SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2004

MATHEMATICS EXTENSION 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks - 84

- Attempt Questions 1 – 7
- All questions are of equal value

Name:____________________________

Teacher:____________________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	**Total**

Marks

Question 1

a) Simplify $\dfrac{1+a^{-1}}{1+a^{-3}}$ **2**

b) Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$ 2

c) Find $\lim\limits_{x \to 0} \dfrac{1-\cos^2 x}{2x^2}$ **2**

d) Use the substitution $u = 1 + x^3$, or otherwise to evaluate $\int_0^1 x^2(1+x^3)^3\ dx$ 4

e) Find the acute angle between the lines $x + y\sqrt{3} = 3$ and $y = 3$ 2

Question 2 (Start a new page)

a) One of the roots of $2x^3 + x^2 - 15x - 18 = 0$ is positive and equal to the product of the other two roots. Find this root. 2

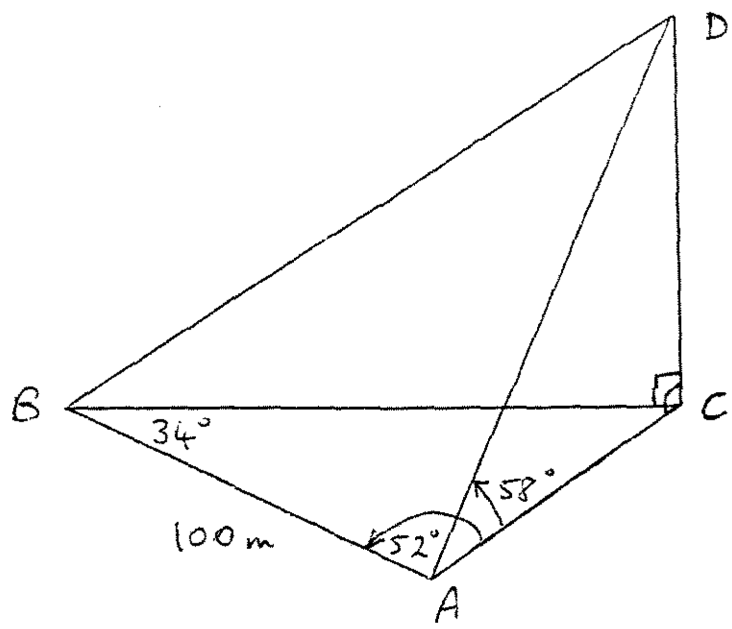
b) If $\dfrac{dy}{dx} = 1 + y$, and when $x = 0, y = 2$; show that $y = 3e^x - 1$ 3

(hint: examine $\dfrac{dx}{dy}$.)

c) Find $\int \dfrac{dx}{\sqrt{16-25x^2}}$ 3

Marks

d)



A pole DC is seen from two points A and B. The angle of elevation from A is $58°$, $\angle CAB$ is $52°, \angle ABC$ is $34°$ and A and B are 100m apart .Find:

(i) How far A is from the foot of the pole, to the nearest metre 3

(ii) The height of the pole, to the nearest metre 1

Question 3 (Start a new page)

a) The equation $\sin\theta + \theta - 2 = 0$ has a root near $\theta = 1.1$. Use this as a first approximation and one application of Newton's Method to find a better approximation of the root correct to 3 decimal places. 3

b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$.

(i) Find the coordinates of M, the midpoint of PQ 1

(ii) If the gradient of PQ is constant, find the equation for the locus of M and show that it is a line parallel to the axis of the parabola. 3

c) Given the function $f(x) = 1 - \tan x$ for the domain $0 \le x \le \frac{\pi}{4}$:

(i) Sketch the graph of $y=f(x)$ 1

(ii) Show that $\int \tan x \, dx = -\ln(\cos x) + c$ 1

(iii) The region in (i) is rotated about the x axis. Find the volume of the solid generated to 2 decimal places. 3

