SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2003

Mathematics Extension 1

General Instuctions

- Reading time - 5 minutes
- Working time - 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 84

- Attempt Questions 1 – 7
- All questions are of equal value

Name :__________________________

Teacher :__________________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	**Total**

Question 1 (12 marks) **Marks**

(a) Fully factorise $x^3 - 3x^2 - 10x + 24$ 2

(b) Differentiate (i) $\cos^{-1} 3x$ 1

(ii) $\cos^3 x$ 1

(c) Solve $\dfrac{x+1}{x-1} \leq 3$ 3

(d) Find, giving your answer to the nearest degree, the acute angle between the lines $x - y - 3 = 0$ and $x - 3y + 1 = 0$. 2

(e) Sketch the function $y = 4\sin^{-1} 2x$ showing clearly the domain and range. 3

Question 2 (12 marks) Start a new page. **Marks**

(a) Find the coordinates of the point P, which divides the interval from $A(-1,8)$ to $B(13,3)$ internally in the ratio $5:2$. 2

(b) Evaluate (i) $\displaystyle\int_0^2 \frac{dx}{\sqrt{16-x^2}}$ 2

(ii $\displaystyle\int_{-1}^0 \frac{2x}{1-2x}\,dx$ using the substitution $u = 1 - 2x$ 3

(c) (i) Sketch $y = |2x - 3|$ 1

(ii) Using the above sketch, or otherwise, determine the values of m for which the equation $|2x-3| = mx$ has 2 solutions 1

(d) Given that $\displaystyle\sum_{r=1}^{n} \sin(2r-1)x = \frac{1-\cos 2nx}{2\sin x}$ 3

find $\displaystyle\int \frac{1-\cos 4x}{2\sin x}\,dx$

Question 3 (12 marks) Start a new page. **Marks**

(a) Prove by Mathematical Induction that 4

$5^n + 2(11)^n$ is a multiple of 3 for all positive integers n.

(b) Consider the curve $y = (1+2x)e^{-2x}$.

(i) Give the coordinates of the y intercept. 1

(ii) Where does the curve cross the x axis ? 1

(iii) Find the coordinates of any stationary points and determine their nature. 3

(iv) Show that there is a point of inflexion where $x = \frac{1}{2}$. 1

(v) Draw a neat sketch of the curve showing all the important features. 2

Question 4 (12 marks) Start a new page. **Marks**

(a) A particle is moving about the origin in Simple Harmonic Motion.

After t seconds its displacement in centimetres from the origin

is given by $x = 5\cos(3t + \frac{\pi}{3})$

(i) By differentiation, show that the acceleration of the particle can be expressed in the form $-n^2x$. 2

(ii) State the amplitude of the motion. 1

(iii) State the period of the motion. 1

(iv) Find the distance travelled by the particle in the first $\frac{\pi}{3}$ seconds. 1

(b) Solve $3\sin\theta - \cos\theta = 2$ for $0 \le \theta \le 360°$ 4

giving your answers to the nearest degree.

(c) At time t minutes, the temperature $T°$ Celsius of an object is given by

$T = 24 - 22e^{-kt}$ where k is a constant.

After 5 minutes the temperature of the object has risen from $2°\ C$ to $13°\ C$.

(i) Find the exact value of k . 2

(ii) Find the temperature of the object after 10 minutes. 1

Question 5 (12 marks) Start a new page. **Marks**

(a) The volume of a cube is increasing at the constant rate of $10\ cm^3/\sec$. 3

Find the rate at which the surface area is increasing when the volume is $125\ cm^3$.

(b) Given that $\sin\theta = \frac{3}{5}$ and θ is acute, find the exact value of $\tan\frac{\theta}{2}$. 3

Justify your answer.

(c) The tangent to the parabola $x^2 = 4ay$ at the variable point $P(2ap, ap^2)$

cuts the x axis at Q and the y axis at R.

