SYDNEY TECHNICAL HIGH SCHOOL

2013

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics

Extension 1

General Instructions

- Reading time – 5 minutes.
- Working Time – 2 hours.
- Write using a blue or black pen.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 14.
- Begin each question on a new page.
- Write your name and your teachers name on the booklet and your multiple choice answer sheet.

Total marks (70)

Section I

10 marks

- Attempt questions 1 – 10.
- Answer on the multiple choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II

60 marks

- Attempt questions 11 – 14
- Answer in the booklet provided and show all necessary working.
- Start a new page for each question and clearly label it.
- Allow about 1 hour 45 minutes for this section.
- Marks are shown beside each question

Section 1 Total marks - 10

1. The smallest positive value of x for which $\tan(2x) = 1$ is

A. 0

B. $\frac{\pi}{8}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

2. The inverse of the function $f(x) = e^{2x+3}$ is

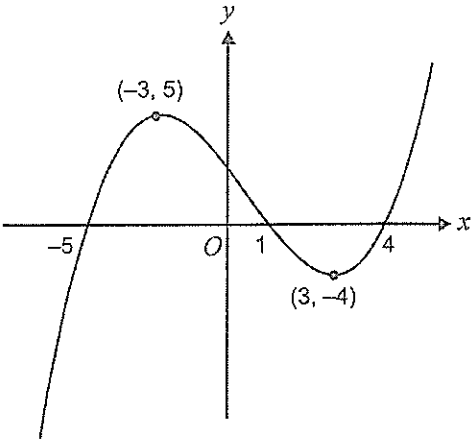
A. $f^{-1}(x) = e^{-2x-3}$

B. $f^{-1}(x) = e^{\frac{x-3}{2}}$

C. $f^{-1}(x) = log_e(\sqrt{x}) - \frac{3}{2}$

D. $f^{-1}(x) = -log_e(2x-3)$

3.



For the graph $y = f(x)$ shown above, $f'(x)$ is negative when

A. $-3 < x < 3$

B. $-3 \leq x \leq 3$

C. $x < -3 \text{ or } x > 3$

D. $x \leq -3 \text{ or } x \geq 3$

4. The solutions to the equation $e^{4x} - 5e^{2x} + 4 = 0$ are

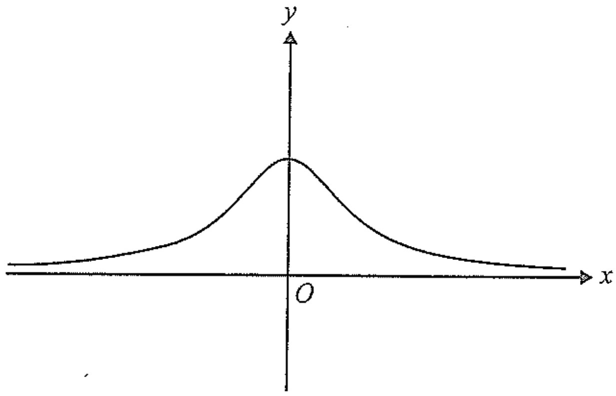
A. $1 \text{ and } 4$

B. $-4 \text{ and } -1$

C. $-\log_e 2, 0, \log_e 2$

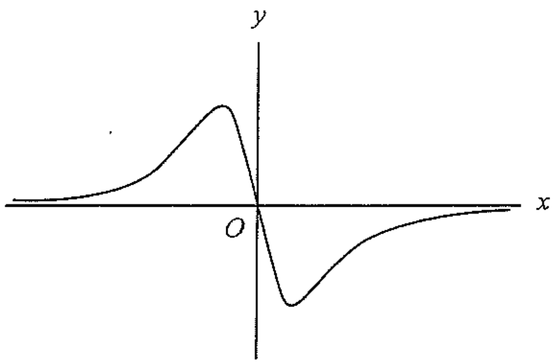
D. $0, \log_e 2$

5. The graph of a function f is shown below

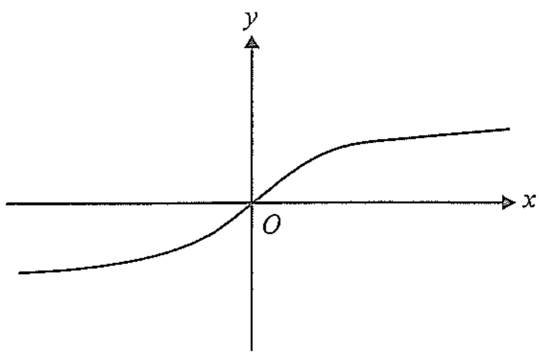


The graph of a primitive function of f could be

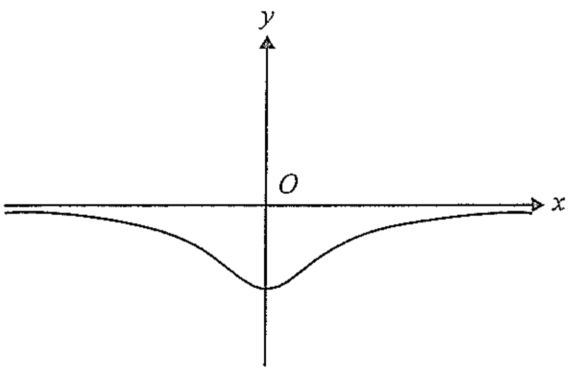
A.