Marks

Question 4 (Start a new page)

a) Find $\int \cos^2 2x \ dx$ 2

b) Prove by Mathematical Induction, that for all positive integers n: 4

$$\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$$

c) The displacement x cm of an object from the origin is given by

$x = \cos t - \sqrt{3}\sin t$

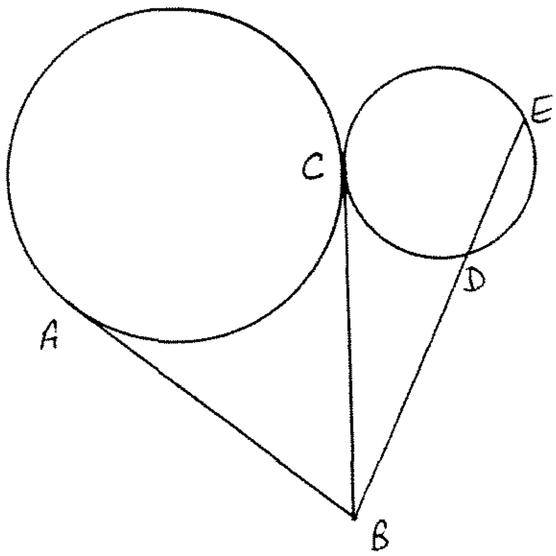
(i) Prove that the object executes simple harmonic motion. 2

(ii) Find an exact time when the object reaches maximum speed 1

(iii) Express the displacement in the form $A\cos(nt+\alpha)$ and state the amplitude. 3

Question 5 (Start a new page)

a)



Not to scale

AB and BC are tangents

and BD = 4 DE

Prove that AB$=2\sqrt{5}$ DE,

giving reasons. 2

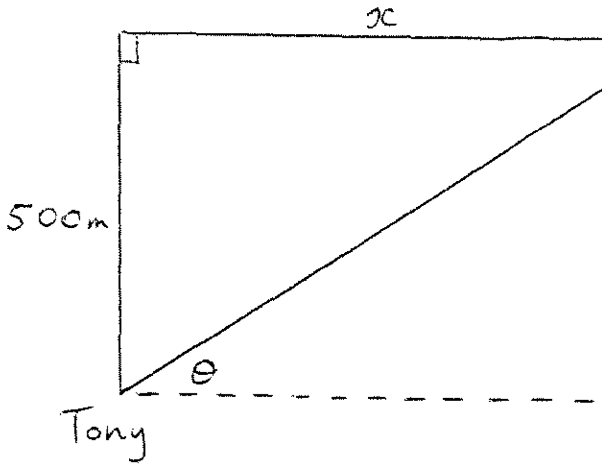
b) The acceleration of a body P is given by $\frac{d^2x}{dt^2} = 18x(x^2+1)$, where x is the displacement of P from 0 at time t. The velocity is v.

Given $t=0, x=0, v=3$ and that $v>0$ throughout the motion:

(i) find v in terms of x 2

(ii) show that $x = \tan 3t$ 2

c)



At 9am, an ultralight aircraft flies directly over Tony's head at a height of 500m. It maintains a constant speed of 20 m/s and a constant altitude.

If x is the horizontal distance travelled by the plane and θ is the angle of elevation from Tony to the plane,:

(i) Show that $\frac{dx}{d\theta} = -500\operatorname{cosec}^2\theta$ 2

(ii) Hence show that $\frac{d\theta}{dt} = \frac{-1}{25}\sin^2\theta$ 2

(iii) Find the rate of change of the angle of elevation at 9.01am (in radians per second) 2

Question 6 (Start a new page)

a) ABCD is a cyclic quadrilateral.

Show that tan A + tan B + tan C + tan D = 0 2

b) A sky-diver opens his parachute when falling at 30 m/s. Thereafter, his acceleration is given by $\frac{dv}{dt} = k(6-v)$ where k is a constant.

(i) Show that this differential equation is satisfied by $v = 6 + Ae^{-kt}$ and find the value of A. 2

(ii) One second after opening his chute, his velocity is 10.7 m/s. Find the value of k to 2 decimal places. 1

(iii) Find his velocity, correct to one decimal place, two seconds after his chute is opened. 1

Marks

c) A soldier is 150 metres from, and on the same horizontal level as, her target. Her weapon can fire with an initial velocity of 50 m/s. Take $g = 10 m/s^2$.

(i) Write the equations of motion for horizontal and vertical displacement. 2

(ii) Find the two possible angles at which she must fire her weapon to hit the target. 4

Question 7 (Start a new page)

a) The function $f(x) = \sec x$ is defined for $0 \le x < \frac{\pi}{2}$.

