SYDNEY TECHNICAL HIGH SCHOOL

(Established 1911)

TRIAL HIGHER SCHOOL CERTIFICATE

2010

Mathematics Extension 1

General Instuctions

- Reading time - 5 minutes
- Working time - 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Diagrams are not drawn to scale

Total marks - 84

- Attempt Questions 1 – 7
- All questions are of equal value

Name :_____________________________

Teacher :_____________________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	**Total**

Question 1 (12 marks)

(a) Simplify $\frac{\sin(x+y)+\sin(x-y)}{\cos(x+y)-\cos(x-y)}$ 2

(b) Differentiate $\cos^{-1}(\frac{2x}{3})$ 2

(c) Solve $\frac{2x}{x-1} < 3$ 3

(d) Find the acute angle between the lines 2

$x - 2y + 3 = 0$ and $3x + y + 6 = 0$

(Answer to the nearest degree)

(e) Evaluate $\int_0^{\sqrt{3}} \frac{4}{x^2+9}\, dx$ 3

Question 2 (12 marks) Start a new page.

(a) Use the substitution $t = tan\frac{A}{2}$ to simplify $1 + tanA\ tan\frac{A}{2}$. 2

(b) Find the value of a if the polynomial $p(x) = x^3 - 2x^2 - ax + 6$ 2

is divisible by $(x + 2)$.

(c) $A(x^2, 12)$ and $B(x, 6)$ are two fixed points for some real value of x. 2

The point $P(1, 10)$ divides the interval AB internally in the ratio 1:2.

Find the possible values of x .

(d) (i) State the domain and range of $y = 4\sin^{-1}\frac{x}{2}$. 2

(ii) Sketch $y = 4\sin^{-1}\frac{x}{2}$. 1

(iii) Find the area bounded by $y = 4\sin^{-1}\frac{x}{2}$, the x axis 3

and the line $x = 1$.

Question 3 (12 marks) Start a new page.

(a) Solve $\sin 4\theta = \cos 2\theta$ for $0 \leq \theta \leq \pi$. 3

(b) $P\ (4p\,, 2p^2)$ is a point on the parabola $x^2 = 8y$.

(i) Find the coordinates of S, the focus of the parabola $x^2 = 8y$. 1

(ii) Find the equation of the tangent at P. 2

(iii) The tangent at P meets the y axis at the point M. 1

Find the coordinates of M.

(iv) The perpendicular from S to the tangent PM meets the 3

tangent at N. Find the coordinates of N.

(v) Find the equation of the locus of the midpoint of 2

the interval MN as the position of P varies.

Question 4 (12 marks) Start a new page.

(a) The volume of a cube is expanding at the constant rate of 5 $mm^3/sec.$ 4

At what rate is the surface area of the cube increasing when

the side length of the cube is 60 centimetres ?

Question 4 continues on page 5

Question 4 (continued)

(b) A particle is moving along the x axis. 2

Its velocity v at *position* x is given by $v = 12 - x^2$.

Find the acceleration of the particle when $x = 4$.

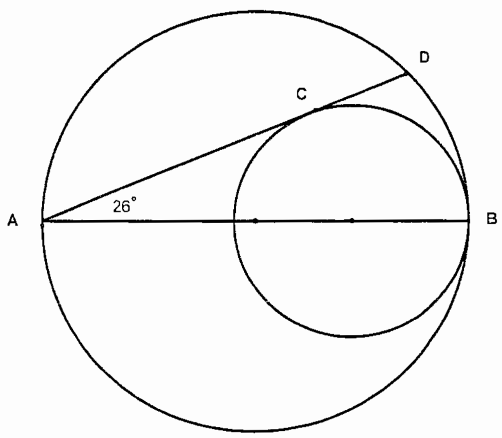
(c) From point A, Sarah observed that the base of a tower is at a bearing of $080°$ and the top of the tower is at an angle of elevation of $9°$. Sarah then walks to point B, 1000m due South of A, and observes that the base of the tower is at a bearing of $065°$. 3

Find the height of the tower above ground level.

