SYDNEY TECHNICAL HIGH SCHOOL

2012
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time - 5 minutes
- Working Time - 2 hours
- Write using a blue or black pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11 - 14
- Begin each question on a new page.
- Write your name and your teacher's name on the booklet and your Multiple Choice answer sheet.

Total marks (70)

Section I

10 marks

- Attempt Questions 1-10
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section

Section II

60 marks

- Attempt questions 11 – 14
- Answer in the booklet provided and show all necessary working.
- Start a new page for each question and clearly label it.
- Allow about 1 hour 45 minutes for this section

Section I

Total marks (10)

Attempt Questions 1-10

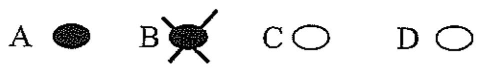
Allow about 15 minutes for this section

Use the multiple choice answer sheet.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

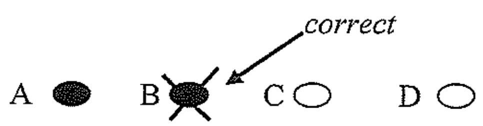
Sample

$2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.



If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:



1. Find the value of a such that $P(x) = x^3 - 2x^2 - ax + 6$ is divisible by $x + 2$.

(A) -5 (B) -3 (C) 3 (D) 5

2. Find the acute angle (to the nearest degree) between the lines $x - y = 2$ and $2x + y = 1$.

(A) 18° (B) 27° (C) 45° (D) 72°

3. Find $\int \frac{dx}{1 + 4x^2}$

(A) $\frac{1}{2}\tan^{-1}2x + c$

(B) $2\tan^{-1}2x + c$

(C) $2\tan^{-1}\frac{x}{2} + c$

(D) $\frac{1}{2}\tan^{-1}\frac{x}{2} + c$

4. Identify the derivative of $x^2 \cos^{-1} x$.

(A) $\frac{x^2}{\sqrt{1-x^2}} - 2x\cos^{-1}x$

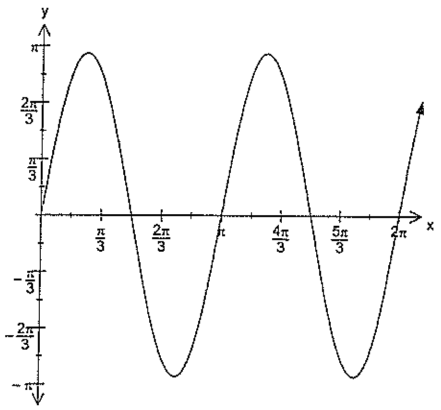
(B) $-\frac{x^2}{\sqrt{1-x^2}} - 2x\cos^{-1}x$

(C) $2x\cos^{-1}x - \frac{x^2}{\sqrt{1-x^2}}$

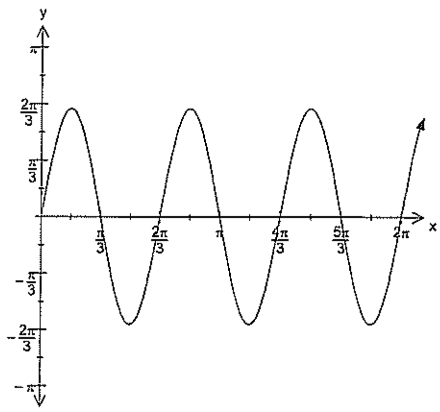
(D) $2x\cos^{-1}x + \frac{x^2}{\sqrt{1-x^2}}$

5. Which graph represents the curve $y = 2\sin 3x$?

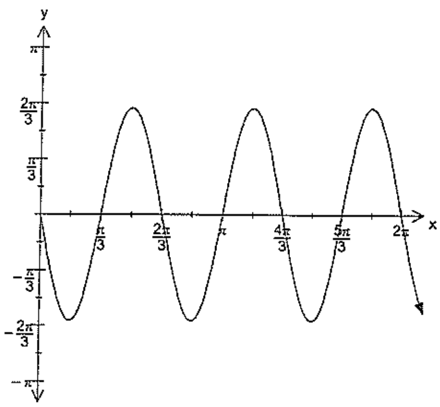
(A)