(i) State the domain of the inverse function $f^{-1}(x)$. 1

(ii) Show that $f^{-1}(x) = \cos^{-1}(\frac{1}{x})$ 1

(iii) Hence find $\frac{d}{dx}\left[f^{-1}(x)\right]$ 2

b) (i) Find all real solutions to the equation $x^4 + x^2 - 1 = 0$, giving your answers correct to three decimal places. 2

(ii) On the same axes, sketch the graphs of $y = \tan^{-1} x$ and $y = \cos^{-1} x$. Label important points. Mark the point P where the two curves intersect. 2

(iii) If $\tan^{-1} x = \cos^{-1} x$ at P, show that $x^4 + x^2 - 1 = 0$ and find the coordinates of P. 4

SOLUTIONS

1 a) $$\frac{1+\frac{1}{a}}{1+\frac{1}{a^3}} = \frac{\frac{a+1}{a}}{\frac{a^3+1}{a^3}} \leftarrow 1$$

$$= \frac{a+1}{a} \times \frac{a^3}{a^3+1}$$

$$= \frac{\cancel{a+1}}{a} \times \frac{a^3}{(\cancel{a+1})(a^2-a+1)} \leftarrow 1$$

$$= \frac{a^2}{a^2-a+1}$$

b) $$\frac{d}{dx}\left[(\cos x)^{-1}\right] = -(\cos x)^{-2} \cdot (-\sin x) \leftarrow 1$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \leftarrow 1$$

$$= \sec x \tan x$$

c) $$= \lim_{x\to 0} \frac{\sin^2 x}{2x^2} \leftarrow 1$$

$$= \frac{1}{2} \lim \left(\frac{\sin x}{x}\right)^2$$

$$= \frac{1}{2} \times 1 \leftarrow 1$$

$$= \frac{1}{2}$$

d) $$\int_0^1 x^2(1+x^3)^3\,dx = \int_1^2 \cancel{x^2}(u^3)\frac{du}{3\cancel{x^2}}$$

$u = 1+x^3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$ 1

$x=0,\ u=1$ 1

$x=1,\ u=2$

$$= \frac{1}{3}\int_1^2 u^3\,du \quad 1$$

$$= \frac{1}{3}\left[\frac{u^4}{4}\right]$$

$$= \frac{1}{3}\left(4-\frac{1}{4}\right)$$

$$= 1\tfrac{1}{4} \quad 1$$

e) $y = -\frac{x}{\sqrt{3}} + 3$, $m_1 = \frac{-1}{\sqrt{3}}$

$m_2 = 0$

$$\tan\theta = \left|\frac{-\frac{1}{\sqrt{3}} - 0}{1+0}\right| \quad 1$$

$$= \frac{1}{\sqrt{3}}$$

$\therefore \theta = 30°$ 1

2 a) Let roots be $\alpha, \beta, \alpha\beta$

and $\alpha \cdot \beta \cdot \alpha\beta = -\frac{d}{a}$ 1

$\therefore \alpha^2\beta^2 = 9$

$\therefore \alpha\beta = 3 \ (>0)$ 1

b) $$\frac{dx}{dy} = \frac{1}{1+y}$$

$\therefore x = \log(1+y) + c$ 1

Sub $x=0, y=2$:

$\therefore 0 = \log 3 + c$

$\therefore c = -\log 3$

$\therefore x = \log(1+y) - \log 3$ 1

$$= \log\left(\frac{1+y}{3}\right)$$

$\therefore e^x = \frac{1+y}{3}$ 1

$\therefore 3e^x = 1+y$

$\therefore y = 3e^x - 1$

c) $\displaystyle\int \frac{dx}{\sqrt{16-25x^2}} = \int \frac{dx}{\sqrt{25\left(\frac{16}{25}-x^2\right)}}$ (1)

$\displaystyle = \frac{1}{5}\int \frac{dx}{\sqrt{\left(\frac{4}{5}\right)^2 - x^2}}$ (1)

$\displaystyle = \frac{1}{5}\sin^{-1}\frac{5x}{4} + c$ (1)

d)(i) $\displaystyle\frac{AC}{\sin 34^\circ} = \frac{100}{\sin 94^\circ}$ (1) for rule (1)

$\displaystyle\therefore AC = \frac{100\sin 34^\circ}{\sin 94^\circ}$

$= 56\text{m}$ (nearest m) (1)

(ii) $\displaystyle\tan 58^\circ = \frac{DC}{56}$

$\therefore DC = 56\tan 58^\circ$

$= 90\text{ m}$ (nearest m) (1)

3 a) $f'(\theta) = \cos\theta + 1$ (1)