(Give answer in metres correct to 1 decimal place.)

(Points A and B are at ground level.)

(d)



The diameter AB of the larger circle is 10 centimetres. 3

The smaller circle touches the larger circle at B and the chord ACD is a tangent to the smaller circle. Angle DAB = $26°$.

Find the radius of the smaller circle, in centimetres correct to 2 decimal places.

Question 5 (12 marks) Start a new page.

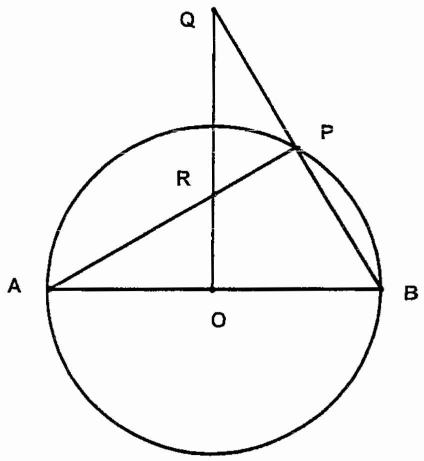
(a) Find $\int cos^2 4x\ dx$ 2

(b) If $f(x) = log_e\sqrt{5-2x}$ find the inverse function $f^{-1}(x)$. 2

(c) O is the centre of the circle.

BPQ , ORQ , ARP and AOB are straight lines.

Angle QOB = $90°$.



(i) Copy the diagram onto your answer sheet.

(ii) Prove that A , O , P and Q are concyclic points. 3

Question 5 continues on page 7

Question 5 (continued)

(d) The temperature of a liquid t minutes after being placed in a freezer is given by the equation $T = -4 + Ae^{-kt}$, where A and k are constants.

(i) Initially the liquid is at a temperature of 40°C. Find the value of A. 1

(ii) When the temperature of the liquid is 26°C the rate of change of the temperature of the liquid is -0.3°C per minute. Show that $k = 0.01$. 2

(iii) Find the time taken for temperature of the liquid to fall from 40°C to 6°C. (Answer in minutes correct to 1 decimal place) 2

Question 6 (12 marks) Start a new page.

(a) Find $\int \frac{dx}{2x\sqrt{1-(\ln x)^2}}$ using the substitution $u = \ln x$. 3

(b) A particle is projected from ground level with a velocity of 32 m/s.

The angle of elevation, θ , is allowed to vary.

You may assume that, if the origin is taken to be the point of projection,

the path of the particle at time t seconds is given by the parametric equations

$$x = 32t\cos\theta$$

$$y = 32t\sin\theta - \frac{1}{2}gt^2$$ where $g\ m/s^2$ is the acceleration due to gravity.

(i) Show that the maximum height reached by the projectile 2

is given by $\frac{512\sin^2\theta}{g}$ metres.

(ii) Find an expression for the maximum distance the particle 3

can land from the point of projection.

(iii) The particle is to be projected so as to hit an object 30 metres above 4

ground level and 64 metres horizontally from the point of projection.

Taking $g = 10\ m/s^2$, calculate the possible angles of projection.

(Give answers correct to the nearest degree)

Question 7 (12 marks) Start a new page.

(a) Use Mathematical Induction to show that

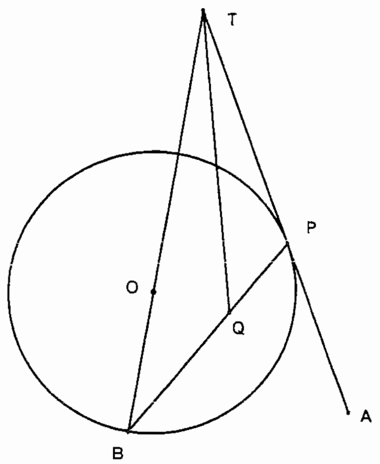
$5^n \geq 1 + 4n$ for all positive integers n. 3

(b) One root of the equation $x^3 + px^2 + qx + r = 0$ 3

equals the sum of the other two roots.

