SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2006

MATHEMATICS EXTENSION 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks - 84

- Attempt Questions 1 - 7
- All questions are of equal value

Name: ______________________

Teacher: ______________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	**Total**
/12	/12	/12	/12	/12	/12	/12	/84

Question 1

a) Find an exact value of $\sin 75°$ **2**

b) Solve $|x-1| > |2-x|$ **2**

c) Find the acute angle between the lines $2x + y = 17$ and $x - y = 3$ (nearest degree) **2**

d) Find the exact value of $\cos(\sin^{-1} \frac{3}{4})$ **2**

e) Solve $\dfrac{x^2 - 9}{x} \geq 0$ **2**

f) Differentiate $\log(xe^x)$ **2**

Question 2

a) Differentiate $\sin^{-1}(\cos x)$ **2**

b) The roots of $x^2 - 6x + k = 0$ differ by 1. Find the value of k **2**

c) Find $\displaystyle\int \frac{dt}{1 + 9t^2}$ **2**

d) Evaluate $\displaystyle\int_0^{\sqrt{8}} \frac{x}{x^2 + 1}\, dx$, giving your answer in simplest exact form **3**

e) Solve $(\log x)^2 - \log(x^2) = 0$ **3**

Question 3

a) Solve $3^{x-1} = 5$. Give your answer correct to 1 decimal place **2**

b) The equation $x^3 - 3x + 1 = 0$ has a root near $x = 1.5$. Use one application of Newton's Method to find a better approximation for the root, correct to 2 decimal places. **2**

c) The polynomial $P(x) = 4x^3 + kx + 6$ has a factor of $x + 3$. Find the value of k and express $P(x)$ in the form $(x+3)Q(x)$ **3**

d) (i) Sketch the curve $y = 3\sin^{-1} 2x$. Clearly indicate values on the axes. **2**

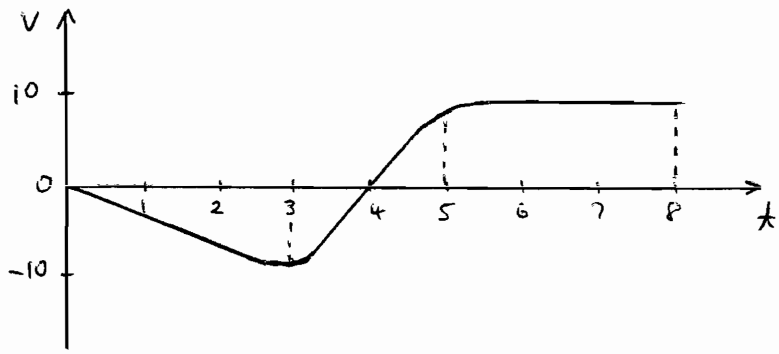
(ii) Find the exact area bounded by the curve $y = 3\sin^{-1} 2x$, the y axis and **3**

Question 4

a) Find $\int \sin^2 4x \; dx$ 2

b) Given $\frac{dx}{dt} = \cos^2 x$ and that t = 0 when $x = \frac{\pi}{4}$, find x as a function of t 3

c) The graph below shows the velocity in metres per second of an object for the first 8 seconds.



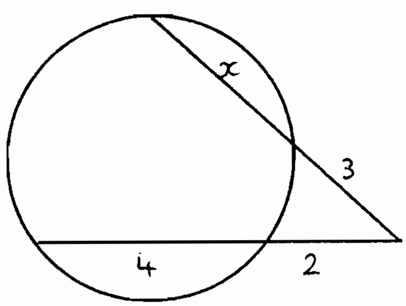
(i) Sketch a graph of its acceleration for $0 \le t \le 8$. Do not show units on the vertical axis. 2

(ii) Find a close approximation for the total distance travelled by the object in the first 8 seconds. 1

d) Use the principle of mathematical induction to prove that $3^{2n} - 1$ is divisible by 8 when n is integral and positive. 4