$\displaystyle\therefore a_2 = 1\cdot 1 - \frac{f(1\cdot 1)}{f'(1\cdot 1)}$

$\displaystyle = 1\cdot 1 - \frac{\sin 1\cdot 1 + 1\cdot 1 - 2}{\cos 1\cdot 1 + 1}$ (1)

$\doteqdot 1\cdot 106$ (1)

b) (i) $\displaystyle\left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$\therefore M$ is $\left[a(p+q), \frac{a}{2}(p^2+q^2)\right]$ (1)

(ii) $M_{pq} = c$

$\displaystyle\therefore \frac{aq^2-ap^2}{2aq-2ap} = c$ (1)

$\displaystyle\therefore \frac{\cancel{a}\cancel{(q-p)}(q+p)}{2\cancel{a}\cancel{(q-p)}} = c$

$\displaystyle\therefore \frac{q+p}{2} = c \Longrightarrow q+p = 2c$ (1)

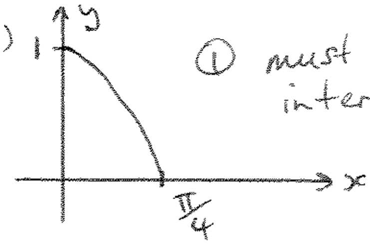
$\therefore M$ has coords

$\left[2ac, \frac{a}{2}(p^2+q^2)\right]$

$\therefore x = 2ac$ (1)

$\therefore M$ has locus eqn $x = 2ac$, which is vertical and parallel to axis of parabola.

c) (i)



(1) must show intercepts

(ii) $\displaystyle\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx$ (1)

$= -\ln(\cos x) + c$

(iii) $\displaystyle V = \pi\int_0^{\pi/4}(1-\tan x)^2\, dx$

$\displaystyle = \pi\int_0^{\pi/4}(1 - 2\tan x + \tan^2 x$

$\displaystyle = \pi\int_0^{\pi/4}(\sec^2 x - 2\tan x)$ (1)

$\displaystyle = \pi\left[\tan x + 2\log(\cos x)\right]_0^{\pi/4}$ (1)

$\doteqdot 0\cdot 96$ (1)

4 a) $\int \frac{1}{2}(1+\cos 4x)\,dx$ ①

$= \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right) + c$ ①

b) Prove true for $n=1$:

LHS $= 1\times 2 = 2$

RHS $= \frac{1.2.3}{3} = 2 =$ LHS ①

$\therefore$ result is true for $n=1$

Assume true for $n=k$:

ie. assume $S_k = \frac{k(k+1)(k+2)}{3}$ ①

Prove true for $n=k+1$:

ie. prove $S_{k+1} = \frac{(k+1)(k+2)(k+3)}{3}$ ①

Now, $S_{k+1} = S_k + T_{k+1}$

$$\left.\begin{aligned} &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\ &= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \end{aligned}\right\}①$$

shown

Since the result is true for $n=1$, then, from above, it must be true for $n=1+1=2$ and $n=2+1=3$ and so on for all positive integers n

c) (i) $\frac{dx}{dt} = -\sin t - \sqrt{3}\cos t$

$\frac{d^2x}{dt^2} = -\cos t + \sqrt{3}\sin t$

$= -x$ ①

which is in the form $\ddot{x} = -n^2x$ for SHM ①

(ii) max. speed when $\ddot{x} = 0$

$\therefore -\cos t + \sqrt{3}\sin t = 0$

$\therefore \sqrt{3}\sin t = \cos t$

$\therefore \tan t = \frac{1}{\sqrt{3}}$ $(\cos t \neq 0)$

$\therefore t = \pi/6$ seconds ①
(or equiv.)

(iii) $A = \sqrt{1+(\sqrt{3})^2}$ and $n=1$ from (i)

$= \sqrt{4}$

$= 2$

$\therefore \cos t - \sqrt{3}\sin t = 2\cos(t+\alpha)$

$\therefore \frac{1}{2}\cos t - \frac{\sqrt{3}}{2}\sin t = \cos(t+\alpha)$

$= \cos t\cos\alpha - \sin t\sin\alpha$

$\left.\begin{aligned}\therefore \cos\alpha &= \tfrac{1}{2} \\ \sin\alpha &= \tfrac{\sqrt{3}}{2}\end{aligned}\right\} \therefore \alpha = \pi/3$

① ①

$\therefore \cos t - \sqrt{3}\sin t = 2\cos\left(t + \frac{\pi}{3}\right)$

$\therefore$ amplitude $= 2$ units. ①

5 a) $BC^2 = BD \cdot BE$
(square of tangent = product of intersecting chords)

