Sydney Technical High School

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION 2011

Mathematics Extension 1

General Instructions

- Reading Time - 5 minutes.
- Working Time - 2 hours.
- Write using a blue or black pen.
- Approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (84)

- Attempt Questions 1-7.
- All questions are of equal value.

NAME ____________________

TEACHER ____________________

Question 1 (12 marks) Marks

a) Find $\frac{d}{dx}(x^2 \cos^{-1} x)$ 1

b) Given that $\log_a x = 2.8$ and $\log_a y = 4.1$, evaluate $log_a\left(\frac{y^2}{x}\right)$ 1

c) Solve $|x-2| < |x|$ 2

d) Solve $\frac{x}{x+2} > 2$ 3

e) Find the coordinates of a point C which divides the interval joining A (-1,2) and B (3,5) 2

externally in the ratio 3:1

f) i) Show that $(\sin A - \cos A)^2 = 1 - sin\ 2A$ 1

ii) Hence find the exact value of $\sin 15° - \cos 15°$ 2

Question 2 (12 marks) START A NEW PAGE.

a) The polynomial $P(x) = x^3 + ax^2 - ax + 2$ has a factor of $(x+2)$.

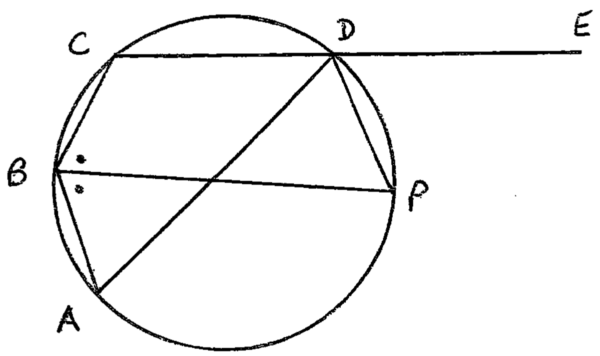
i) Find the value of a. 2

ii) Fully factorise P(x). 2

b) Using the substitution $u = x - 2$, or otherwise, find $\int_{e+2}^{3} \frac{x}{x-2}\,dx$ 4

c) Find $\int \frac{1}{\sqrt{1-4x^2}}\,dx$ 1

d)



ABCD is a cyclic quadrilateral. CD is produced to E.

P is a point on the circle such that $\angle ABP = \angle PBC$.

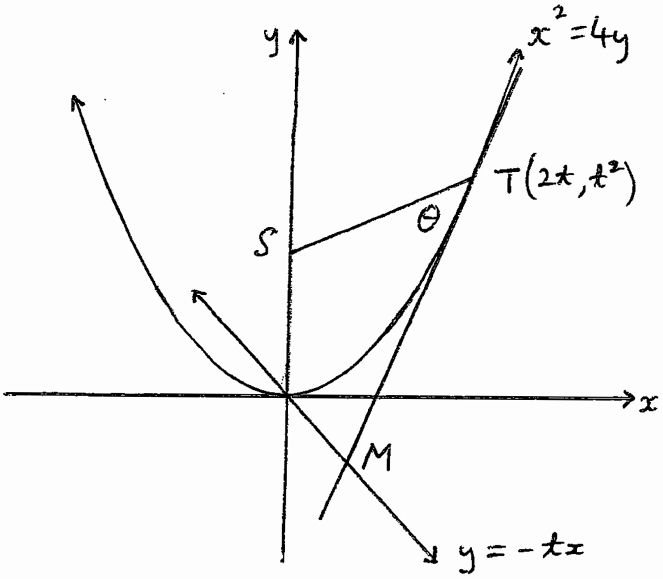
i) Copy the above diagram into your answer booklet.

ii) Explain why $\angle ABP = \angle ADP$. 1

iii) Prove that PD bisects $\angle ADE$. 2

Question 3 (12 marks) START A NEW PAGE.

a)



The parabola $x^2 = 4y$, with focus S, has a tangent at T $(2t, t^2)$.