Question 5

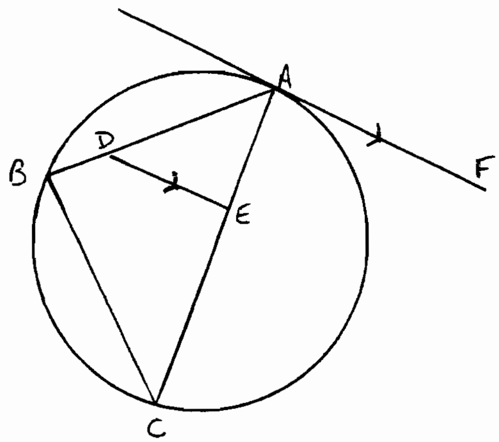
a)



Find x 1

Not to scale

b)

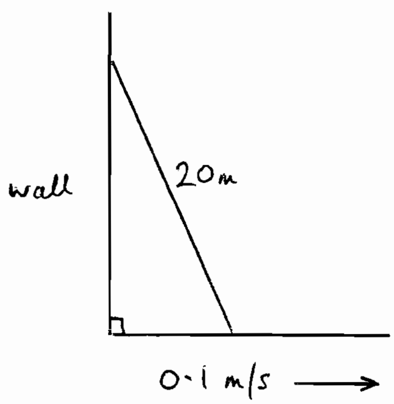


A, B, C are 3 points on the circle and DE is parallel to tangent AF 3

(i) Copy this diagram onto your answer page

(ii) Prove that BDEC is a cyclic quadrilateral

c)



A 20 metre long ladder is resting against a wall. Its base begins to slip along the ground at a rate of 0.1 m/s.
Find the rate at which the top of the ladder is descending when it is 16 metres above the ground. 4

d) Newton's Law of Cooling states that the rate at which a body loses heat is proportional to the difference between the temperature of the body T and room temperature R.

ie $\frac{dT}{dt} = -k(T-R)$

(i) Show that $T = R + Ce^{-kt}$, where C is a constant, is a solution of this differential equation. 1

(ii) A cup of coffee cools from $90°$ C to $50°$ C in 20 minutes in a room whose temperature is $22°$ C. Find the temperature of the coffee after 1 hour, to the nearest degree. 3

Question 6

(a) Solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \leq x \leq 2\pi$ 3

(b) The speed v m/s of a particle moving along the x - axis is given by

$v^2 = 24 - 6x - 3x^2$, where x metres is the particle's displacement form the origin.

(i) Show that the particle is executing Simple Harmonic Motion. 2

(ii) Find the amplitude and period of the motion. 3

(c) P and Q are points on the parabola $x^2 = 4ay$ with parameters p and q.

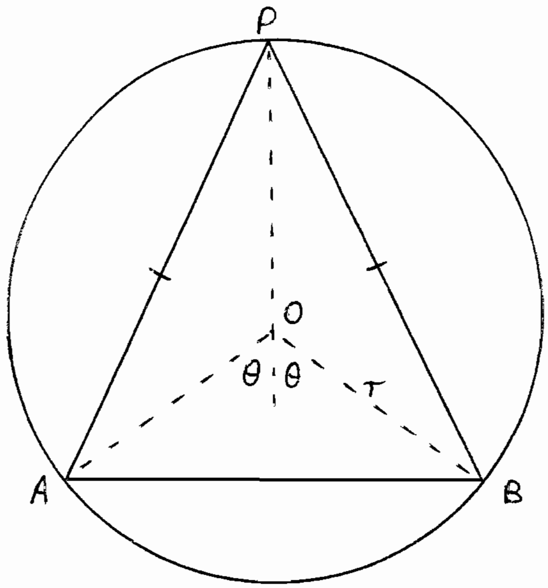
(i) Find the coordinates of M, the midpoint of PQ. 1

(ii) If PQ subtends a right angle at the origin, show that pq = -4. 1

(iii) Find the cartesian equation of the locus of M 2

Question 7

(a)



A, P ,B are points on a circle, centre O and radius r. Chord AB is such that AP = BP and *PO* produced bisects $\angle AOB$.