$= 4DE \times 5DE$ ①

$= 20DE^2$

$\therefore BC = \sqrt{20DE^2}$

$= 2\sqrt{5}\,DE$

and $AB = BC$ (equal tangents to a circle) ①

$\therefore AB = 2\sqrt{5}\,DE$

b)(i) $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 18x^3 + 18x$

$\therefore \frac{1}{2}v^2 = \frac{18x^4}{4} + 9x^2 + c$

$\therefore v^2 = 9x^4 + 18x^2 + k$ (1)

$(x = 0,\ v = 3)$:

$\therefore 9 = 0 + 0 + k \ (k = 9)$

$\therefore v^2 = 9x^4 + 18x^2 + 9$

$= (3x^2 + 3)^2$

$\therefore v = 3x^2 + 3 \ (> 0)$ (1)

(ii) $v = \frac{dx}{dt} = 3x^2 + 3$

$\therefore \frac{dt}{dx} = \frac{1}{3x^2 + 3}$

$\therefore t = \frac{1}{3}\int \frac{1}{x^2+1}\,dx$ (1)

$= \frac{1}{3}\tan^{-1}x + c$

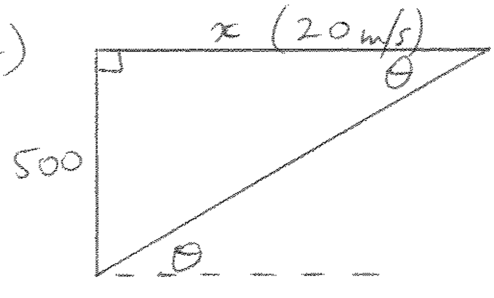
$(t = 0,\ x = 0)$:

$\therefore 0 = 0 + c \ (c = 0)$

$\therefore 3t = \tan^{-1}x$ (1)

$\therefore \tan 3t = x$ as reqd.

c)



(i) $\tan\theta = \frac{500}{x}$

$\therefore x = \frac{500}{\tan\theta}$ (1)

$\therefore \frac{dx}{d\theta} = \frac{-\sec^2\theta \times 500}{\tan^2\theta}$

$= \frac{-500}{\cancel{\cos^2\theta}} \times \frac{\cancel{\cos^2\theta}}{\sin^2\theta}$ (1)

$= \frac{-500}{\sin^2\theta}$

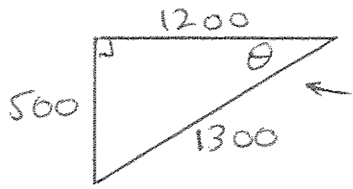
$= -500\,\text{cosec}^2\theta$

(ii) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}$ (1)

$= \frac{\sin^2\theta}{-500} \times 20$ (1)

$= -\frac{1}{25}\sin^2\theta$

(iii) At 9.01 am, $t = 60$, $x = 1200$m



(1)

$\therefore \frac{d\theta}{dt} = -\frac{1}{25} \times \left(\frac{5}{13}\right)^2$

$= -\frac{1}{\cancel{25}} \times \frac{\cancel{25}}{169}$

(1)

$= -\frac{1}{169}$ radians/second

(0.006)

6 a) opposite angles supplementary

$\therefore \tan A + \tan B + \tan C + \tan D$ (1)

$= \tan A + \tan B + \tan(180° - A) + \tan(180° -$

$= \cancel{\tan A} + \cancel{\tan B} - \cancel{\tan A} - \cancel{\tan B}$ (1)

$= 0$

b) (i) $v = 6 + Ae^{-kt}$

$\therefore \frac{dv}{dt} = Ae^{-kt} \times (-k)$

$= -kAe^{-kt}$

$= -k(6 + Ae^{-kt} - 6)$

$= -k(v-6)$ ① for process

$= k(6-v)$

When $t = 0$, $v = 30$:

$\therefore 30 = 6 + Ae^{0}$

$\therefore A = 24$ ①

(ii) $v = 6 + 24e^{-kt}$

When $t = 1$, $v = 10.7$

$\therefore 10.7 = 6 + 24e^{-k}$

$\therefore \frac{4.7}{24} = e^{-k}$

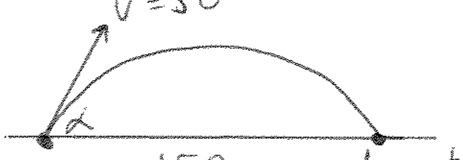
$\therefore -k = \log\left(\frac{4.7}{24}\right)$