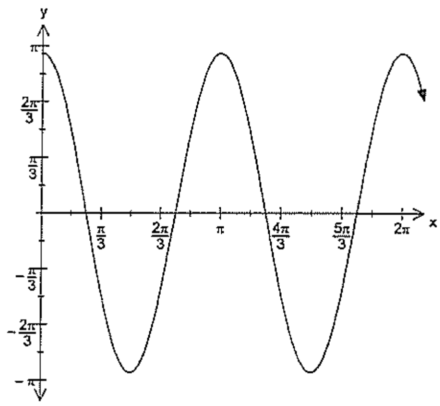
(B)



(C)



(D)



6. If $f(x) = \dfrac{2}{x+1}$, what is $f^{-1}(x)$?

(A) $y = \dfrac{x+1}{2}$

(B) $y = \dfrac{2-x}{x}$

(C) $y = \dfrac{2-x}{2}$

(D) $y = \dfrac{2+x}{x}$

7. Given that $log_a 2 = x$, find an expression for a^{3x}.

(A) 8

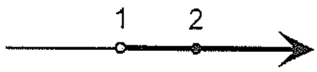
(B) x^{6x}

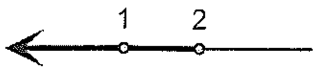
(C) 2^{3x^2}

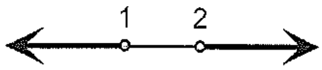
(D) a^{3a^2}

8. For what values of x is $\frac{x+4}{x-1} < 6$?

(A) 

(B) 

(C) 

(D) 

9. Evaluate $\lim_{x \to 0} \frac{5x \cos 2x}{\sin x}$

(A) -10

(B) -5

(C) 5

(D) 10

10. Identify the domain and range of $f(x) = \sin^{-1} 2x$.

(A) Domain $\rightarrow \{x: -\frac{1}{2} \leq x \leq \frac{1}{2}\}$ and Range $\rightarrow \{y: -\frac{\pi}{4} \leq \sin^{-1}2x \leq \frac{\pi}{4}\}$

(B) Domain $\rightarrow \{x: -\frac{1}{2} \leq x \leq \frac{1}{2}\}$ and Range $\rightarrow \{y: -\frac{\pi}{2} \leq \sin^{-1}2x \leq \frac{\pi}{2}\}$

(C) Domain $\rightarrow \{x: -2 \leq x \leq 2\}$ and Range $\rightarrow \{y: -\frac{\pi}{2} \leq \sin^{-1}2x \leq \frac{\pi}{2}\}$

(D) Domain $\rightarrow \{x: -2 \leq x \leq 2\}$ and Range $\rightarrow \{y: -\pi \leq \sin^{-1}2x \leq \pi\}$

End of Section 1

Section II

Total marks (60)
Attempt Questions 11 - 14
Allow about 1 hour 45 minutes for this section.

Answer all questions, starting each question on a new sheet of paper with your name and question number at the top of the page. Do not write on the back of sheets.

Question 11 **(15 Marks)** Use a Separate Sheet of paper **Marks**

a) Find $\int e^{\frac{x}{4}} dx$ **1**

b) Find the exact value of $\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{2dx}{\sqrt{9-x^2}}$ **3**

c) Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 x\, e^{\tan x}\, dx$ using the substitution $u = \tan x$. **3**

d) Find the largest possible domain of $y = \ln(\sin^{-1}x)$. **2**

e) Solve for x: $\left(x+\frac{1}{x}\right)^2 - 5\left(x+\frac{1}{x}\right) + 6 = 0$ **2**

f) AB is the diameter and AC a chord of a circle.
The bisector of $\angle BAC$ cuts the circle at D.

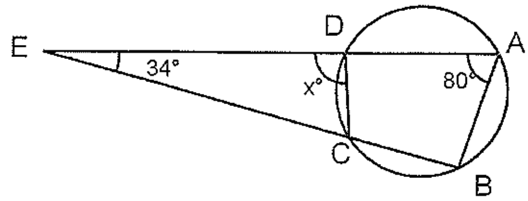
(i) Construct a diagram showing all of this information. **1**

(ii) The tangent at D meets AC produced at E. Prove that the tangent is perpendicular to AE. **3**