The acute angle θ is the angle between ST and the tangent at T.

i) Show that the tangent at T has gradient t. 1

ii) Find $\tan\theta$, in simplest form, in terms of t. 3

iii) The tangent at T intersects the line $y = -tx$ at M. Find the locus of M as T moves on the parabola. 2

b) Differentiate $\sin^{-1}(tan^2 x)$ with respect to x. 2

c) Simplify $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta}$ 2

d) The polynomial $P(x) = x^3 + 2x^2 + x - 2$ has roots α, β, γ. 2

Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

Question 4 (12 marks) START A NEW PAGE.

a) Consider the function $f(x) = x - 2e^{-x} + 1$ for $x \geq 0$.

i) Show that the function is increasing and that its graph is concave down. 2

ii) Examine $f(x)$ as $x \to \infty$ and sketch the graph of $y = f(x)$. Clearly show the equations of any asymptotes. 2

iii) On the same diagram, sketch the graph of the inverse function $y = f^{-1}(x)$. 1

iv) Find the x value of the point of intersection of the two curves. 1

b) A hot metal bar cools when placed into a room of constant temperature according to the rule $\frac{dT}{dt} = \cdot k(T - T_{0)}$ where T is the temperature of the bar after time t and T_0 is room temperature.

i) Verify that $T = T_0 + Ae^{\ kt}$, where A and k are constants, is a solution to the above differential equation. 1

ii) A metal bar with temperature 175°C is placed in a room at temperature 25°C and cools to 75°C after 10 minutes. 3

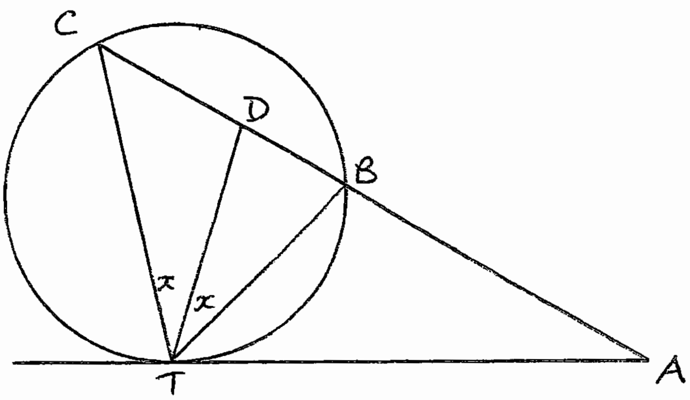
How long will it take for the metal bar to cool to 40°C. (Give your answer correct to the nearest minute).

c) Find the value of $\sin\left[\cos^{-1}\left(\frac{-1}{3}\right)\right]$ in exact form. 2

Question 5 (12 marks) START A NEW PAGE.

a) Solve $cos^2x - \sin 2x = 0$, for $0 \leq x \leq 2\pi$, correct to 2 decimal places where applicable. 3

b)



TA is a tangent to the circle and CDBA is a straight line.

TD bisects $\angle BTC$ as indicated.

i) Copy the diagram into your answer booklet.

ii) Prove that $AT = AD$. 2

c) A particle is moving in a straight line. At time t seconds it has displacement x cm from a fixed point O, with velocity v cm/s and acceleration $a = 2x^3 + 2x$ cm/s^2.

Initially, the particle is 1 cm to the right of O with a velocity of 2 cm/s.

i) Prove the result $\frac{dv}{dt} = \frac{d}{dx}(\frac{1}{2}v^2)$ 1

ii) Show that $v = x^2 + 1$. Justify your answer. 3

iii) Find the expression for the displacement x in terms of time t. 2

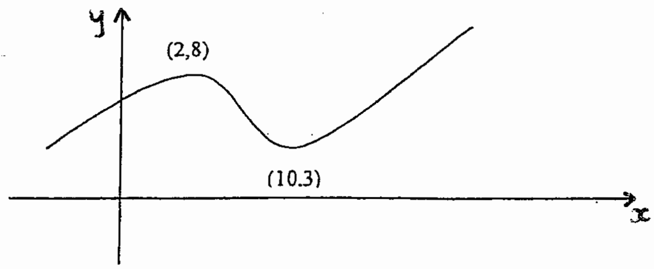
iv) Find the time taken for the particle to reach $x = 3$. 1

Question 6 (12 marks) START A NEW PAGE.

