Name:______________________ Teacher:______________

SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

2009

EXTENSION 1 MATHEMATICS

Instructions:

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start each question on a new page

Total Marks

- Attempt Questions 1 – 7
- All questions are of equal value

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

Question 1 **Marks**

a) Simplify $\frac{4^n}{4^{n+1}-4^n}$ 1

b) Evaluate $\lim\limits_{x\to 0} \frac{sin4x}{3x}$ 1

c) The polynomial $P(x) = x^4 + ax^3 + 2x - 4$ has a remainder of -7 when divided by $x + 2$. Find the value of a. 2

d) Find the coordinates of the point P which divide the interval from $A(-1,5)$ to $B(6,-4)$ externally in the ratio $3:2$. 2

e) Find to the nearest degree, the acute angle between the lines $x - y = 2$ and $3x + y = 5$. 2

f) Find $\int x\sqrt{1-x}\ dx$ using the substitution $u = 1 - x$ 2

g) Solve for x: $\frac{2x-3}{x-2} \geq 1$ 2

Question 2 (Start a new page)

a) Differentiate with respect to x

(i) $y = ln\left(\frac{2x-3}{3x+2}\right)$ 2

(ii) $y = tan^3(3x+5)$ 2

(iii) $y = cos^{-1}(sinx)$ 2

b) Find

(i) $\int \frac{dx}{3+4x^2}$ 2

(ii) $\int \frac{2}{\sqrt{1-16x^2}} dx$ 2

(iii) $\int sin^2 \frac{x}{2}\ dx$ 2

Question 3 **Marks**

a) Prove the identity $\frac{cosx - cos2x}{sin2x + sinx} = cosecx - cotx$ 3

b) $P(x)$ is an odd polynomial of degree 3. It has (x-2) as a factor and when it is divided by $(x-4)$, the remainder is 96. Find $P(x)$. 3

c) Solve $\sqrt{3}\,cos\theta - sin\theta = -\sqrt{3}$ over the domain $0 \le \theta \le 2\pi$ 3

d) Sketch $y = -2sin^{-1}\frac{x}{3}$ showing the domain and range on your diagram. 3

Question 4

a) Let T be the temperature inside a room at time t and let R be the constant outside air temperature. Newton's law of cooling states that the rate of change of the temperature T is proportional to (T-R).

(i) Show that the function $T = R + Ae^{-kt}$ (where A and k are constants) is a solution of the differential equation $\frac{dT}{dt} = -k(T - R)$ 1

(ii) A metal baking dish is removed from an oven at 200°C. If the dish takes one minute to cool to 170°C, and the room temperature is $20°C$, find the values of A and k, correct to 2 decimal places if necessary 2

(iii) Find the time that it takes the dish to cool to $50°C$. 1

b) A particle is oscillating in simple harmonic motion about a fixed point. Its displacement $x cm$ at a time t seconds is given by

$x = 2cos3t + 4$.

(i) Explain why $2 \le x \le 6$ is the interval in which the particle moves. 1

(ii) Write down the amplitude and centre of motion. 2

(iii) Find $\ddot{x}$ as a function of t. 1

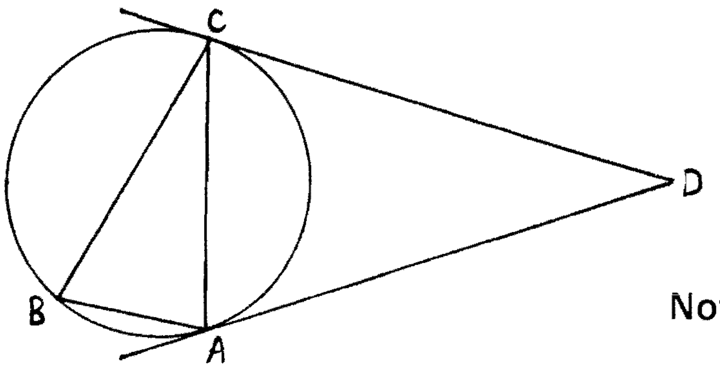
(iv) Show that $\ddot{x} = -9(x - 4)$ 1

(v) Show that $v^2 = -9x^2 + 72x - 108$ 2

(vi) Find the greatest speed of the particle. 1

Question 5 (Start a new page) **Marks**

a)



AD and CD are tangents to a circle. B is a point on the circle such that $\angle CBA$ and $\angle CDA$ are equal and are both double $\angle BCA$.