End of Question 11

Question 12 (15 Marks) Use a separate sheet of paper **Marks**

a) Find the value of x, giving reasons for your answer. **2**



(Not to scale)

b) i) Without using calculus, sketch $y = (x-1)(x^2-4)$ **2**

ii) Hence, solve the inequality $(x-1)(x^2-4) < 0$ **1**

c) A spherical balloon leaks air such that the radius decreases at a rate of $0.5\ cms^{-1}$. Calculate the rate of change of the volume of the balloon when the radius is $10\ cm$. **3**

d) Find the exact value of $\int_0^1 \frac{x\,dx}{1+x^2}$ **2**

e) Evaluate $\lim_{x \to \infty} \frac{x^2-2}{3x^2-x+1}$ **1**

f) Prove by mathematical induction that (for n a positive integer) **4**

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

End of Question 12

Question 13 (15 Marks) Use a Separate Sheet of paper **Marks**

a) A particle moves such that its displacement x cm from the origin, O after time t seconds is given by:

$$x = \sqrt{3}\cos 3t - \sin 3t$$

(i) Show that the particle moves in Simple Harmonic Motion. **2**

(ii) Evaluate the period of motion. **1**

(iii) Find the time when the particle first passes through the origin. **2**

(iv) Find the velocity when the particle is 1 cm from the origin on its first oscillation. **2**

b) (i) Prove $\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ **1**

(ii) Prove $\dfrac{d}{dx}(x\ln x) = 1 + \ln x$ **1**

(iii) The acceleration of a particle moving in a straight line and starting from rest at 1 cm on the positive side of the origin is given by:

$$\frac{d^2x}{dt^2} = 1 + \ln x$$

(α) Derive the equation relating v and x. **2**

(β) Hence, evaluate v when $x = e^2$. **1**

c) The region bounded by $y = \ln x$, $x = 2$, $x = 5$ and the x-axis is rotated about the x-axis.

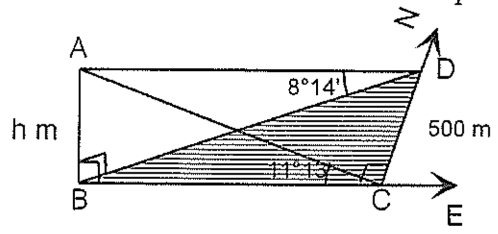
(i) Write an integral expression for the volume formed in terms of y and dx. Do not evaluate this integral. **1**

(ii) Use the trapezoidal rule with four function values (3 strips) to find an approximation to this volume (2 decimal places). **2**

End of Question 13

Question 14 (15 Marks) Use a Separate Sheet of paper **Marks**

a) A is the top of a vertical mast AB standing on level ground. Two points C and D are on horizontal ground such that C is due East of B and D is $500\ m$ due North of C. The angles of elevation of A from C and D respectively are $11°\ 13'$ and $8°\ 14'$. **4**



Calculate the height, h of the tower to the nearest metre.

b) The polynomial equation $8x^3 - 36x^2 + 22x + 21 = 0$ has roots which form an arithmetic progression. Find the roots. **3**

c) For the arithmetic sequence $\log_{10}(x-2)$, $\log_{10}(x-2)^2$, $\log_{10}(x-2)^3$, …, show that the sum of n terms is $\frac{n}{2}\log_{10}(x-2)^{n+1}$. **3**

d) One hundred grams of sugar cane in water are being converted into dextrose at a rate which is proportional to the amount at any time. That is, if M *grams* are converted in t *minutes*, then $\frac{dM}{dt} = k(100 - M)$ where k is a constant.

(i) Show that $M = 100 + Ae^{-kt}$, where A is a constant, satisfies the differential equation. **2**

(ii) Find A, given that when $t = 0$, $M = 0$. **1**

(iii) If *40 grams* are converted in the first *10 minutes*, find how many grams are converted in the first *30 minutes*. **2**

End of Examination

STHS Trial HSC Examination – Mathematics Extension 1 2012

Multiple Choice Answer Sheet

Name ______________________________

Completely fill the response oval representing the most correct answer.