(i) Show that the area of ΔAPB is given by A = $r^2 \sin\theta(1+\cos\theta)$ 2

(ii) Show that $\dfrac{dA}{d\theta} = r^2(\cos\theta + \cos 2\theta)$ 2

(iii) Find the value of θ in radians that will maximize the area of ΔAPB 2

(b) A projectile is fired from a point on a horizontal plane with a velocity of 80 m/s and at an angle of 60° to the horizontal. Take $g = 10\,m/s^2$.

(i) State the vertical and horizontal equations for displacement 2

(ii) Find the time taken for the projectile to reach maximum height 1

(iii) Find the speed of the projectile (to 1 decimal place) and the acute angle to the horizontal (to nearest degree) that the projectile makes one second before impact with the ground. 3

SOLUTIONS (EXT 1 Trial HSC 2006)

1

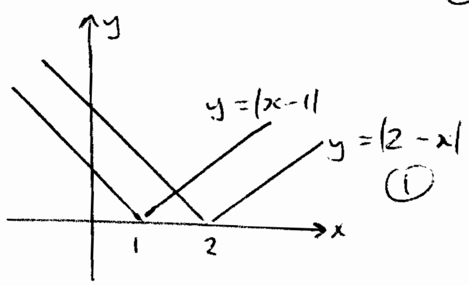
a) $\sin 75^\circ = \sin(45^\circ + 30^\circ)$ (1)

$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$ (1)

b)



$\therefore x > 1\frac{1}{2}$ (1)

c) $m_1 = -2$, $m_2 = 1$

$\tan\theta = \left|\frac{-2-1}{1+(-2)\times 1}\right|$ ← (1)

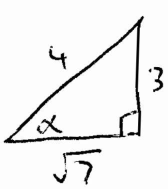
$= \left|\frac{-3}{-1}\right|$

$= 3$

$\therefore \theta \doteqdot 72^\circ$ ← (1)

d) let $\alpha = \sin^{-1}\frac{3}{4}$

$\therefore \sin\alpha = \frac{3}{4}$ ← (1)



$\therefore \cos\left(\sin^{-1}\frac{3}{4}\right)$

$= \cos\alpha$

$= \frac{\sqrt{7}}{4}$ ← (1)

e) $\frac{x^2-9}{x} \times x^2 \geq 0 \times x^2 \quad (x \neq 0)$

$\therefore x(x-3)(x+3) \geq 0$ (1) method

f) $\log(xe^x) = \log x + \log(e^x)$

$= \log x + x$ (1)

$\therefore \frac{dy}{dx} = \frac{1}{x} + 1$ (1)

2

a) Let $y = \sin^{-1} u$ where $u = \cos x$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{\sqrt{1-u^2}} \times (-\sin x)$

$= \frac{-\sin x}{\sqrt{1-\cos^2 x}}$ ← (1)

$= \frac{-\sin x}{\sqrt{\sin^2 x}}$

$= \frac{-\sin x}{|\sin x|} = \pm 1$ (1)

b) Roots α and $\alpha + 1$

$\alpha + \alpha + 1 = -\frac{b}{a}$, $\alpha(\alpha+1) = \frac{c}{a} = k$

$2\alpha + 1 = 6$ $\quad \therefore 2\frac{1}{2} \times 3\frac{1}{2} = k$

$\therefore \alpha = 2\frac{1}{2}$ (1) $\quad \therefore k = 8\frac{3}{4}$ (1)

c) $\int \frac{1}{1+9t^2}\,dt = \int \frac{1}{9\left(\frac{1}{9}+t^2\right)}\,dt$

$= \frac{1}{9}\int \frac{1}{\left(\frac{1}{3}\right)^2 + t^2}\,dt$

$= \frac{1}{9} \times \frac{1}{\frac{1}{3}} \tan^{-1}\left(\frac{t}{\frac{1}{3}}\right) + c$