(i) Copy the diagram into your answer booklet.

(ii) Let $\angle CDA = \alpha$ and derive $\angle CAD$ in terms of α (give reasons). 2

(iii) Prove that BC is a diameter of the circle (give reasons). 2

b) The equation $x^3 - 2x^2 + 4x - 5 = 0$ has roots α, β, γ. Find the values of

(i) $\alpha\beta\gamma$ 1

(ii) $\alpha\beta + \beta\gamma + \alpha\gamma$ 1

(iii) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ 2

c) (i) Differentiate $x\cos^{-1}x - \sqrt{1 - x^2}$ 2

(ii) Hence evaluate $\int_0^1 \cos^{-1}x\,dx$ 2

Question 6 (Start a new page)

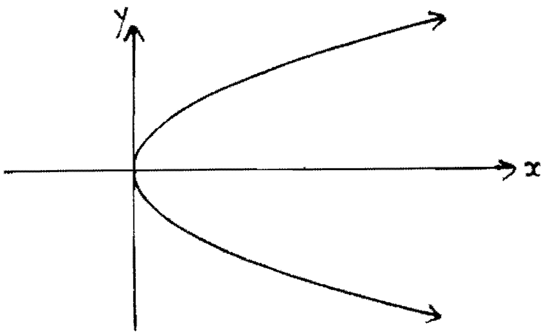
a) Prove by Mathematical Induction for n a positive integer, that 4

$1 \times 2^{\circ} + 2 \times 2^1 + 3 \times 2^2 + \ldots\ldots\ldots\ldots\ldots. + n \times 2^{n-1} = 1 + (n-1)2^n$

b) (i) Find the equation of the normal at $P(at^2, 2at)$ on the parabola $y^2 = 4ax$. 2

(ii) The normal intersects the $x - axis$ at point Q. Find the coordinates of Q and hence find the coordinates of R where R is the midpoint of PQ. 2

(iii) Hence find the Cartesian equation of the locus of R. 1



c) Find $\int \frac{cosxsinx}{2-sin^2x}dx$ using the substitution $u = sinx$. 3

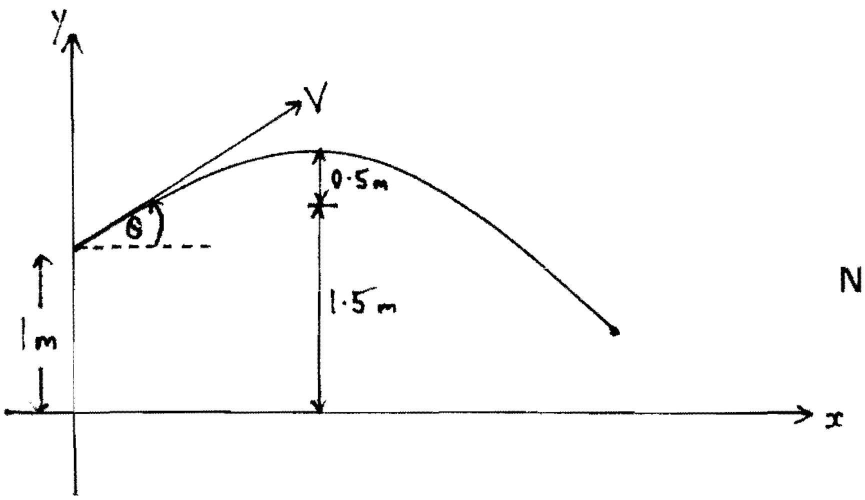
Question 7 **Marks**

a) (i) Show that $cos^{-1}\frac{2}{\sqrt{5}} = tan^{-1}\frac{1}{2}$ 1

(ii) Hence prove that $tan^{-1}\frac{2}{3} + cos^{-1}\frac{2}{\sqrt{5}} = tan^{-1}\frac{7}{4}$ 3

b) When the polynomial $P(x)$ is divided by $x+4$ the remainder is 5 and when $P(x)$ is divided by $(x-1)$ the remainder is 9. Find the remainder when $P(x)$ is divided by $(x-1)(x+4)$. 3

c)



Not to scale.

A boy throws a ball and projects it with a speed of Vm/s from a point 1m above the ground. The ball lands on top of a flowerpot in a neighbour's yard. The angle of projection is θ and indicated in the diagram. The equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -10$. It has been found that $y = Vtsin\theta - 5t^2 + 1$.