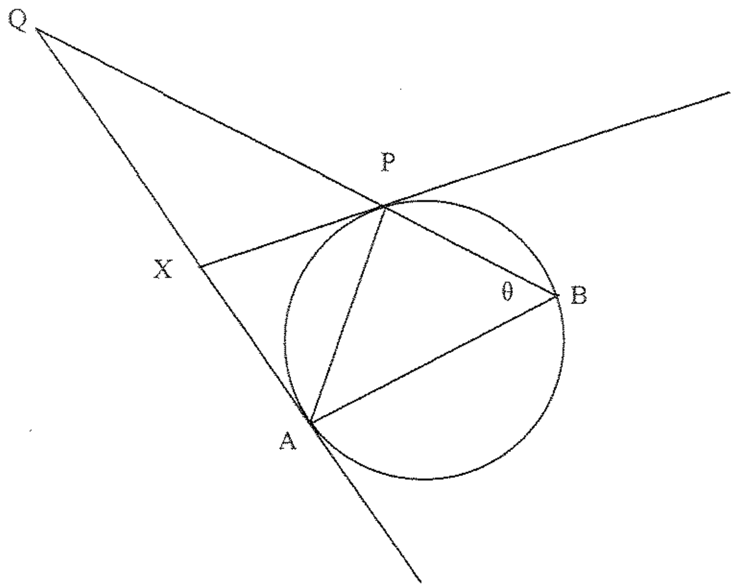
(i) Show that the equation of the tangent at P is given by $y - px + ap^2 = 0$. 2

(ii) Find the coordinates of Q and R. 2

(iii) Find the cartesian equation of the locus of M, the midpoint of QR. 2

Question 6 (12 marks) Start a new page. **Marks**

(a)



In the diagram AB is a diameter of the circle.

The chord BP of the circle is produced to meet, at Q, the tangent to the circle at A.

The tangent to the circle at P meets AQ at X.

(i) If $\angle ABP = \theta$, show that $\angle XPQ = \frac{\pi}{2} - \theta$. 3

(ii) Show that X is the midpoint of AQ. 3

(b) Find $\int \sin^2 10x \, dx$ 3

(c) The function $f(x)$ is defined by $f(x) = x - \frac{2}{x}$, for $x > 0$. 3

Evaluate $f^{-1}(2)$

Question 7 (12 marks) Start a new page. **Marks**

(a) The velocity of a particle moving in a straight line is given by

$\frac{dx}{dt} = 1 + x^2$ where x is the displacement, in metres, from the origin

and t is the time in seconds. Initially the particle is at $x = 1$.

(i) Find an expression for the acceleration of the particle in terms of x. 2

(ii) Find an expression for the displacement of the particle in terms of t. 3

(b) A projectile is fired from ground level with an initial velocity $V\ m/s$ at an angle of elevation of α to the horizontal. The only force acting on the projectile is gravity which equals $g\ m/s^2$.

(i) Derive expressions for the horizontal and vertical components of displacement from the point of projection as functions of t, where t is the time in seconds since the projectile was fired. 3

(ii) Derive an expression for the time of flight assuming the projectile lands at ground level. 1

(iii) At the instant the projectile is fired, a target moving at a constant speed of $A\ m/s$ at a position b metres horizontally from the point of projection, starts moving away from the point of projection in the same horizontal direction that the projectile is moving. Show that for the projectile to hit the target V and α must satisfy the equation $V^2 \sin 2\alpha - 2AV \sin\alpha - bg = 0$ 3

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

STHS 2003 EXT 1 THSC

QUESTION 1

a) $P(2) = 8 - 12 - 20 + 24$

$= 0$

$\therefore x - 2$ is a factor

$\therefore x^3 - 3x^2 - 10x + 24$

$= (x-2)(x^2 - x - 12)$

$= (x-2)(x-4)(x+3)$

b) i) $\frac{d}{dx}(\cos^{-1} 3x) = \frac{-3}{\sqrt{1-9x^2}}$

ii) $\frac{d}{dx}(\cos^3 x) = -3\cos^2 x \sin x$

c) $x \neq 1$

$x + 1 = 3(x-1)$

$x = 2$

test ✓ ⊕ × ● ✓ (1, 2)

$\therefore x < 1$ or $x \geq 2$

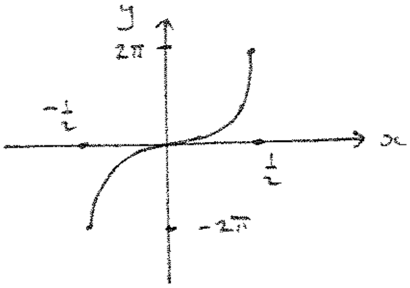
d) $m_1 = 1 \quad m_2 = \frac{1}{3}$

$\therefore \tan\theta = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}}$

$= \frac{1}{2}$

$\therefore \theta = 27^\circ$

e)