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

2012 - Extension 1 Trial HSC SOLUTIONS

Multiple Choice

(1) D	(3) A	(5) B	(7) A	(9) C
(2) D	(4) C	(6) B	(8) D	(10) B

Q11

a) $\int e^{\frac{x}{4}}\,dx = 4e^{\frac{x}{4}} + C$

b) $\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{2\,dx}{\sqrt{9-x^2}} = \left[2\sin^{-1}\frac{x}{3}\right]_{\frac{3\sqrt{3}}{2}}^{3}$

$= \frac{\pi}{3}$

c) $\int_0^{\frac{\pi}{4}} \sec^2 x\, e^{\tan x}\,dx =$ $u = \tan x$

$\therefore \frac{du}{dx} = \sec^2 x$

$\therefore du = \sec^2 x\,dx$

$= \int_0^1 e^u\,du$

$= e - 1$

d) Domain of $\sin^{-1}x \Rightarrow -1 \le x \le 1$

Domain of $\ln x \Rightarrow x > 0$

$\therefore$ Domain of $\ln(\sin^{-1}x) \Rightarrow 0 < x \le 1$

e) $\left(x+\frac{1}{x}\right)^2 - 5\left(x+\frac{1}{x}\right) + 6 = 0$

let $u = x + \frac{1}{x}$

$\therefore u^2 - 5u + 6 = 0$

$\therefore (u-2)(u-3) = 0$

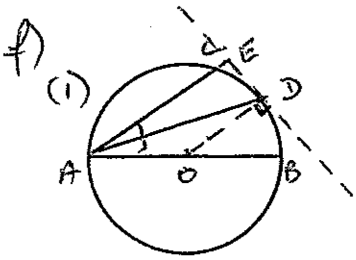
$\therefore x + \frac{1}{x} = 2$ or $x + \frac{1}{x} = 3$

$\therefore x^2 - 2x + 1 = 0$ $\quad$ $x^2 - 3x + 1 = 0$

$\therefore (x-1)^2 = 0$ $\quad$ $\therefore x = \frac{3 \pm \sqrt{9-4}}{2}$

$\therefore x = 1$ $\quad$ $= \frac{3 \pm \sqrt{5}}{2}$

f) (i)



Dotted lines are constructions for part (ii)

(ii) Construct: OD (radius)
Extend AC to E on tangent.

Proof: ΔOAD is isosceles (OA and OD radii)

$O\hat{A}D = D\hat{A}C$ (data)

$O\hat{D}A = D\hat{A}O$ (base $\angle$s of isosceles Δ)

$\therefore O\hat{D}A = D\hat{A}C$

$\therefore AE \parallel OD$ (alternate $\angle$s)

$OD \perp DE$ (radius at point of contact $\perp$ tangent)

$\therefore AE \perp DE$ QED

(Not the only approach e.g. using alternate angle)

Q 12

a) $D\hat{C}B = 100°$ (opposite ∠s in cyclic quadrilateral are supplementary)

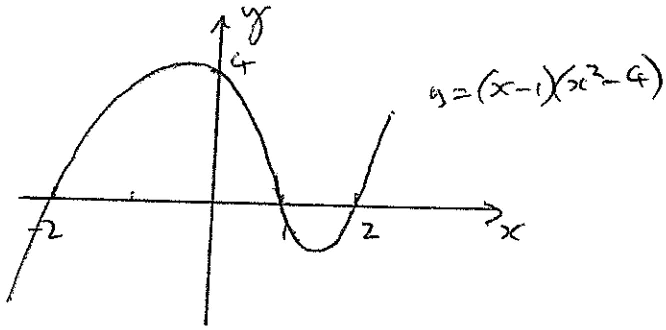
$\therefore x + 34 = 100$ (external ∠ equal to sum of opposite internal ∠s)

$\therefore x = 66°$

b) (i) $y = (x-1)(x^2-4)$

$= (x-1)(x-2)(x+2)$

leading coefficient $= 1 > 0$



(ii) $(x-1)(x^2-4) < 0$

$\therefore x < -2,\ 1 < x < 2$

c) $\frac{dr}{dt} = -0.5\ cm\,s^{-1}$

$V = \frac{4}{3}\pi r^3$

$\therefore \frac{dV}{dr} = 4\pi r^2$

Now $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$= 4\pi r^2 \times -0.5$