(i) Show that $x = Vtcos\theta$ 1

(ii) When the ball is at its maximum height above the ground, it passes directly above a 1.5 metre high fence and clears it by 0.5m.

Show that $V = \frac{\sqrt{20}}{sin\theta}$ 3

(iii) Find the value of V given $\theta = tan^{-1}\frac{9}{40}$, giving your answer in exact form. 1

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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S.T.H.S. 2009 Ext. 1 Trial Solutions

Question 1

a) $\dfrac{4^n}{4^{n+1}-4^n}$

$= \dfrac{\cancel{4^n}}{\cancel{4^n}(4-1)}$

$= \dfrac{1}{3}$ 1

b) $\lim\limits_{x \to 0} \dfrac{\sin 4x}{3x}$

$= \dfrac{4}{3} \lim\limits_{x \to 0} \dfrac{\sin 4x}{4x}$

$= \dfrac{4}{3}$ 1

c) $P(x) = x^4 + ax^3 + 2x - 4$

$-7 = (-2)^4 + a(-2)^3 - 4 - 4$ 1

$1 = 16 - 8a$

$8a = 15$

$a = \frac{15}{8}$ 1

d) $A(-1, 5)$ $B(6, -4)$

$m : n = 3 : -2$ (external)

$x = \dfrac{3 \times 6 + -2 \times -1}{3 + -2}$ 1 $y = \dfrac{3 \times -4 + -2 \times 5}{3 + -2}$

$x = \dfrac{20}{1}$ $y = \dfrac{-22}{1}$

$\therefore$ P is $(20, -22)$ 1

e) $y = x - 2$ $m_1 = 1$

$y = -3x + 5$ $m_2 = -3$

$\tan\theta = \left|\dfrac{1 - -3}{1 + 1 \times -3}\right|$ 1

$= \left|\dfrac{4}{-2}\right|$

$\tan\theta = 2$ 1

$\theta = 63°$ (nearest °)

f) $\int x\sqrt{1-x}\,dx$ $u = 1 - x$, $du = -dx$

$\int (1-u)\sqrt{u} \cdot -du$

$-\int u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$ 1

$= -\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right)$

$= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$ 1

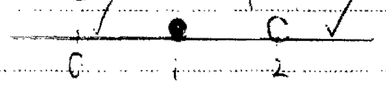
g) $\dfrac{2x-3}{x-2} \geq 1$

Critical pts. $x \neq 2$

$\dfrac{2x-3}{x-2} = 1$ 1

$2x - 3 = x - 2$

$x = 1$



$x \leq 1,\ x > 2$ 1

Question 2

a) (i) $y = \ln\left(\dfrac{2x-3}{3x+2}\right)$

$= \ln(2x-3) - \ln(3x+2)$ 1

$y' = \dfrac{2}{2x-3} - \dfrac{3}{3x+2}$ 1

(ii) $y = \tan^3(3x+5)$ 1

$y' = 3 \times 3\tan^2(3x+5) \times \sec^2(3x+5)$

$= 9\tan^2(3x+5)\sec^2(3x+5)$

1

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(iii) $y = \cos^{-1}(\sin x)$

$y' = \frac{-1}{\sqrt{1-\sin^2 x}} \times \cos x$ (1)

$= \frac{-\cos x}{\sqrt{1-\sin^2 x}}$ (1)

b (i) $\int \frac{dx}{3+4x^2}$

$\frac{1}{4}\int \frac{dx}{\left(\frac{\sqrt{3}}{2}\right)^2 + x^2}$ (1)

$\frac{1}{4} \times \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)$

$\frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)$ or

$\underline{\frac{\sqrt{3}}{6} \tan^{-1}\left(\frac{2\sqrt{3}\,x}{3}\right) + C}$ (1)

(ii) $\int \frac{2}{\sqrt{1-16x^2}}\, dx$

$\frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{1}{4}\right)^2 - x^2}}$ (1)

$\underline{\frac{1}{2} \sin^{-1} 4x + c}$ (1)

(iii) $\int \sin^2 \frac{x}{2}\, dx$

$\int \frac{1-\cos x}{2}\, dx$ (1)

$\underline{\frac{x}{2} - \frac{\sin x}{2} + C}$ (1)