$= \frac{1}{3}\tan^{-1} 3t + c$

d) $= \frac{1}{2}\int_0^{\sqrt{8}} \frac{2x}{x^2+1}\,dx$

$= \frac{1}{2}\left[\log(x^2+1)\right]_0^{\sqrt{8}}$ ①

$= \frac{1}{2}(\log 9 - \log 1)$

$= \frac{1}{2}\log 9$ ①

$= \log 3$ ①

e) $(\log x)^2 - 2\log x = 0$ ①

$\therefore \log x(\log x - 2) = 0$

$\therefore \log x = 0$ or $\log x = 2$ ① both

$\therefore x = 1$ or e^2 ① both

3 a) $\log(3^{x-1}) = \log 5$

① $(x-1)\log 3 = \log 5$

$x\log 3 - \log 3 = \log 5$

$\therefore x = \frac{\log 5 + \log 3}{\log 3}$

$\doteqdot 2.5$ ①

b) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(1.5) = 1.5^3 - 4.5 + 1$

$= -0.125$

$f'(1.5) = 3(1.5)^2 - 3$

$= 3.75$

$\therefore x_2 = 1.5 - \left(\frac{-0.125}{3.75}\right)$

$\doteqdot 1.53$

c) $P(-3) = 0$

$\therefore 4(-27) - 3k + 6 = 0$

$\therefore 3k = -102$

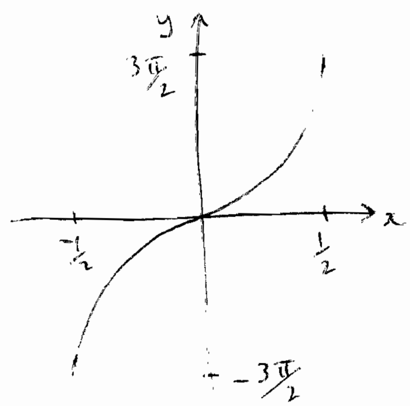
$\therefore k = -34$ ①

$$\begin{array}{r l}
 & 4x^2 - 12x + 2 \\
x+3 \,\big) & 4x^3 \qquad\quad -34x + 6 \\
 & \cancel{4x^3} + 12x^2 \\
\hline
 & -12x^2 \\
 & -\cancel{12x^2} - 36x \\
\hline
 & 2x + 6
\end{array}$$

① method

$\therefore P(x) = (x+3)(4x^2 - 12x + 2)$ ①

d) (i)



① shape
① axes

(ii) $A = \int f(y)\,dy$

$y = 3\sin^{-1}2x$

$\frac{y}{3} = \sin^{-1}2x$

$\sin\frac{y}{3} = 2x$

$\therefore x = \frac{1}{2}\sin\frac{y}{3}$ ①

$\therefore$ Area $= \int_0^{3\pi/2} \frac{1}{2}\sin\frac{y}{3}\,dy$

$= \frac{1}{2}\times(-3)\left[\cos\frac{y}{3}\right]_0^{3\pi/2}$ ①

$= -\frac{3}{2}\left(\cos\frac{\pi}{2} - \cos 0\right)$

$= -\frac{3}{2}(0 - 1)$

Question 4

a) $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\sin^2 4x = \frac{1}{2}(1-\cos 8x)$