Prove that $p^3 = 4pq - 8r$.

(c)



P is a point on a circle. TP is a tangent to a circle, centre O. 3

TOB and TPA are straight lines. QT bisects angle BTP .

Let angle $PTQ = x$.

Copy the diagram onto your answer sheet and then

find an expression for angle APB , giving reasons for each step.

(d) Find all real x such that $|\, 2x - 1 \,| > \sqrt{x(2-x)}$ 3

End of Paper. ✈

STHS EXT 1 TRIAL SOLUTIONS

Question 1

a) $$\frac{\sin x\cos y + \sin y\cos x + \sin x\cos y - \sin y\cos x}{\cos x\cos y - \sin x\sin y - \cos x\cos y - \sin x\sin y}$$

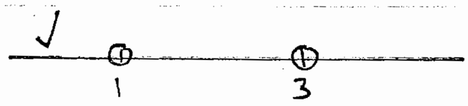
$$= \frac{2\sin x\cos y}{-2\sin x\sin y}$$

$$= -\cot y$$

b) $$\frac{-2}{\sqrt{9-4x^2}}$$

c) $x \neq 1$ $\qquad 2x = 3x - 3$

$x = 3$



$\therefore \quad x < 1 \quad$ or $\quad x > 3$

d) $m_1 = \frac{1}{2} \qquad m_2 = -3$

$$\tan\theta = \left|\frac{-3-\frac{1}{2}}{1+(-3)(\frac{1}{2})}\right|$$

$\therefore \quad \theta = 82^\circ$

e) $$\left[\frac{4}{3}\tan^{-1}\frac{x}{3}\right]_0^{\sqrt{3}}$$

$$= \frac{4}{3}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right)$$

$$= \frac{4}{3}\cdot\frac{\pi}{6}$$

$$= \frac{2\pi}{9}$$

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Question 2

a) $1 + \tan A \tan \frac{A}{2}$

$= 1 + \frac{2t}{1-t^2} \cdot t$

$= \frac{1-t^2+2t^2}{1-t^2}$

$= \frac{1+t^2}{1-t^2}$

$= \sec A$

b) $P(-2) = 0$

$\therefore -8 - 8 + 2a + 6 = 0$

$a = 5$

c) $\frac{2 \times x^2 + 1 \times x}{3} = 1$

$2x^2 + x - 3 = 0$

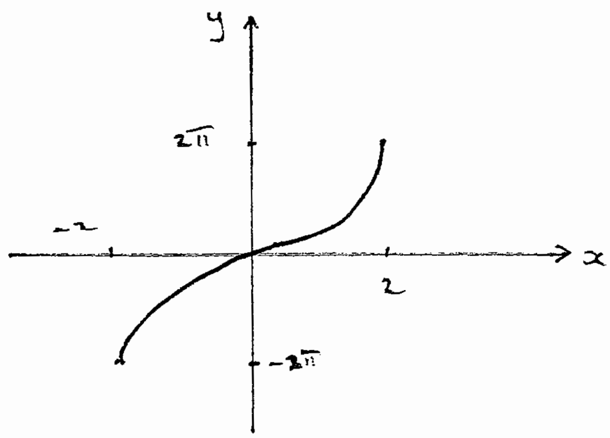
$(2x+3)(x-1) = 0$

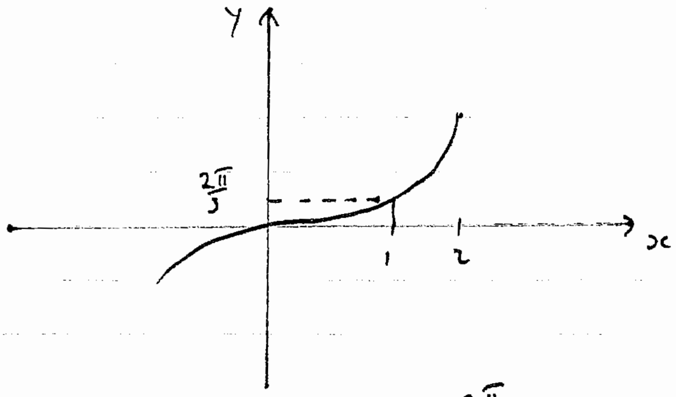
$x = -\frac{3}{2}, +1$

d) i) D: $-2 \le x \le 2$

R: $-2\pi \le y \le 2\pi$

ii)