$\therefore k \doteqdot 1.63$ ①

(iii) $v = 6 + 24e^{-3.26}$

$\doteqdot 6.9$ m/s ①

c) (i)



$x = 50\cos\alpha\, t$ ①

$y = 50\sin\alpha\, t - 5t^2$ ①

(ii) At target: $x = 150$, $y = 0$

$\therefore 150 = 50\cos\alpha\, t$

$\therefore \cos\alpha\, t = 3$

$\therefore t = \frac{3}{\cos\alpha}$ ①

$\therefore 0 = 50\sin\alpha \cdot \frac{3}{\cos\alpha} - 5\left(\frac{3}{\cos\alpha}\right)^2$

$= 150\tan\alpha - \frac{45}{\cos^2\alpha}$ ① for process

$= 150\tan\alpha - 45\sec^2\alpha$

$= 150\tan\alpha - 45(1+\tan^2\alpha)$

$\therefore 45\tan^2\alpha - 150\tan\alpha + 45 = 0$

$\therefore 3\tan^2\alpha - 10\tan\alpha + 3 = 0$

$(3\tan\alpha - 1)(\tan\alpha - 3) = 0$ ① for eqn

$\therefore \tan\alpha = \frac{1}{3}$ or 3

$\therefore \alpha = 18^\circ 26'$ or $71^\circ 34'$ ①

$\therefore$ soldier can hit target with either angle above.

7 a) (i) For $f(x) = \sec x$:

$D: 0 \le x < \frac{\pi}{2}$

$R: y \geq 1$

$\therefore$ for $f^{-1}(x)$, $D: x \geq 1$ (1)

(ii) $y = \sec x$

$\therefore f^{-1}(x): x = \sec y$ (1) for process

$= \frac{1}{\cos y}$

$\therefore \cos y = \frac{1}{x}$

$\therefore y = \cos^{-1}\left(\frac{1}{x}\right)$

(iii) $\frac{d}{dx}\left[\cos^{-1}\left(\frac{1}{x}\right)\right] = \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \left(-x^{-2}\right)$

$= \frac{1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{1}{x^2}$ (1) (1)

$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$

$\left(\text{or } \frac{1}{\sqrt{x^4-x^2}} \text{ or } \frac{1}{x\sqrt{x^2-1}}\right)$

b) (i) Solve $m^2 + m - 1 = 0$ $(m = x^2)$

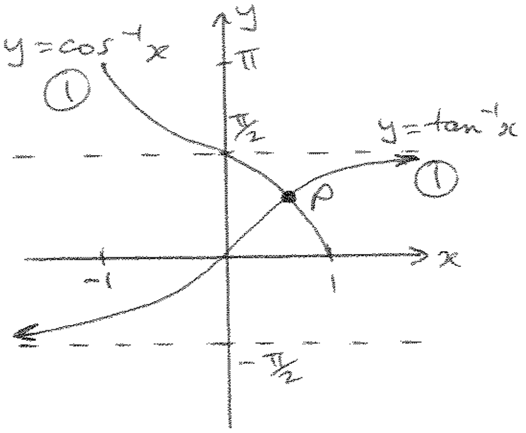
$\therefore m = \frac{-1 \pm \sqrt{1-4\times1\times-1}}{2}$

$= \frac{-1 \pm \sqrt{5}}{2}$ (1)

$\therefore x^2 = \frac{-1+\sqrt{5}}{2}$ only (as $x^2 \geq 0$)

$\therefore x \doteqdot \pm 0.786$ (1)

(ii) $y = \cos^{-1}x$ (1)



(−1 if x/y points missing)

(iii) At P, $\tan^{-1}x = y = \cos^{-1}x$

$\therefore x = \tan y$ (1), $x = \cos y$

$\therefore \cos y = \frac{1}{\sqrt{x^2+1}}$, $\tan y = \frac{\sqrt{1-x^2}}{x}$ (1)

Now, since $\tan y = \cos y$ $(= x)$

then either : $\frac{1}{\sqrt{x^2+1}} = \frac{\sqrt{1-x^2}}{x}$

or $\frac{1}{\sqrt{x^2+1}} = x$ (1)

or $\frac{\sqrt{1-x^2}}{x} = x$

which all give $x^4 + x^2 - 1 = 0$

Coordinates of P are

$(0.786, 0.666)$

(1)