true for $n = k+1$

ie $\cos(x + (k+1)\pi) = (-1)^{k+1} . \cos x$

LHS $= \cos(x + (k+1)\pi)$

$= \cos[(x + \pi k) + \pi)]$

$= \cos(x+\pi k)\cos\pi - \sin(x+\pi k)\sin\pi$ (0)

$= \cos(x + \pi k) \times (-1)$

$= (-1)^k . \cos x . (-1)$ from *

$= (-1)^{k+1} . \cos x$

$=$ RHS

Since true for $n=1$ and if true for $n=k$ $\therefore$ true for $n=k+1$ $\therefore$ true for all +ve integers (4)

b) i) $N = 5000 + Ae^{kt}$

$\frac{dN}{dt} = kAe^{kt}$

since $Ae^{kt} = N - 5000$

$\therefore$ $\frac{dN}{dt} = k(N - 5000)$ (1)

ii) $t = 0$ $N = 15,000$

$15000 = 5000 + Ae^0$

$\therefore$ $A = 10,000$

$t = 2$ $N = 20,000$

$20,000 = 5000 + 10000e^{k \times 2}$

$15,000 = 10,000e^{2k}$ (2)

$\ln\left(\frac{3}{2}\right) = 2k$

$k = \frac{1}{2}\ln(3/2)$ $k \doteqdot .2027$

iii) $N = 5000 + 10{,}000\, e^{7 \times \frac{1}{2}\ln(3/2)}$

$N \doteqdot 46335$ ①

c) $\frac{dr}{dt} = 12$ m/min

want $\frac{dh}{dt}$ when $h = 5$

$h^2 = r^2 - 9$

$h = \sqrt{r^2 - 9}$

$\frac{dh}{dr} = \frac{r}{\sqrt{r^2 - 9}}$

$\frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dt}$

$= \frac{r}{\sqrt{r^2-9}} \cdot 12$

when $h = 5$ $\quad r = \sqrt{34}$ using pyth

$\therefore \frac{dh}{dt} = \frac{12\sqrt{34}}{\sqrt{25}}$

$= 13.99$ m/s

OR $\frac{12}{5}\sqrt{34}$ m/s ④

Question 7

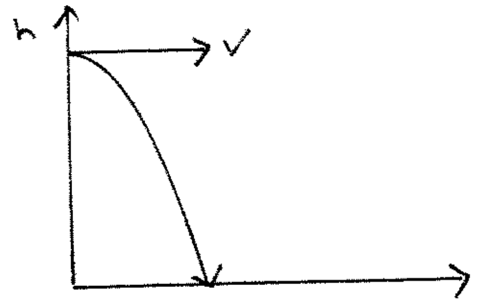
a) i) $\alpha + \beta + \gamma = 4$ ①

ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{0}{-1}$

$= 0$ ②

b)



i) $t = 0 \quad x = 0 \quad y = h \quad \dot{x} = V \quad \dot{y} = 0$

$\ddot{x} = 0$ — ①

$\dot{x} = c_1$

$c_1 = V$

$\dot{x} = V$ — ②

$x = Vt + c_2$

$c_2 = 0$

$x = Vt$ — ③

$\ddot{y} = -g$ — ④

$\dot{y} = -gt + k_1$

$k_1 = 0$

$\dot{y} = -gt$ — ⑤

$y = -\frac{gt^2}{2} + k_2$

$k_2 = h$

$\therefore y = -\frac{gt^2}{2} + h$ — ⑥

④ mks

ii) reach ground when $y = 0$

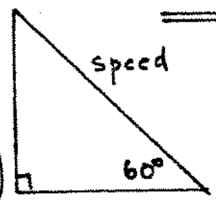
sub into eqn ⑥

$0 = -\frac{gt^2}{2} + h$

$gt^2 = 2h$

$t = \sqrt{\frac{2h}{g}}$ seconds ①

iii)



$\dot{y} = -g\left(\sqrt{\frac{2h}{g}}\right)$

$\dot{x} = V$

$\tan 60° = \frac{g\sqrt{\frac{2h}{g}}}{V}$

$V\sqrt{3} = g\sqrt{\frac{2h}{g}}$

$\therefore 3V^2 = g.2h$ ②

speed $= \sqrt{V^2 + g^2 \frac{2h}{g}}$

$= \sqrt{V^2 + 2gh}$

$= \sqrt{V^2 + 3V^2}$

speed $= 2V$ m/s ②