QUESTION 2

a) $P\left(\frac{-1\times 2 + 13\times 5}{7}, \frac{8\times 2 + 3\times 5}{7}\right)$

$\therefore P\left(9, \frac{31}{7}\right)$

b) i) $\int_0^2 \frac{dx}{\sqrt{16-x^2}}$

$= \sin^{-1}\frac{x}{4}\Big]_0^2$

$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$

$= \frac{\pi}{6}$

ii) $\int_{-1}^0 \frac{2x}{1-2x}\,dx$

$u = 1 - 2x$

$du = -2\,dx$

$x = 0 \quad u = 1$

$x = -1 \quad u = 3$

$= -\frac{1}{2}\int_{-1}^0 \frac{2x}{1-2x} \cdot -2\,dx$

$= -\frac{1}{2}\int_3^1 \frac{1-u}{u}\,du$

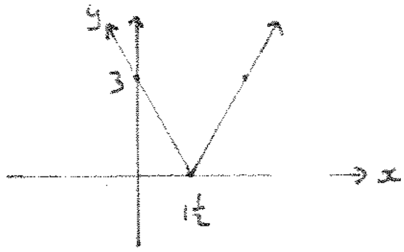
$= \frac{1}{2}\int_1^3 \frac{1}{u} - 1\,du$

$= \frac{1}{2}\left[\ln u - u\right]_1^3$

$= \frac{1}{2}\left[(\ln 3 - 3) - (\ln 1 - 1)\right]$

$= \frac{1}{2}(\ln 3 - 2)$

c) i)



ii) $0 < m < 2$

d) $\sum_{r=1}^{n} \sin(2r-1)x = \frac{1-\cos 2nx}{2\sin x}$

when $n=2$

$\sin x + \sin 3x = \frac{1-\cos 4x}{2\sin x}$

$\therefore \int \frac{1-\cos 4x}{2\sin x}\,dx$

$= \int \sin x + \sin 3x \; dx$

$= -\cos x - \frac{1}{3}\cos 3x + c$

QUESTION 3

a) test $n=1$

$5^1 + 2\times 11^1 = 27$

which is a multiple of 3

$\therefore$ true for $n=1$

assume true for $n=k$

i.e. $5^k + 2(11)^k = 3M$

(where M is an integer)

test for $n=k+1$

$5^{k+1} + 2(11)^{k+1}$

$= 5(5^k) + 22\times 11^k$

$= 5(3M - 2\times 11^k) + 22\times 11^k$

from assumption

$= 15M + 12\times 11^k$

$= 3(5M + 4\times 11^k)$

which is a multiple of 3

$\therefore$ true for $n=k+1$ if true for $n=k$

As true for $n=1$ also true for $n=2$

As true for $n=2$ also true for $n=3$

and so on for all positive integers n.

b) i) $x=0 \quad y=1$

ii) $(-\frac{1}{2}, 0)$

iii) $y' = 2e^{-2x} + -2(1+2x)e^{-2x}$

$= -4xe^{-2x}$

st. pts when $y'=0$

$-4xe^{-2x} = 0$

$x = 0$

when $x=0$, $y=1$

test nature

$y'' = -4e^{-2x} + 8xe^{-2x}$

$= e^{-2x}(8x-4)$

when $x=0 \quad y'' < 0$

$\Rightarrow$ max at $(0,1)$

iv) inflexion when $y''=0$

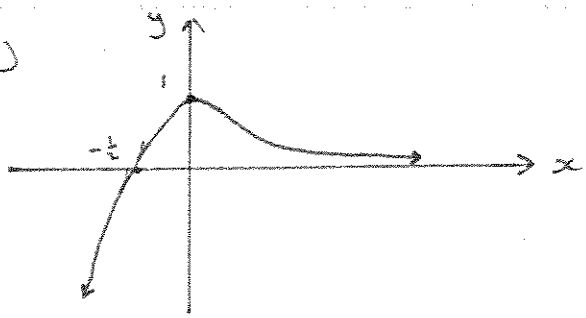
$e^{-2x}(8x-4) = 0$

$8x - 4 = 0$

$x = \frac{1}{2}$

(3)

v)