iii) 

$$y = 4\sin^{-1}\frac{x}{2}$$

$$x = 2\sin\frac{y}{4}$$

$$\text{Area} = \frac{2\pi}{3} - \int_0^{\frac{2\pi}{3}} 2\sin\frac{y}{4}\,dy$$

$$= \frac{2\pi}{3} + \left[8\cos\frac{y}{4}\right]_0^{\frac{2\pi}{3}}$$

$$= \frac{2\pi}{3} + 8\cdot\frac{\sqrt{3}}{2} - 8$$

$$= \frac{2\pi}{3} + 4\sqrt{3} - 8 \quad \text{sq units}$$

Question 3

a) $2\sin 2\theta\cos 2\theta - \cos 2\theta = 0$

$\cos 2\theta(2\sin 2\theta - 1) = 0$

$\cos 2\theta = 0 \qquad \sin 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$

b) i) $S(0, 2)$

ii) $y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{x}{4}$ when $x = 4p$

$m_T = p$

$\therefore\ y - 2p^2 = p(x - 4p)$

$y = px - 2p^2$

iii) sub $x = 0$ $\quad y = -2p^2$

$\therefore \quad M(0, -2p^2)$

iv) $m_{\perp} = -\frac{1}{p}$

$\therefore \quad y - 2 = -\frac{1}{p}(x - 0)$

$py - 2p = x$

$y = 2 - \frac{1}{p}x$

Solve simulteneously with tangent

$\therefore$

$2 - \frac{1}{p}x = px - 2p^2$

$x\left(p + \frac{1}{p}\right) = 2(p^2 + 1)$

$$x = \frac{2(p^2+1)}{\frac{p^2+1}{p}}$$

$x = 2p$

$\therefore \quad y = 2p^2 - 2p$

$= 0$

$\therefore \quad N(2p, 0)$

v) midpoint $(p, -p^2)$

$\therefore \quad x = p \quad , \quad y = -p^2$

$\therefore \quad y = -x^2$

Question 4

a) $V = x^3$ $\qquad A = 6x^2$

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$

$$\therefore \quad 5 = 3x^2 \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{5}{3x^2}$$

$$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$$

$$= 12x \times \frac{5}{3x^2}$$

$$= \frac{20}{x}$$

when $x = 600$ mm

$$\frac{dA}{dt} = \frac{1}{30} \text{ mm}^2/\text{sec}$$

b) $\frac{1}{2}v^2 = \frac{1}{2}(12 - x^2)^2$

$$\therefore \quad a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= -2x(12 - x^2)$$