a) i) Sketch the graph of $y = 2\cos^{-1} 2x$, clearly showing the coordinates of endpoints. 2

ii) The region bounded by the above curve and the coordinate axes is rotated about the y axis. Find the exact value of the solid of revolution formed. 4

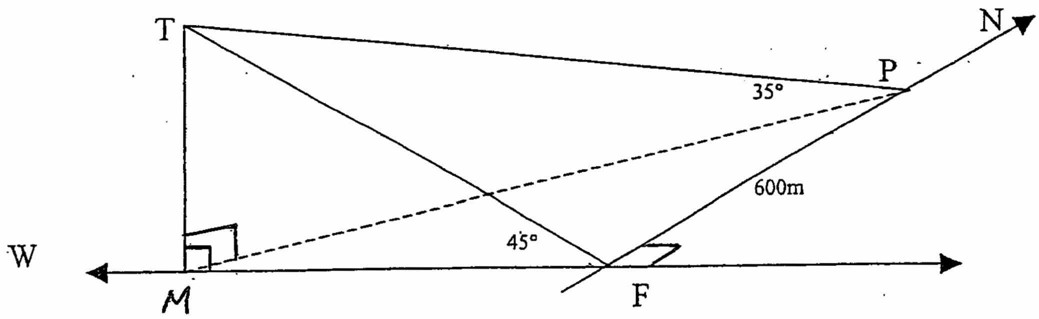
b) A cubic function $y = f(x)$ is sketched below. There are stationary points at (2,8) and (10,3).



i) If $g(x) = f'(x)$, sketch $y = g(x)$. 1

ii) Hence find the area of the region bounded by $y = g(x)$ and the x axis. (Do not find the equation of the function $y = f(x)$). 2

c) At the Mount Snow Gum ski fields there are two chairlifts that meet at Top Station (T). The first chairlift starts in the valley below from Wombat Flat (F) and runs due west as it rises at an angle of 45°. The second lift starts from Pot Hole Creek (P), 600m horizontally due north of Wombat Flat in the same valley. It rises at an angle of 35° to Top Station.



How high is Top Station above the valley below? (Give your answer correct to the nearest metre) 3

Question 7 (12 marks) START A NEW PAGE.

a) Use mathematical induction to prove that $5^n > 3^n + 4^n$ for all positive integers $n \geq 3$. 3

b) A spherical model of the Earth is being inflated so that its volume is increasing at a constant rate of 25 cm^3/s. Find the rate at which the surface area is increasing when the radius is 10 cm. 3

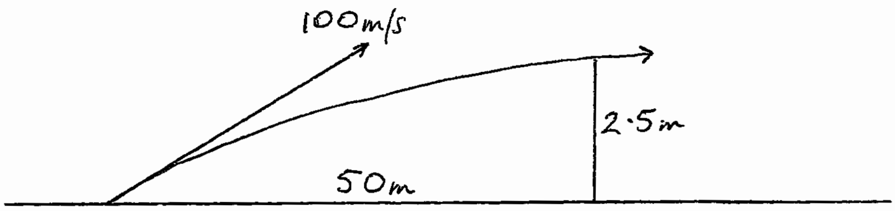
$$\left(V = \frac{4}{3}\pi r^3 \ , \ SA = 4\pi r^2 \right)$$

c) An object is projected at an initial velocity $v\ m/s$, from ground level at an angle θ to the horizontal. You may take the horizontal and vertical components of the position of the object as $x = vt\cos\theta$ and $y = -5t^2 + vt\sin\theta$.

i) Find an expression for the maximum height attained by the object. Simplify your answer. 2

ii) Derive an expression for the Cartesian equation for the motion, i.e. y in terms of x. 1

iii)



The object is projected with an initial velocity of 100 m/s and just clears a 2.5 m high wall at a distance 50 m from the point of projection. The base of the wall is at the same level as the point of projection.

Calculate the angle(s) of projection, correct to the nearest minute. 3

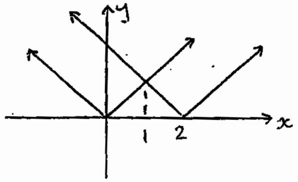
END OF PAPER.