$= -2\pi r^2$

$= -200\pi\ cm^3 s^{-1}$

d) $\int_0^1 \frac{x\,dx}{1+x^2}$

$= \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$

$= \frac{1}{2}\left[\ln(1+x^2)\right]_0^1$

$= \frac{1}{2}(\ln 2 + \ln 1)$

$= \frac{\ln 2}{2}$

e) $\lim_{x\to\infty} \frac{x^2-2}{3x^2-x+1}$

$= \lim_{x\to\infty} \frac{1-\frac{2}{x^2}}{3-\frac{1}{x}+\frac{1}{x^2}}$

$= \frac{1}{3}$

f) RTS $\frac{1}{1.2}+\frac{1}{2.3}+\cdots\frac{1}{n(n+1)} = \frac{n}{n+1}$

For n=1,

LHS $= \frac{1}{1.2} = \frac{1}{2}$

RHS $= \frac{1}{1+1} = \frac{1}{2}$ = LHS

$\therefore$ true for n=1.

Assume true for n=k.

$\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{k(k+1)} = \frac{k}{k+1}$

For n=k+1,

We'd expect $\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

LHS $= \frac{1}{1.2}+\frac{1}{2.3}+\cdots\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)}$

$= \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{k+1}{k+2}$

= RHS

$\therefore$ If true for n=k, then true for n=k+1.

Now, True for n=k=1

$\therefore$ true for n=k+1=2

True for n=k=2

then true for n=k+1=3,..

$\therefore$ true for all integral values of n, n≥

Q13

a) (i) $x = \sqrt{3}\cos 3t - \sin 3t$

$\frac{dx}{dt} = -3\sqrt{3}\sin 3t - 3\cos 3t$

$\frac{d^2x}{dt^2} = -9\sqrt{3}\cos 3t + 9\sin 3t$

$= -9(\sqrt{3}\cos 3t - \sin 3t)$

$= -9x$

$\therefore$ SHM with $n^2 = 9$

(ii) Period of motion $T = \frac{2\pi}{n}$

$= \frac{2\pi}{3}$

b) (i) RTS $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\cdot\frac{dv}{dx}$

$= v\frac{dv}{dx}$

$= \frac{dx}{dt}\cdot\frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \frac{d^2x}{dt^2}$ QED

(ii) $\frac{d}{dx}(x\ln x)$

let $u = x$

$\frac{du}{dx} = 1$

$v = \ln x$

$\frac{dv}{dx} = \frac{1}{x}$

$= vu' + uv'$

$= \ln x \cdot 1 + x \cdot \frac{1}{x}$

$= \ln x + 1$

(iii) a) $\frac{d^2x}{dt^2} = 1 + \ln x$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1 + \ln x$

$\therefore \int \frac{d}{dx}\left(\frac{1}{2}v^2\right)dx = \int 1 + \ln x \, dx$

$\therefore \frac{1}{2}v^2 = \int \frac{d}{dx}(x\ln x)\,dx$

$= x\ln x + c$

$\therefore v^2 = 2x\ln x + c$

when $x = 1$, $v = 0$ $\therefore c = 0$

$\therefore v^2 = 2x\ln x$

(b) when $x = e^2$, $v^2 = 2e^2\ln e^2$
$= 4e^2$ $\therefore v = 2e$ cm s^{-1}

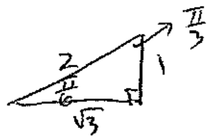
(iii) $\sqrt{3}\cos 3t - \sin 3t = 0$

$\therefore \sqrt{3} - \tan 3t = 0$

$\therefore \tan 3t = \sqrt{3}$

$\therefore 3t = \frac{\pi}{3}$

$\therefore t = \frac{\pi}{9}$ s



(iv) Let $\sqrt{3}\cos 3t - \sin 3t = R\cos(3t + \alpha)$

$= R(\cos 3t\cos\alpha - \sin 3t\sin\alpha)$

$\therefore R\cos\alpha = \sqrt{3}$

$R\sin\alpha = 1$

$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \frac{\pi}{6}$

Now, let $2\cos\left(3t + \frac{\pi}{6}\right) = 1$

$\therefore \cos\left(3t + \frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore 3t + \frac{\pi}{6} = \frac{\pi}{3}$

$\therefore 3t = \frac{\pi}{6}$

$\therefore t = \frac{\pi}{18}$ s

When $t = \frac{\pi}{18}$ s,

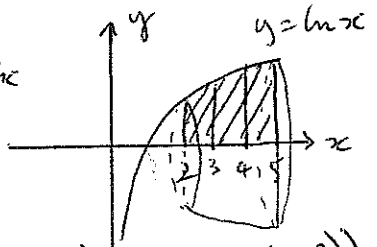
$V = -3\sqrt{3}\sin\frac{\pi}{6} - 3\cos\frac{\pi}{6}$