when $x = 4$

$$a = -8(12 - 16)$$

$$= 32$$

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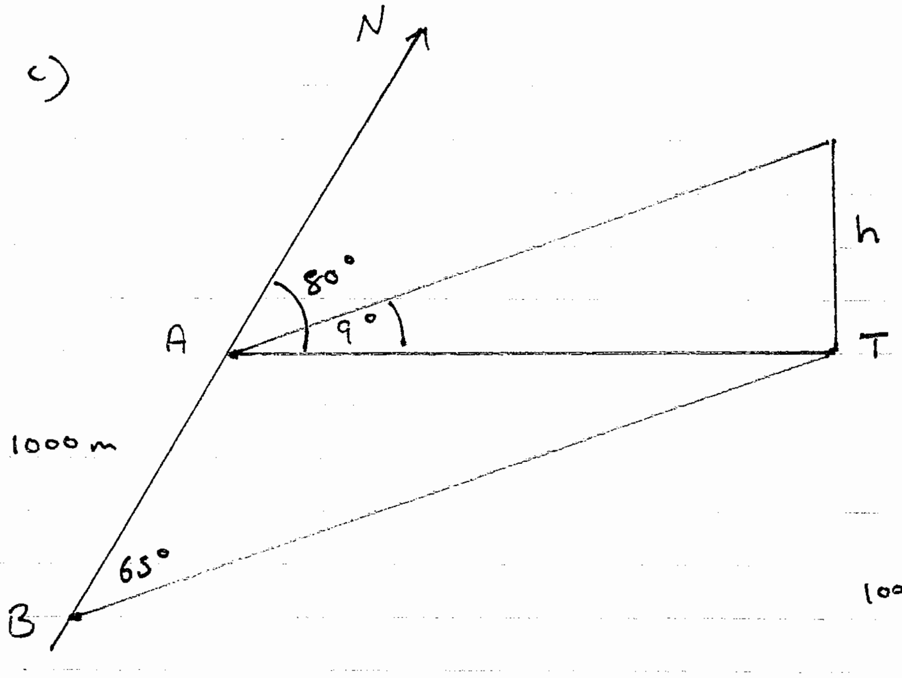
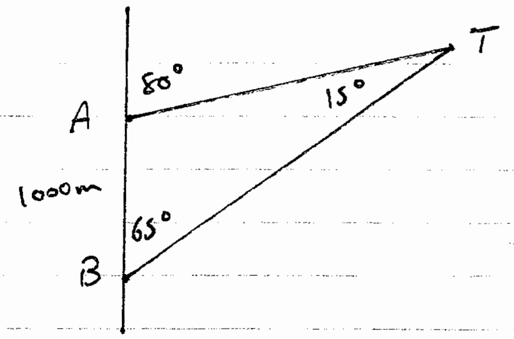
c)





$$\tan 9^\circ = \frac{h}{AT}$$

$$AT = h \tan 81^\circ$$

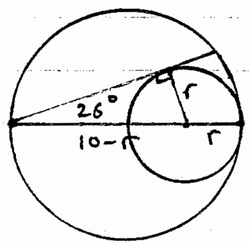
$$\therefore \quad \frac{h \tan 81^\circ}{\sin 65^\circ} = \frac{1000}{\sin 15^\circ}$$

$$h = \frac{1000 \sin 65^\circ}{\sin 15^\circ \tan 81^\circ}$$

$$= 554.6 \text{ m}$$

d)

$$\sin 26^\circ = \frac{r}{10-r}$$



$$10 \sin 26^\circ = r(1 + \sin 26^\circ)$$

$$r = \frac{10 \sin 26^\circ}{1 + \sin 26^\circ}$$

$$= 3.05 \text{ cm}$$

Question 5

a) $\int \cos^2 4x \, dx$

$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$= \frac{1}{2}\int 1 + \cos 8x \, dx$

$\cos^2 4x = \frac{1}{2}(1 + \cos 8x)$

$= \frac{1}{2}\left(x + \frac{1}{8}\sin 8x\right) + c$

b) $y = \log_e (5-2x)^{\frac{1}{2}}$

$\therefore$ inverse is $x = \log_e (5-2y)^{\frac{1}{2}}$

$x = \frac{1}{2}\log_e (5-2y)$

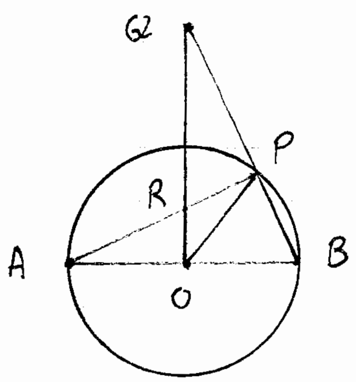
$2x = \log_e (5-2y)$

$e^{2x} = 5-2y$

$y = \frac{1}{2}(5 - e^{2x})$

$\therefore f^{-1}(x) = \frac{1}{2}(5 - e^{2x})$

c) i)



ii) let $\angle APO = x$

$\therefore \angle PAO = x$ (equal angles in isosceles triangle)