Solutions

1 a) $y' = 2x\cos^{-1}x - \frac{x^2}{\sqrt{1-x^2}}$

b) $2\log_a y - \log_a x = 8\cdot2 - 2\cdot8$
$= \underline{\underline{5\cdot4}}$

c)



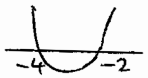
$\underline{\underline{x>1}}$

d) $\frac{x}{\cancel{x+2}} \times (x+2)^{\cancel{2}} > 2x(x+2)^2$

$x^2+2x > 2x^2+8x+8$

$x^2+6x+8<0$

$(x+4)(x+2)<0$



$\underline{\underline{-4<x<-2}}$

e) $(3,-1) \Rightarrow x = \frac{-1\times-1+3\times3}{2}$

m n

$= \frac{10}{2}$

$= 5$

$y = \frac{-1\times2+3\times5}{2}$

$= \frac{13}{2}$

$= 6\frac12$ ∴ $\underline{\underline{P \text{ is } (5, 6\frac12)}}$

f) i) LHS $= \sin^2A - 2\sin A\cos A + \cos^2A$

$= \sin^2A + \cos^2A - 2\sin A\cos A$

$= 1 - 2\sin A\cos A$

$= 1 - \sin 2A$

$=$ RHS

ii) $(\sin15-\cos15)^2 = 1-\sin(2\times15)$

∴ $\sin15-\cos15 = \pm\sqrt{1-\sin30}$

(take -ve root only as $\sin15<\cos15$

∴ $\sin15-\cos15 = -\sqrt{1-\frac12}$

$= -\sqrt{\frac12}$

$= \frac{-1}{\sqrt2}$

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2 a) i) $P(-2)=0$

$-8+4a+2a+2=0$

$6a=6$

$\underline{\underline{a=1}}$

ii)

$$\begin{array}{r l}
 & x^2 - x + 1 \\
x+2 \,\big)\!\!\! & \overline{x^3+x^2-x+2} \\
 & \underline{x^3+2x^2} \\
 & -x^2-x \\
 & \underline{-x^2-2x} \\
 & x+2
\end{array}$$

∴ $\underline{\underline{P(x) = (x+2)(x^2-x+1)}}$

b) $\int_{e+2}^{3} \frac{x}{x-2}\,dx = \int_e^1 \frac{u+2}{u}\,du$

$u = x-2$
$\frac{du}{dx} = 1$
$du = dx$
$x = e+2,\ u = e$
$x = 3,\ u = 1$

$= \int_e^1 \left(1+\frac2u\right)du$

$= \left[u + 2\log u\right]_e^1$

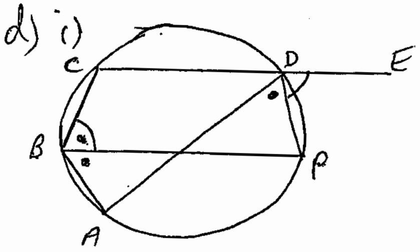
$= (1+0)-(e+2)$

$= \underline{\underline{-e-1}}$

c) $\int \frac{1}{\sqrt{4(\frac14-x^2)}}\,dx$

$= \underline{\underline{\frac12\sin^{-1}2x + c}}$

d) i)



ii) common arc AP makes equal angles at the circumference.

iii) $\angle PDE = \angle CBP$ (ext. angle of cyclic quad. CDPB)
and $\angle ADP = \angle ABP$ (from ii)
and since $\angle ABP = \angle PBC$ (given in question)
then $\angle PDE = \angle ADP$ as reqd.

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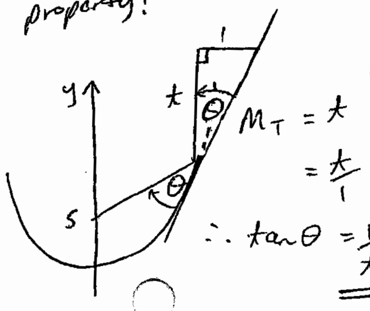
3 a) i) $y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{2x}{4}$

$= \frac{x}{2}$

When $x = 2t$, $M_T = \frac{2t}{2}$
$= t$

ii) (Ideally)
using the reflective property!