<u>Question 3</u>

a) $\frac{\cos x - \cos 2x}{\sin 2x + \sin x} = \operatorname{cosec} x - \cot x$

$\frac{\cos x - (2\cos^2 x - 1)}{2\sin x \cos x + \sin x}$ (1)

$\frac{-2\cos^2 x + \cos x + 1}{\sin x(2\cos x + 1)}$ (1)

$\frac{\cancel{(2\cos x + 1)}(1-\cos x)}{\sin x\cancel{(2\cos x + 1)}}$ (1)

$\operatorname{cosec} x - \cot x = RHS$

b) $P(x)$ is odd. It must pass through $(0,0)$ and if $(x-2)$ is a factor so is $(x+2)$ (1)

$\therefore P(x) = ax(x-2)(x+2)$ (1)

$P(-4) = -4a(-6)(-2) = 96$

$a = -2$

$\underline{\therefore P(x) = -2x(x^2-4)}$ (1)

c) $\sqrt{3}\cos\theta - \sin\theta = -\sqrt{3}$

$\sqrt{3} \times \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} = -\sqrt{3}$

(1)

$\sqrt{3}(1-t^2) - 2t = -\sqrt{3}(1+t^2)$

$\sqrt{3} - \cancel{\sqrt{3}t^2} - 2t = -\sqrt{3} - \cancel{\sqrt{3}t^2}$

$2t = 2\sqrt{3}$

$\tan\frac{\theta}{2} = \sqrt{3}$

$\therefore \frac{\theta}{2} = \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$ (1)

Test $\theta = \pi$

$-\sqrt{3} - \sin\pi = -\sqrt{3}$ ✓

$\therefore$ Solutions are

$\underline{\theta = \pi, \frac{2\pi}{3}}$ (1)

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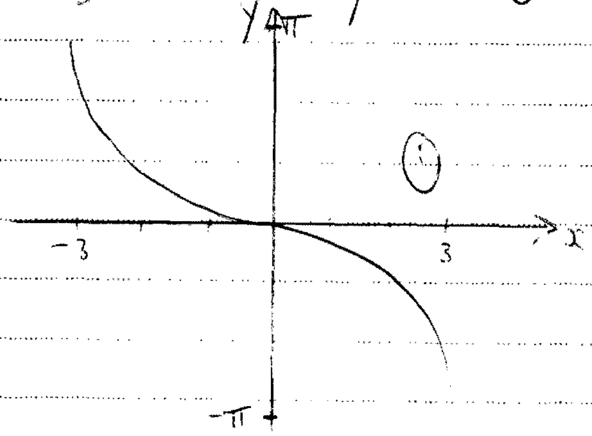
d) $y = -2\sin^{-1}\left(\frac{x}{3}\right)$

Domain: $-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$ (1)

Range: $-\pi \le y \le \pi$ (1)



Question 4

a) (i) $T = R + Ae^{kt}$

$\frac{dT}{dt} = Ake^{kt}$

$\frac{dT}{dt} = k \times Ae^{kt}$

$\frac{dT}{dt} = k(T - R)$ (1)

(ii) $170 = 20 + Ae^{k\times 1}$

also $200 = 20 + Ae^{0}$

$180 = A$ (1)

$\therefore 170 = 20 + 180e^{k}$

$e^{k} = \frac{150}{180}$

$k = -0.18$ (1)

(iii) $T = 20 + 180e^{-0.18t}$

$50 = 20 + 180e^{-0.18t}$

$\frac{1}{6} = e^{-0.18t}$

$\ln\frac{1}{6} = -0.18t$

$t = \frac{\ln\frac{1}{6}}{-0.18}$

$t = 10.0$ minutes (1)

b) (i) Because $-1 \le \cos 3t \le 1$

$\therefore 2 \le x \le 4$ (1)

(ii) Since motion is simple harmonic, centre of motion is halfway between 2 and 6

ie: $x = 4$ (1)

and amplitude is 2. (1)

(iii) $x = 2\cos 3t + 4$

$\dot{x} = -2\times 3\sin 3t$

$\ddot{x} = -18\cos 3t$ (1)

(iv) $\ddot{x} = -18\cos 3t$

$= -9(2\cos 3t + 4 - 4)$

$\ddot{x} = -9(x - 4)$ (1)

(v) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9(x-4)$

$\frac{1}{2}v^2 = -\frac{9x^2}{2} + 36x + C_1$

$v^2 = -9x^2 + 72x + C_2$ (1)

when $x = 2$, $v = 0$

$0 = -9\times 2^2 + 72\times 2 + C_2$

$C_2 = -108$

$\therefore v^2 = -9x^2 + 72x - 108$ (1)

(vi) Greatest speed occurs at centre of motion $x = 4$.