QUESTION 4

a) i) $x = 5\cos(3t + \frac{\pi}{3})$

$\dot{x} = -15\sin(3t + \frac{\pi}{3})$

$\ddot{x} = -45\cos(3t + \frac{\pi}{3})$

$= -9 \times 5\cos(3t + \frac{\pi}{3})$

$= -9x$

ii) amplitude = 5 cm

iii) period = $\frac{2\pi}{n}$

$= \frac{2\pi}{3}$ sec

iv) distance = 10 cm.

b) $3\sin\theta - \cos\theta = 2$

$\sqrt{10}\sin(\theta - \alpha) = 2$

where $\tan\alpha = \frac{1}{3}$

$\alpha = 18.43^\circ$

$\therefore \sin(\theta - \alpha) = \frac{2}{\sqrt{10}}$

$\therefore \theta - \alpha = 39.23^\circ, 140.77^\circ$

$\therefore \theta = 58^\circ, 159^\circ$

c) i) $T = 24 - 22e^{-kt}$

when $t = 5$ $T = 13$

$\therefore 13 = 24 - 22e^{-5k}$

$\therefore e^{-5k} = \frac{1}{2}$

$-5k = \ln\frac{1}{2}$

$k = -\frac{1}{5}\ln\frac{1}{2}$ or

$k = \frac{1}{5}\ln 2$

ii) $T = 24 - 22e^{-10k}$

$= 18.5^\circ C$

QUESTION 5

a) $V = x^3$ $S = 6x^2$

$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$

$\therefore 10 = 3x^2 \cdot \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = \frac{10}{3x^2}$

$\therefore \frac{dS}{dt} = \frac{dS}{dx} \cdot \frac{dx}{dt}$

$= 12x \cdot \frac{10}{3x^2}$

$= \frac{40}{x}$

when $x = 5$ cm

$\frac{dS}{dt} = \frac{40}{5}$

$= 8$ cm²/sec

(4)

b) $\sin\theta = \frac{3}{5}$

$\sin\theta = \frac{2t}{1+t^2}$

where $\tan\frac{\theta}{2} = t$

$\therefore \frac{3}{5} = \frac{2t}{1+t^2}$

$3t^2 - 10t + 3 = 0$

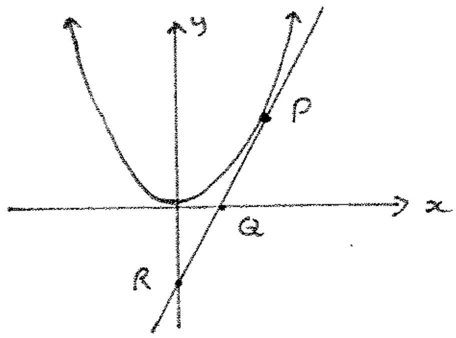
$(3t-1)(t-3) = 0$

$t = \frac{1}{3},\ 3$

$\therefore \tan\frac{\theta}{2} = \frac{1}{3}$

(it is not $\tan\frac{\theta}{2} = 3$ as θ would have to be obtuse)

c)



i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

when $x = 2ap$

$\frac{dy}{dx} = p$

$\therefore$ equation is

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y - px + ap^2 = 0$

ii) when $y = 0$ $\quad px = ap^2$

$x = ap$

$\therefore Q(ap, 0)$

when $x = 0$ $\quad y = -ap^2$

$\therefore R(0, -ap^2)$

iii) midpoint $= \left(\frac{ap}{2}, -\frac{ap^2}{2}\right)$

$\therefore$ locus is

$x = \frac{ap}{2}, \quad y = \frac{-ap^2}{2}$

eliminate $p \Rightarrow p = \frac{2x}{a}$

$\therefore y = -\frac{a}{2}\left(\frac{2x}{a}\right)^2$

$\therefore x^2 = -\frac{1}{2}ay$

QUESTION 6

a) i) $\angle APB = \frac{\pi}{2}$ (angle in semi circle)

$\angle XPA = \theta$ (alternate segment theorem)

$\therefore \angle XPQ = \pi - \frac{\pi}{2} - \theta$ (straight angle)

$= \frac{\pi}{2} - \theta$

ii) $\angle QAB = 90°$ (angle between tangent and radius)

$\therefore \angle AQB = \frac{\pi}{2} - \theta$ (angle sum of Δ)

$\therefore \Delta XPQ$ is isosceles (2 angles equal)

$\therefore XQ = XP$ (opposite equal angles)

but $XP = XA$ (tangents equal in length)

$\therefore XQ = XA$

$\therefore X$ is midpoint.