1

$\therefore \int \sin^2 4x\,dx = \frac{1}{2}\int (1-\cos 8x)\,dx$

$= \frac{1}{2}\left(x - \frac{\sin 8x}{8}\right) + c$

$= \frac{x}{2} - \frac{\sin 8x}{16} + c$

1

b) $\frac{dt}{dx} = \frac{1}{\cos^2 x}$

$= \sec^2 x$

$\therefore t = \int \sec^2 x\,dx$

$= \tan x + c$ 1

$t=0,\ x=\frac{\pi}{4} \Rightarrow 0 = 1 + c\ (c=-1)$

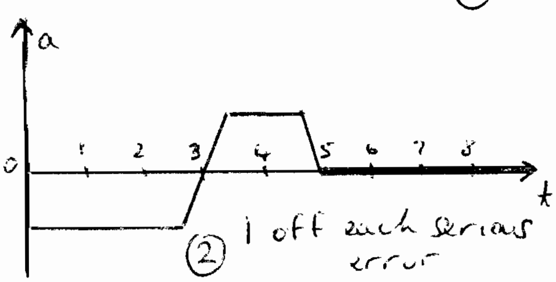
$\therefore t = \tan x - 1$ 1

$\therefore \tan x = t+1$

$\therefore x = \tan^{-1}(t+1)$

1

c i)



2 1 off each serious error

(ii) total distance travelled

= total area enclosed by vel. graph

$= 15 + 5 + 5 + 30$

$= 55$ metres

1

d) Prove true for $n=1$:

$3^2 - 1 = 8$ which is divis by 8. 1

Assume true for $n=k$:

i.e. that $3^{2k} - 1 = 8P$ for some integer P. 1

Prove true for $n=k+1$:

i.e. that $3^{2(k+1)} - 1 = 8Q$ for some integer Q.

Now, $3^{2(k+1)} - 1 = 3^{2k+2} - 1$

$= 3^{2k} \times 3^2 - 1$

$= 9 \times 3^{2k} - 1$

(2 marks)

$= 9(3^{2k} - 1) + 8$

(from above) $= 9 \times 8P + 8$

$= 8(9P + 1)$

$= 8Q$, since $9P+1$ is integral.

Since the result was true for $n=1$, then from above it must be true for $n = 1+1 = 2$, then $n = 2+1 = 3$ and so on for all positive, integral n.

1 mark off if very poor attempt

Question 5

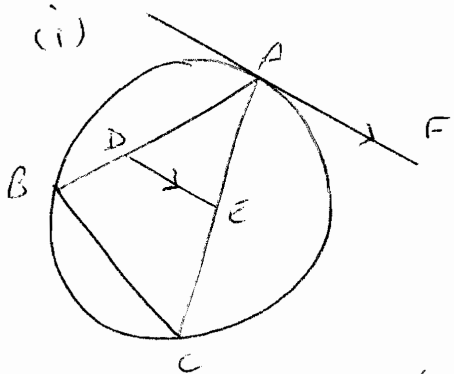
a) $3(3+x) = 2 \times 6$

$9 + 3x = 12$

$3x = 3$

$x = 1 \leftarrow ①$

b) (i)



(ii) $\angle FAE = \angle ABC$ (angle between chord AC and tangent equals angle in alternate segment)

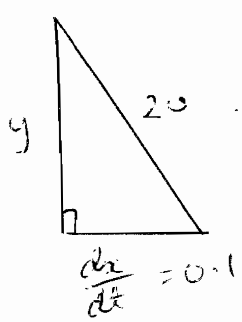
1 mark off each serious error

and $\angle FAE = \angle AED$ (alternate angles $AF \parallel DE$)

$\therefore \angle ABC = \angle AED$

$\therefore$ BDEC is a cyclic quad. since exterior angle equals opposite interior angle.

c)



$\frac{dx}{dt} = 0.1$

$y^2 = 20^2 - x^2$

$\therefore y = (400 - x^2)^{\frac{1}{2}} \leftarrow ①$

$\frac{dy}{dx} = \frac{1}{2}(400 - x^2)^{-\frac{1}{2}} \times (-2x)$

$= \frac{-x}{\sqrt{400 - x^2}} \leftarrow ①$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= \frac{-x}{\sqrt{400 - x^2}} \times 0.1$

When $y = 16$, $x = 12$

$\therefore \frac{dy}{dt} = \frac{-12 \times 0.1}{\sqrt{400 - 144}}$

$= \frac{-1.2}{16}$

$= -0.075$ or ①

$\therefore$ ladder is descending at 0.075 m/s.

d) (i) $T = R + Ce^{-kt}$

$\therefore \frac{dT}{dt} = Ce^{-kt} \times (-k)$ ① fully correct method

$= -kCe^{-kt}$

$= -k(R + Ce^{-kt} - R)$

$= -k(T - R)$ as req'd.