$v^2 = -9\times 4^2 + 72\times 4 - 108$

$V_{max} = 6$ cm/s (1)

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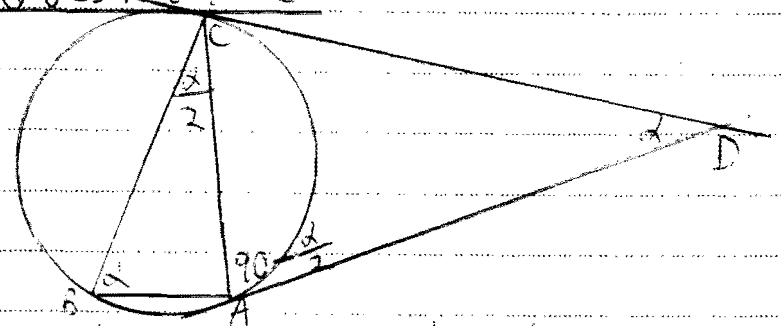
Question 5



(i)

(ii) ΔCAD is isosceles

$\angle CAD = 90 - \frac{\alpha}{2}$ (1)

(equal base angles of an isosceles triangle)

(iii) $\angle ACD = 90 - \frac{\alpha}{2}$ from (i)

$\angle BCA = \frac{\alpha}{2}$ (given)

$\therefore \angle BCD = \frac{\alpha}{2} + 90 - \frac{\alpha}{2}$

$= 90^\circ$ (1)

$\therefore$ BC is a diameter as angle between radius and pt. of contact of tangent CD is 90° (1)

b(i) $x^3 - 2x^2 + 4x - 5 = 0$ has roots α, β, γ

$\therefore \alpha\beta\gamma = -\frac{d}{a}$

$= 5$ (1)

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= 4$ (1)

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ (1)

$= \frac{4}{5}$ (1)

c.(i) $\frac{d}{dx}\left(x\cos^{-1}x - (1-x^2)^{\frac{1}{2}}\right)$

$x \times \frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$ (1)

$= \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x + \frac{x}{\sqrt{1-x^2}}$

$= \cos^{-1}x$ (1)

(ii) $\int_0^1 \cos^{-1}x\,dx$

$= \int_0^1 \frac{d}{dx}\left[x\cos^{-1}x - \sqrt{1-x^2}\right]dx$

$= \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$ (1)

$= 1 \times \cos^{-1}1 - 0 - (0 - \sqrt{1})$

$= 1$ (1)

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Question 6

a) Step 1 - Show result is true for $n=1$ ① for steps 1, 2

$1 \times 2^{1-1} = 1 + (1-1)2^1$

$1 = 1$ ✓

Step 2 - Assume result is true for $n=k$, k an integer > 1.

ie: $1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \dots + k \times 2^{k-1} = 1 + (k-1)2^k$

Step 3 - Show result is true for $n = k+1$

ie: $1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \dots + k \times 2^{k-1} + (k+1) \times 2^k$ ①

$1 + (k-1)2^k + (k+1) \times 2^k$ using Step 2 ①

$1 + [k-1+k+1]2^k$

$1 + [2k] \times 2^k$

$1 + k \times 2^{k+1}$ as required ①

Step 4

Since result is true for $n=1$, it must also be true for $n = 1+1 = 2$, $n = 2+1 = 3$ and hence for all positive integral values of n.

b)(i) $m_{tangent} = \frac{dy}{dx}$

$= \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$= \frac{2a}{2at}$

$= \frac{1}{t}$

$\therefore m_{normal} = -t$ ①

$y - 2at = -t(x - at^2)$

$tx + y = 2at + at^3$ is the eq'n of normal ①

(ii) At Q, $y = 0$

$\therefore tx = 2at + at^3$

$x = 2a + at^2$

$\therefore Q(2a + at^2, 0)$ ①

$P(at^2, 2at)$. R is midpoint of PQ

ie $\left(\frac{2a + at^2 + at^2}{2}, \frac{0 + 2at}{2}\right)$

R is $(a + at^2, at)$ ①

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(iii) $R(a + at^2, at)$

$y = at, \quad x = a(1+t^2)$

$t = \frac{y}{a}$

$\therefore x = a\left(1 + \frac{y^2}{a^2}\right)$

$ax = a^2 + y^2$ (1)

c) $\int \frac{\cos x \sin x}{2 - \sin^2 x}\, dx$

Let $u = \sin x$

$du = \cos x\, dx$

$\int \frac{u\, du}{2 - u^2}$ (1)