$M_T = t$

$= \frac{t}{1}$

∴ $\tan\theta = \underline{\underline{\frac1t}}$

or

(alternatively) $a=1 \Longrightarrow S$ is $(0,1)$

∴ $M_{ST} = \frac{t^2-1}{2t}$ and $M_T = t$

Using $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\frac{\frac{t^2-1}{2t} - t}{1+\frac{t^2-1}{2t}\times t}\right|$

$= \left|\frac{\frac{t^2-1-2t^2}{2t}}{\frac{2+t^2-1}{2}}\right|$

$= \left|\frac{\cancel{-t^2-1}}{\cancel{2}t}\times\frac{\cancel{2}}{(-1)\cancel{1+t^2}}\right|$

$= \left|\frac{-1}{t}\right|$

$= \underline{\underline{\frac1t}}$

iii)

For M, solve $y = tx - at^2$, $y = -tx$

$\therefore -tx = tx - at^2$

$-2tx = -at^2$

$2x = t$

$x = t/2 \Rightarrow y = -\frac{t^2}{2}$

$\therefore t = 2x \Rightarrow y = -\frac{4x^2}{2}$

$\therefore \underline{\underline{y = -2x^2}}$

b) $y' = \frac{1}{\sqrt{1-(\tan^2 x)^2}} \times 2\tan x \sec^2 x$

$= \frac{2\tan x \sec^2 x}{\sqrt{1-\tan^4 x}}$

c)

$= \frac{\sin 3\theta \cos\theta - \cos 3\theta \sin\theta}{\sin\theta\cos\theta}$

$= \frac{\sin(3\theta - \theta)}{\sin\theta\cos\theta}$

$= \frac{2\cancel{\sin\theta\cos\theta}}{\cancel{\sin\theta\cos\theta}}$

$= \underline{\underline{2}}$

d) $(\alpha+\beta+\gamma)^2 = \alpha^2 + \alpha\beta + \alpha\gamma + \beta^2 + \beta\alpha + \beta\gamma + \gamma\alpha + \gamma\beta + \gamma^2$

$\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= (-2)^2 - 2(1)$

$= \underline{2}$

4) a) i) $f'(x) = 1 + 2e^{-x}$ which is pos. for $x \geq 0$

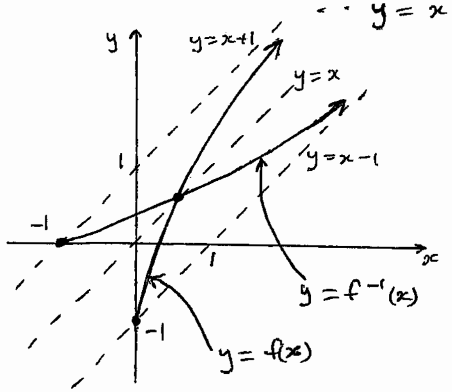
$\therefore$ fn is increasing.

$f''(x) = -2e^{-x}$ which is neg. for $x \geq 0$

$\therefore$ graph is concave down

ii) $x = 0, y = -1$. As $x \to +\infty$, $y \to \infty - \frac{2}{\infty} + 1$

$\therefore y = x+1$ is oblique asymptote.



iii) see graph

iv) Solve $\cancel{x} - 2e^{-x} + 1 = \cancel{x}$

$2e^{-x} = 1$

$e^{-x} = \frac{1}{2}$

$-x = \log\frac{1}{2}$

$\therefore \underline{\underline{x = \log 2}}$

b) ii) $T = 25 + Ae^{kt}$

when $t = 0$, $175 = 25 + Ae^0$

$\therefore A = 150$

$\therefore \underline{T = 25 + 150e^{kt}}$

i) If $T = T_0 + Ae^{kt}$

$\frac{dT}{dt} = kAe^{kt}$

$= k(T_0 + Ae^{kt} - T_0)$

$= k(T - T_0)$ as req'd

$75 = 25 + 150e^{10k}$

$\frac{50}{150} = e^{10k}$

$\ln\frac{1}{3} = 10k$

$\therefore \underline{k = \frac{1}{10}\ln\frac{1}{3}}$

$40 = 25 + 150e^{\frac{1}{10}\ln\frac{1}{3}t}$

$\frac{15}{150} = e^{\frac{1}{10}\ln\frac{1}{3}t}$

$\ln\frac{1}{10} = \frac{1}{10}\ln\frac{1}{3}t$

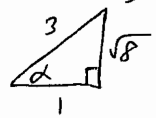
$\therefore t = \frac{\ln\frac{1}{10}}{\frac{1}{10}\ln\frac{1}{3}}$