(ii) $t = 0$, $T = 90$, $R = 22$:

$\therefore 90 = 22 + Ce^0$

$= 22 + C$

$\therefore C = 68$

$\therefore T = 22 + 68e^{-kt}$ ①

$t = 20$, $T = 50$

$\therefore 50 = 22 + 68e^{-20k}$

$28 = 68e^{-20k}$

$\frac{28}{68} = e^{-20k}$

$\log\left(\frac{28}{68}\right) = -20k$

$\therefore k = \frac{\log\left(\frac{28}{68}\right)}{-20}$ ①

When $t = 60$ mins,

$T = 22 + 68e^{3\log\left(\frac{28}{68}\right)}$

$\doteqdot 27°$ ①

Question 6

a) Let $\sqrt{3}\cos x - \sin x = A\cos(x+\alpha) = 1$

$A = \sqrt{3+1}$
$= 2$

$\therefore \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \cos(x+\alpha) = \frac{1}{2}$ 1

$= \cos x\cos\alpha - \sin x\sin\alpha = \frac{1}{2}$

$\therefore \cos\alpha = \sqrt{3}/2$, $\sin\alpha = \frac{1}{2}$ $\therefore \alpha$ is in 1st quadrant $\therefore \alpha = \pi/6$ 1

$\therefore \cos(x + \pi/6) = \frac{1}{2}$

$\therefore x + \pi/6 = \pi/3$ (1st, 4th quad)
$= \pi/3$ or $5\pi/3$

$\therefore x = \pi/6$ or $3\pi/2$ 1 both

b) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ 1

$= \frac{d}{dx}\left(12 - 3x - \frac{3}{2}x^2\right)$

$= -3 - 3x$

$= -3(x+1)$

which is in the form $\ddot{x} = -n^2(x-b)$ for SHM 1

(ii) $n^2 = 3$

$\therefore n = \sqrt{3}$ $(n > 0)$ 1

$\therefore$ period $= \frac{2\pi}{\sqrt{3}}$ secs.

amplitude (max x) when $v^2 = 0$

$\therefore 24 - 6x - 3x^2 = 0$ 1

$\therefore x^2 + 2x - 8 = 0$

$(x-2)(x+4) = 0$

$\therefore x = 2, -4$

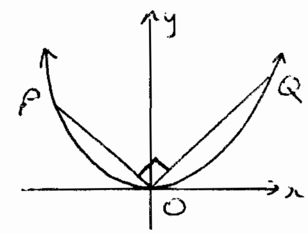
$\therefore$ amplitude = 3 units.

c) (i) $P(2ap, ap^2)$
$Q(2aq, aq^2)$

$M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left[a(p+q), \frac{a(p^2+q^2)}{2}\right]$ 1

(ii)



$m_{OP} = \frac{ap^2}{2ap}$, similarly
$= \frac{p}{2}$, $m_{OQ} = \frac{q}{2}$ 1

$OP \perp OQ \Rightarrow m_{OP} \times m_{OQ} = -1$

$\therefore \frac{p}{2} \times \frac{q}{2} = -1$

$\therefore pq = -4$

(iii) $x = a(p+q)$, $y = \frac{a(p^2+q^2)}{2}$

$\therefore p+q = \frac{x}{a} \Rightarrow y = \frac{a\left[(p+q)^2 - 2pq\right]}{2}$

1 method

$= \frac{a\left[\left(\frac{x}{a}\right)^2 + 8\right]}{2}$

$\therefore 2y = \frac{x^2}{a} + 8a$

$\therefore y = \frac{x^2}{2a} + 4a$ (or $x^2 = 2ay - 8a^2$)