$\angle APB = 90°$ (angle in a semi-circle)

$\angle PBA = 90° - x$ (angle sum of triangle)

$\angle OQB = x$ (angle sum of triangle)

$\therefore A, O, P, Q$ concyclic (OP subtends equal angles at A and Q)

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d) i) $T = -4 + Ae^{-kt}$

when $t=0$ $T=40$

$40 = -4 + Ae^{0}$

$A = 44$

ii) $\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T+4)$

$\therefore \quad -0.3 = -k(26+4)$

$k = \frac{0.3}{30}$

$k = 0.01$

iii) $T = -4 + 44e^{-0.01t}$

$6 = -4 + 44e^{-0.01t}$

$\frac{10}{44} = e^{-0.01t}$

$\ln \frac{10}{44} = -0.01t$

$t = \frac{-1}{0.01} \ln \frac{10}{44}$

$= 148.2$ minutes

Question 6

a) $$\int \frac{dx}{2x\sqrt{1-(\ln x)^2}}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2}\sin^{-1} u$$

$$= \frac{1}{2}\sin^{-1}(\ln x) + c$$

b) i) max height when $\dot{y} = 0$

$$\dot{y} = 96\sin\theta - gt$$

$$0 = 96\sin\theta - gt$$

$$t = \frac{96\sin\theta}{g}$$

$$\therefore\ y = 96\left(\frac{96\sin\theta}{g}\right)\sin\theta - \frac{g}{2}\left(\frac{96\sin\theta}{g}\right)^2$$

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Question 6

a) $\displaystyle\int \frac{dx}{2x\sqrt{1-(\ln x)^2}}$ $\qquad u = \ln x$, $\quad du = \frac{1}{x}\,dx$

$= \frac{1}{2}\displaystyle\int \frac{du}{\sqrt{1-u^2}}$

$= \frac{1}{2}\sin^{-1} u$

$= \frac{1}{2}\sin^{-1}(\ln x) + c$

b) i) max height when $\dot{y} = 0$

$\dot{y} = 32\sin\theta - gt$

$0 = 32\sin\theta - gt$

$t = \dfrac{32\sin\theta}{g}$

$\therefore$ max height $= 32\left(\dfrac{32\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\dfrac{32\sin\theta}{g}\right)^2$

$= \dfrac{1024\sin^2\theta}{g} - \dfrac{1024\sin^2\theta}{2g}$

$= \dfrac{512\sin^2\theta}{g}$

ii) max distance when $\theta = 45^\circ$ and $y = 0$

$0 = 32t\sin 45^\circ - \frac{1}{2}gt^2$

$0 = t\left(\frac{32}{\sqrt{2}} - \frac{1}{2}gt\right)$

$t = \cancel{0}, \ \dfrac{64}{\sqrt{2}g}$

$$\therefore \text{ max distance } = 32\left(\frac{64}{\sqrt{2}g}\right)\cos 45^\circ$$

$$= \frac{1024}{g} \text{ metres}$$

iii) $x = 32t\cos\theta \longrightarrow t = \dfrac{x}{32\cos\theta}$

$$\therefore y = 32\left(\frac{x}{32\cos\theta}\right)\sin\theta - 5\left(\frac{x}{32\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{5x^2}{1024}\left(1 + \tan^2\theta\right)$$

when $x = 64$, $y = 30$

$$30 = 64\tan\theta - \frac{5 \times 64^2}{1024}\left(1 + \tan^2\theta\right)$$

$$20\tan^2\theta - 64\tan\theta + 50 = 0$$

$$\tan\theta = \frac{64 \pm \sqrt{64^2 - 4 \times 20 \times 50}}{40}$$

$$\theta = 62^\circ, 54^\circ$$

Question 7

a) Step 1 : show true for $n = 1$

$LHS = 5^1$
$= 5$

$RHS = 1 + 4$
$= 5$

$LHS \geq RHS$
$\therefore$ true for $n = 1$

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Step 2 : assume result true for $n = k$