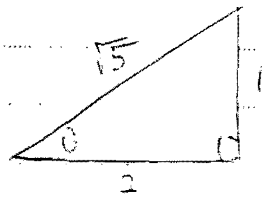
$\frac{1}{-2}\int \frac{-2u}{2-u^2}\, du$

$-\frac{1}{2}\log_e(2-u^2) + c$ (1)

$= -\frac{1}{2}\log_e(2 - \sin^2 x) +$ (1)

Question 7

a) (i)



Let $\theta = \cos^{-1}\frac{2}{\sqrt{5}}$

From Δ above

$\theta = \tan^{-1}\frac{1}{2}$

$\therefore \cos^{-1}\frac{2}{\sqrt{5}} = \tan^{-1}\frac{1}{2}$ (1)

(ii) Prove

$\tan^{-1}\frac{2}{3} + \cos^{-1}\frac{2}{\sqrt{5}} = \tan^{-1}\frac{7}{4}$

$\tan^{-1}\frac{2}{3} + \tan^{-1}\frac{1}{2} = \tan^{-1}\frac{7}{4}$

from part (i)

Let $\alpha = \tan^{-1}\frac{2}{3} \quad \beta = \tan^{-1}\frac{1}{2}$

$\therefore \tan\alpha = \frac{2}{3}, \quad \tan\beta = \frac{1}{2}$ (1)

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{\frac{2}{3} + \frac{1}{2}}{1 - \frac{2}{3}\times\frac{1}{2}}$ (1)

$= \frac{\frac{7}{6}}{\frac{2}{3}}$

$\tan(\alpha+\beta) = \frac{7}{4}$ (1)

$\therefore \alpha + \beta = \tan^{-1}\frac{7}{4}$

$\therefore$ RHS = LHS

b) Let $P(x) = (x-1)(x+4)Q(x) + ax + b$ (1)

$P(-4) = 0 - 4a + b = 5$) Solve

$P(1) = 0 + a + b = 9$) simultaneously

$5a = 4 \quad a = \frac{4}{5}, \quad b = 8\frac{1}{5}$

$\therefore$ Remainder is $\frac{4}{5}x + 8\frac{1}{5}$ (1) (1)

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c) (i) $\ddot{x} = 0$

$\dot{x} = c$

When $t = 0$ $\dot{x} = V\cos\theta$

$\therefore V\cos\theta = c$

$\dot{x} = V\cos\theta$

$x = Vt\cos\theta + c$

when $t = 0$ $x = 0$ $\therefore c = 0$

$\therefore x = Vt\cos\theta$ (1)

(ii) At max. height

$y = 2$ when $\dot{y} = 0$

$\dot{y} = V\sin\theta - 10t$

$0 = V\sin\theta - 10t$

$t = \frac{V\sin\theta}{10}$ when $y = 2$.

$y = Vt\sin\theta - 5t^2 + 1$ (1)

$2 = V\sin\theta \times \frac{V\sin\theta}{10} - 5 \times \frac{V^2\sin^2\theta}{100} + 1$ (1)

$1 = \frac{V^2\sin^2\theta}{10} - \frac{5V^2\sin^2\theta}{100}$

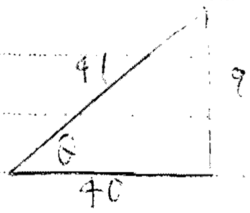
$1 = \frac{10V^2\sin^2\theta - 5V^2\sin^2\theta}{100}$

$1 = \frac{V^2\sin^2\theta}{20}$

$V^2 = \frac{20}{\sin^2\theta}$

$\therefore V = \frac{\sqrt{20}}{\sin\theta}$ (1)

(iii)



$V = \frac{\sqrt{20}}{\sin\theta}$

$= \frac{\sqrt{20}}{9/41}$

$= \frac{41\sqrt{20}}{9}$

$= \frac{82\sqrt{5}}{9}$ m/s (1)