(5)

b) $\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\sin^2 10x = \frac{1}{2}(1 - \cos 20x)$

$\therefore \int \sin^2 10x \, dx$

$= \frac{1}{2}\int 1 - \cos 20x \, dx$

$= \frac{1}{2}\left(x - \frac{1}{20}\sin 20x\right) + c$

) $f(x) = x - \frac{2}{x}, \quad x > 0$

$\therefore x - \frac{2}{x} = 2$

$x^2 - 2x - 2 = 0$

$x = \frac{2 \pm \sqrt{12}}{2}$

$= 1 \pm \sqrt{3}$

answer must be positive

$\therefore f^{-1}(2) = 1 + \sqrt{3}$

QUESTION 7

a) i) $v = 1 + x^2$

$\frac{1}{2}v^2 = \frac{1}{2}(1+x^2)^2$

$\therefore a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= 2x(1+x^2)$

ii) $\frac{dx}{dt} = 1 + x^2$

$\therefore \int \frac{dx}{1+x^2} = \int dt$

$\tan^{-1} x = t + c$

when $t = 0 \quad x = 1$

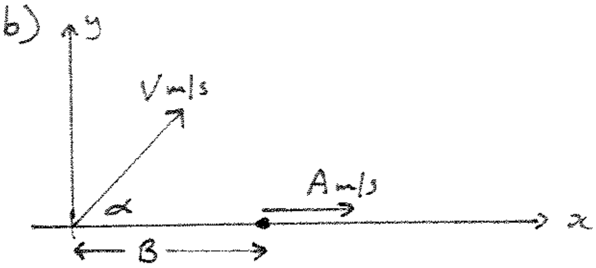
$\therefore \tan^{-1} 1 = c$

$c = \frac{\pi}{4}$

$\tan^{-1} x = t + \frac{\pi}{4}$

$x = \tan\left(t + \frac{\pi}{4}\right)$

b)



initial conditions: when $t=0$

$x=0,\ y=0,\ \dot{x} = V\cos\alpha,\ \dot{y} = V\sin\alpha,$

$\ddot{x} = 0,\ \ddot{y} = -g$.

i) horizontal   vertical

$\ddot{x} = 0 \qquad \ddot{y} = -g$

integrating

$\dot{x} = c_1 \qquad \dot{y} = -gt + c_2$

when $t=0 \quad \dot{x} = V\cos\alpha,\ \dot{y} = V\sin\alpha$

$\Rightarrow c_1 = V\cos\alpha \qquad c_2 = V\sin\alpha$

$\therefore \dot{x} = V\cos\alpha \qquad \dot{y} = V\sin\alpha - gt$

integrating

$x = Vt\cos\alpha + c_3 \qquad y = Vt\sin\alpha - \frac{1}{2}gt^2 + c_4$

when $t=0 \quad x=0 \quad y=0$

$\Rightarrow c_3 = 0, \quad c_4 = 0$

$\therefore x = Vt\cos\alpha \qquad y = Vt\sin\alpha - \frac{1}{2}gt^2$

⑥

ii) when $y = 0$

$vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$t(v\sin\alpha - \frac{1}{2}gt) = 0$

$t = 0 , \frac{2v\sin\alpha}{g}$

$\therefore$ time of flight $= \frac{2v\sin\alpha}{g}$

iii) Range of projectile $= v\left(\frac{2v\sin\alpha}{g}\right)\cos\alpha$

$= \frac{v^2\sin 2\alpha}{g}$

distance travelled by target in the time of flight equals

$A \times \frac{2v\sin\alpha}{g}$

$\therefore$ displacement of target from origin is

$B + A \times \frac{2v\sin\alpha}{g}$

which must equal range of projectile

$\therefore \frac{v^2\sin 2\alpha}{g} = B + \frac{2AV\sin\alpha}{g}$

$V^2\sin 2\alpha - 2AV\sin\alpha - Bg = 0$