i.e. $5^k \geqslant 1 + 4k$

Step 3 : show result is true for $n = k+1$

$$5^{k+1} = 5 \times 5^k$$

$$\geqslant 5(1 + 4^k)$$

$$= 5 + 5.4^k$$

$$\geqslant 1 + 4.4^k$$

$$= 1 + 4^{k+1}$$

which is the required result

$\therefore$ true for $n = k+1$ if true for $n = k$

Step 4 : as true for $n=1$, also true for $n = 1+1$, $n = 2$
as true for $n = 2$, also true for $n = 2+1$, $n = 3$
and so on for all positive integers n.

b) let roots be $\alpha, \beta, \alpha + \beta$

$$\therefore \quad 2\alpha + 2\beta = -p$$
$$\alpha\beta + \alpha(\alpha+\beta) + \beta(\alpha+\beta) = q$$
$$\alpha\beta(\alpha+\beta) = -r$$

or

$$\alpha + \beta = \frac{-p}{2} \qquad (1)$$

$$\alpha^2 + \beta^2 + 3\alpha\beta = q \qquad (2)$$

$$\alpha\beta(\alpha+\beta) = -r \qquad (3)$$

sub (1) into (3)

$$\alpha\beta\left(\frac{-p}{2}\right) = -r$$

$$\alpha\beta = \frac{2r}{p} \qquad (4)$$

rearrange (2)

$$(\alpha+\beta)^2 + \alpha\beta = q$$

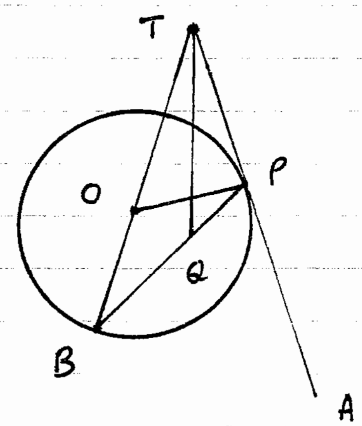
sub in (1) and (4)

$$\left(\frac{-p}{2}\right)^2 + \frac{2r}{p} = q$$

$$\frac{p^2}{4} + \frac{2r}{p} = q$$

$$p^3 + 8r = 4pq$$

c)



$\angle QTP = x$

$\angle BTQ = x$ (given QT bisects $\angle BTP$)

$\angle TPO = 90^\circ$ (angle between radius and tangent)

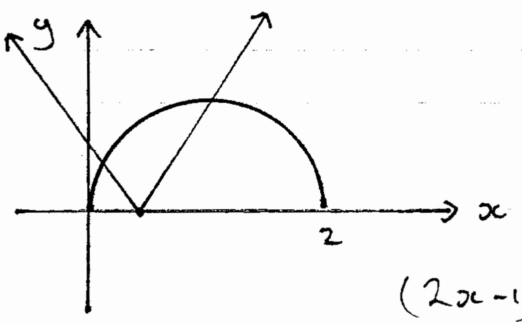
$\angle TOP = 90^\circ - 2x$ (angle sum of triangle)

$\angle TBP = 45^\circ - x$ (angle at centre is double at circumference)

$\angle OPB = 45^\circ - x$ (equal angles in isosceles triangle)

$\therefore \angle APB = 45^\circ + x$ (right angle)

d)



$$(2x-1)^2 = x(2-x)$$

$$5x^2 - 6x + 1 = 0$$

$$(5x-1)(x-1) = 0$$

$$x = \tfrac{1}{5}, 1$$

$$\therefore \quad 0 \le x < \tfrac{1}{5}, \qquad 1 < x \le 2$$