1 answer

Question 7

a) (i) $A = \text{area } \Delta OAP \times 2 + \text{area } \Delta OAB$

1 $\rightarrow$ $= \frac{1}{2}r^2 \sin(180° - \theta) \times 2 + \frac{1}{2}r^2 \sin 2\theta$

1 $\rightarrow$ $= r^2 \sin\theta + \frac{1}{2}r^2(2\sin\theta\cos\theta)$

$= r^2\sin\theta + r^2\sin\theta\cos\theta$

$= r^2\sin\theta(1+\cos\theta)$

(ii) $\frac{dA}{d\theta} = r^2\cos\theta(1+\cos\theta) - \sin\theta(r^2\sin\theta)$ ← 1

$= r^2\cos\theta + r^2\cos^2\theta - r^2\sin^2\theta$

$$1\left\{\begin{array}{l} = r^2\cos\theta + r^2(\cos^2\theta - \sin^2\theta) \\ = r^2\cos\theta + r^2\cos 2\theta \end{array}\right.$$

$= r^2(\cos\theta + \cos 2\theta)$

(iii) max area when $\frac{dA}{d\theta} = 0$

$$\left.\begin{array}{l} \text{ie } \cos\theta + \cos 2\theta = 0 \\ \therefore \cos\theta + 2\cos^2\theta - 1 = 0 \\ \therefore 2\cos^2\theta + \cos\theta - 1 = 0 \\ (2\cos\theta - 1)(\cos\theta + 1) = 0 \\ \therefore \cos\theta = \frac{1}{2} \text{ or } -1 \end{array}\right\} 1$$

$\therefore \theta = \frac{\pi}{6}, \cancel{\frac{5\pi}{6}}, \cancel{\pi}$

1 ↗ not possible

θ	$\frac{\pi}{6}^-$	$\frac{\pi}{6}$	$\frac{\pi}{6}^+$
$\frac{dA}{d\theta}$	+	0	−

or show that $\frac{d^2A}{d\theta^2} < 0$ when $\theta = \frac{\pi}{6}$

$\therefore$ a maximum area of ΔAPB occurs when $\theta = \frac{\pi}{6}$

(−1 mark if A' or A'' proof method not done)

b) (i) $x = vt\cos\alpha$

$= 80t \times \frac{1}{2}$

$= 40t$ ← 1

$y = vt\sin\alpha - 5t^2$

$= 80t \times \frac{\sqrt{3}}{2} - 5t^2$

$= 40\sqrt{3}\,t - 5t^2$ ← 1

(ii) max. height when $\dot{y} = 0$

$\therefore \dot{y} = 40\sqrt{3} - 10t = 0$

$\therefore t = \frac{40\sqrt{3}}{10}$

1 → $= 4\sqrt{3}$ seconds.

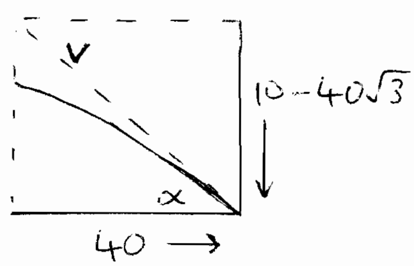
(iii) time of flight $= 8\sqrt{3}$ seconds

When $t = 8\sqrt{3} - 1$,

$\dot{y} = 40\sqrt{3} - 10(8\sqrt{3} - 1)$

$= 40\sqrt{3} - 80\sqrt{3} + 10$

$$\left.\begin{array}{l} \therefore \dot{y} = 10 - 40\sqrt{3} \\ \text{and } \dot{x} = 40 \end{array}\right\} 1$$



$v^2 = \sqrt{(10 - 40\sqrt{3})^2 + 40^2}$

$\therefore v \doteqdot 71.5$ m/s ← 1

and $\tan\alpha = \left|\frac{10 - 40\sqrt{3}}{40}\right|$