$\doteqdot 21$ minutes.

c) $\sin\left[\pi - \cos^{-1}\left(\frac{1}{3}\right)\right] = \sin(\pi - \alpha)$ $\qquad \alpha = \cos^{-1}\left(\frac{1}{3}\right)$

$= \sin\alpha$ $\qquad \cos\alpha = \frac{1}{3}$

$= \frac{\sqrt{8}}{3}$



5 a) $\cos^2 x - 2\sin x\cos x = 0$

$\cos x(\cos x - 2\sin x) = 0$

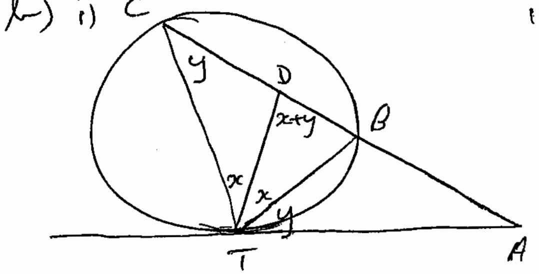
$\cos x = 0$ or $2\sin x = \cos x$

$\tan x = \frac{1}{2}$

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}, 0.464, \pi + 0.464$

$= \frac{\pi}{2}, \frac{3\pi}{2}, 0.46, 3.61$

b) i)



ii) $\angle BTA = \angle BCT\ (= y)$ (angle in alternate segment)

$\angle BDT = x + y$ (ext. angle of $\triangle CDT$)

$\therefore \angle BDT = \angle DTA\ (= x+y)$

$\therefore AD = AT$ (equal sides opposite equal angles in $\triangle DAT$)

c)

i)

$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

$= \frac{dv}{dx} \cdot v$

$= \frac{dv}{dx} \cdot \frac{d}{dv}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 2x$

$\frac{1}{2}v^2 = \frac{2x^4}{4} + x^2 + c$

$v^2 = x^4 + 2x^2 + k$

$(x=1, v=2)$ $4 = 1 + 2 + k$ $(k = 1)$

$v^2 = x^4 + 2x^2 + 1$

$\therefore v = \pm\sqrt{(x^2+1)^2}$

$= +(x^2+1)$ since satisfies $x = 1, v = 2$

iii) $v = \frac{dx}{dt} = x^2 + 1$

$\frac{dt}{dx} = \frac{1}{x^2+1}$

$\therefore t = \tan^{-1} x + c$

$(t=0, x=1)$ $0 = \tan^{-1} 1 + c$ $(c = -\frac{\pi}{4})$

$\therefore t = \tan^{-1} x - \frac{\pi}{4}$

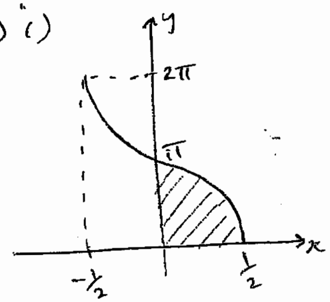
$\therefore t + \frac{\pi}{4} = \tan^{-1} x$

$\therefore x = \tan\left(t + \frac{\pi}{4}\right)$

iv) $t = \tan^{-1} 3 - \frac{\pi}{4}$

$\doteqdot 0.46$ secs.