$= -3\sqrt{3}\cdot\frac{1}{2} - 3\frac{\sqrt{3}}{2}$

$= -3\sqrt{3}$ cm s^{-1}

e) (i) $V = \pi\int_2^5 (\ln x)^2 dx$

$= \pi\int_2^5 y^2 dx$



(ii) $V \simeq \pi\,\frac{1}{2}\left(((\ln 2)^2 + (\ln 5)^2) + 2((\ln 3)^2 + (\ln 4)^2)\right)$

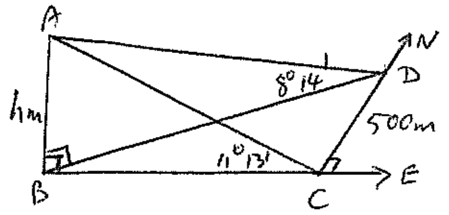
$= \frac{\pi}{2}\left[\ln^2 2 + \ln^2 5 + 2\ln^2 3 + 2\ln^2 4\right]$

$\simeq 14.6528\ldots$

$\simeq 14.65\ u^3$ to 2 dec. pl.

Q14

a)



$\frac{h}{BD} = \tan 8°14'$

$\therefore BD = \frac{h}{\tan 8°14'}$

Also $BC = \frac{h}{\tan 11°13'}$

Now $BD^2 - BC^2 = 500^2$

$\therefore \frac{h^2}{\tan^2 8°14'} - \frac{h^2}{\tan^2 11°13'} = 500^2$

$\therefore h^2\left(\frac{1}{\tan^2 8°14'} - \frac{1}{\tan^2 11°13'}\right) = 500^2$

$\therefore h^2 = \frac{500^2}{\left(\frac{1}{\tan^2 8°14'} - \frac{1}{\tan^2 11°13'}\right)}$

$\doteqdot 11193.7458$

$\therefore h = 105.8$

≈ 106 m

b) $8x^3 - 36x^2 + 22x + 21 = 0$

Let roots be $\alpha - d, \alpha, \alpha + d$

Now $\Sigma\alpha = 3\alpha = \frac{36}{8}$

$\therefore \alpha = \frac{3}{2}$

$\Pi\alpha = \alpha(\alpha^2 - d^2)$

$= \frac{3}{2}\left(\frac{9}{4} - d^2\right)$

$= \frac{-21}{8}$

$\therefore \frac{9}{4} - d^2 = \frac{-7}{4}$

$\therefore d^2 = 4$

$\therefore d = \pm 2$

$\therefore$ roots are $-\frac{1}{2}, 1\frac{1}{2}, 3\frac{1}{2}$

c) $a = \log_{10}(x-2)$

$d = \log_{10}(x-2)^2 - \log_{10}(x-2)$

$= \log_{10}(x-2)$

$\therefore S_n = \frac{n}{2}\left[2\log_{10}(x-2) + (n-1)\log_{10}(x-2)\right]$

$= \frac{n}{2}\left[(n+1)\log_{10}(x-2)\right]$

$= \frac{n}{2}\left[\log_{10}(x-2)^{n+1}\right]$ QED

d) (i) $M = 100 + Ae^{-kt}$

$\frac{dM}{dt} = -Ake^{-kT}$

$= k\left(100 - (100 + Ae^{-kt}\right)$

$\therefore \frac{dM}{dt} = k(100 - M).$

(ii) When $t = 0$, $M = 0$

$\therefore 0 = 100 + Ae^0$

$\therefore A = -100$

(iii) $M = 100 - 100e^{-kt}$

When $t = 10$, $M = 40$

$\therefore 40 = 100 - 100e^{-10k}$

$\therefore 100e^{-10k} = 60$

$\therefore e^{-10k} = 0.6$

$\therefore \ln e^{-10k} = \ln 0.6$

$\therefore -10k = \ln 0.6$

$\therefore k = \frac{\ln 0.6}{-10}$

$\therefore k \doteqdot 0.05108$

when $t = 30$,

$M = 100 - 100e^{-30 \times 0.05108}$

$= 78.4$ gm