6 a) i)



ii) $V = \pi\int_0^{\pi}\left(\frac{1}{2}\cos\frac{1}{2}y\right)^2 dy$

$= \frac{\pi}{4}\int_0^{\pi}\cos^2\frac{1}{2}y\, dy$

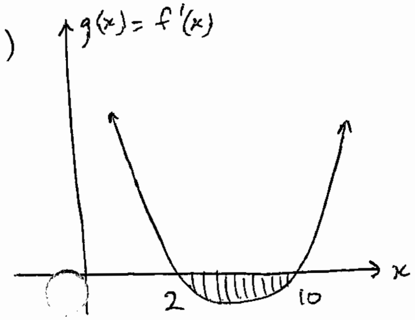
$= \frac{\pi}{4} \times \frac{1}{2}\int_0^{\pi}(1 + \cos y)\, dy$

$= \frac{\pi}{8}\left[y + \sin y\right]_0^{\pi}$

$= \frac{\pi}{8}(\pi + 0 - 0 - 0)$

$= \frac{\pi^2}{8}\ u^3.$

b) i)



ii) $A = \left|\int_2^{10} g(x)\, dx\right|$

$= \left|\left[f(x)\right]_2^{10}\right|$

$= |3 - 8|$

$= |-5|$

$= 5$

c) MT = MF $\Rightarrow$ find MF (use ΔMFP)

$$\tan 35 = \frac{MT}{MP}$$

$$\therefore MP = \frac{MT}{\tan 35}$$

Now $MP^2 = MF^2 + 600^2$

$$\frac{MF^2}{\tan^2 35} = MF^2 + 600^2$$

$$\therefore MF^2 = MF^2 \tan^2 35 + 600^2 \tan^2 35$$

$$\therefore MF^2 - MF^2 \tan^2 35 = 600 \tan^2 35$$

$$\therefore MF^2(1-\tan^2 35) = 600^2 \tan^2 35$$

$$\therefore MF = \sqrt{\frac{600^2 \tan^2 35}{1-\tan^2 35}}$$

$\therefore$ tower is approx. 588 m high.

7

a) Test $n=3$, $5^3 > 3^3+4^3$

$125 > 27+64$ ✓

$\therefore$ result is true for $n=3$

Assume true for $n=k$, ie. assume $5^k > 3^k+4^k$

Prove true for $n=k+1$, ie. prove that $5^{k+1} > 3^{k+1}+4^{k+1}$

Now, $5^{k+1} = 5 \times 5^k$

$> 5(3^k+4^k)$ from assumption

$= 5\times 3^k + 5\times 4^k$

$> 3\times 3^k + 4\times 4^k$

$= 3^{k+1} + 4^{k+1}$ (proven)

Since true for $n=3$, then from above it must be true for $n=3+1=4$ and so on for all $n \geq 3$

b) $\frac{dV}{dt} = 25$, $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$

$\frac{dV}{dr} = 4\pi r^2$, $\frac{dA}{dr} = 8\pi r$

Now, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$

$= 8\pi r \times \frac{1}{4\pi r^2} \times 25$

$= \frac{2}{r} \times 25$

When $r=10$,

$\frac{dA}{dt} = 5\ \text{cm}^2/\text{s}$.

c) i) Max y when $y' = -10t + v\sin\theta = 0 \Rightarrow t = \frac{v\sin\theta}{10}$

$\therefore y_{max} = \frac{-5v^2\sin^2\theta}{100} + \frac{v^2\sin^2\theta}{10}$

$= \frac{5v^2\sin^2\theta}{100}$

$= \frac{v^2\sin^2\theta}{20}$

ii) $t = \frac{x}{v\cos\theta} \Rightarrow y = \frac{-5x^2}{v^2\cos^2\theta} + \frac{\cancel{v}\, x\sin\theta}{\cancel{v}\cos\theta}$

$\therefore y = \frac{-5x^2}{v^2}\sec^2\theta + x\tan\theta$ (or equivalent)

iii) $v=100$, $(x,y) = (50, 2.5)$

$\therefore 2.5 = \frac{-5\times 50^2}{10000}(1+\tan^2\theta) + 50\tan\theta$

$25000 = -12500(1+\tan^2\theta) + 500000\tan\theta$

$2 = -(1+\tan^2\theta) + 40\tan\theta$

$\therefore \tan^2\theta - 40\tan\theta + 3 = 0$

$\therefore \tan\theta = \frac{40 \pm \sqrt{1600-12}}{2}$

$= 39.9$ or 0.075

$\therefore \theta = 88°34'$ or $4°18'$