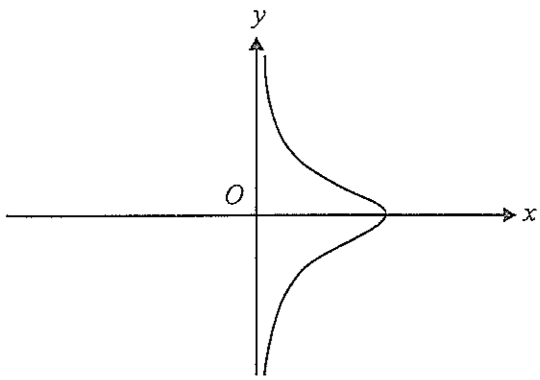
B.



C.



D.



6. The derivative of $log_e\ (2f(x))$ with respect to x is

A. $\frac{f'(x)}{f(x)}$

B. $2\ \frac{f'(x)}{f(x)}$

C. $\frac{f'(x)}{2f(x)}$

D. $log_e\ (2f'(x))$

7. The normal to the curve with equation $y = x^{\frac{3}{2}} + x$ at the point (4,12) is parallel to the straight line with equation

A. $4x = y$

B. $4y + x = 7$

C. $y = \frac{x}{4} + 1$

D. $x - 4y = -5$

8. The function with rule $f\ (x) = -3\sin\ (\frac{\pi x}{5})$ has period

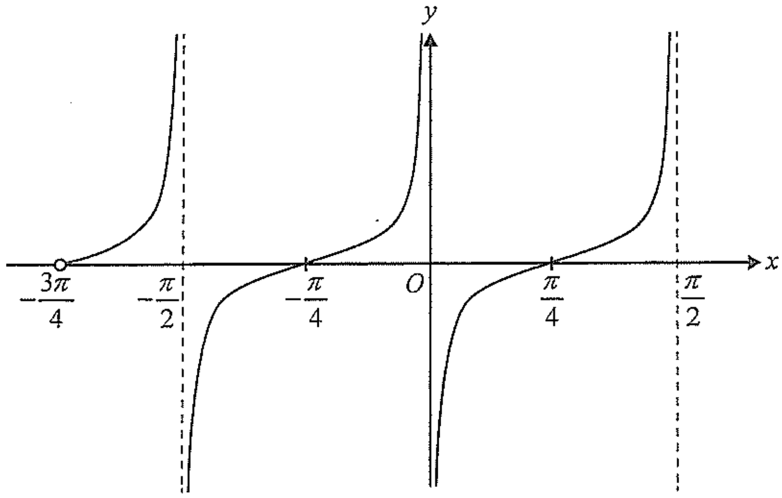
A. 3

B. 5

C. 10

D. $\frac{\pi}{5}$

9. A section of the graph of f is shown below:



The equation of f could be

A. $f(x) = \tan x$

B. $f(x) = \tan\left(x - \frac{\pi}{4}\right)$

C. $f(x) = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$

D. $f(x) = \tan\left[2\left(x - \frac{\pi}{2}\right)\right]$

10. The equation of the chord of contact of the tangents to the parabola $x^2 = 8y$ from the point (3,-2) is;

A. $3x - 4y + 8 = 0$

B. $3x - 8y + 16 = 0$

C. $3x - 8y - 8 = 0$

D. $3x - 4y + 16 = 0$

Section 2 Total marks - 60

Answer all questions starting each question on a new side of paper with your name and question number at the top of the page

Question 11 (15 marks)

A. Find the coordinates of the point P which divides the interval from A(-3,6) to B(12, -4) in the ratio of -2:3 **2**

B. Find the value $cos105°$ in simplest exact form with a rational denominator. **3**

C. Solve the inequality $\frac{2x+1}{x-1} \geq 3$ and graph your solution on a number line **3**

D. Find $\lim_{x \to 0} \frac{\sin 6x}{7x}$ **1**

E. Use the substitution u = t + 1 or otherwise to evaluate $\int_0^1 \frac{t}{\sqrt{t+1}} dt$ **3**
(Leave your answer in exact form)

F. Find the acute angle, to the nearest degree, between the lines **3**
$y = 3x + 1$ and $y = -x + 6$

Question 12 (15 marks) (Start a new page)

A. (i) Show that the equation of the tangent at $T\ (-2t,\ t^2\ )$ on the parabola $y = \frac{1}{4}x^2$ is given by $y + tx + \ t^2\ = 0$ — 2

(ii) The point $M\ (x, y)$ is the midpoint of the interval TA where A is the x intercept of the equation of the tangent at T. Find the equation of the locus of M as T moves on the parabola. — 3

B. Find $\int \frac{dx}{4 + x^2}$ — 1

C. Given $f\ (x) = sin^{-1} 2x$

(i) Write down the domain and range of $f(x)$ — 2

(ii) Sketch the curve — 1

D. A spherical balloon is expanding so that its volume $V\ mm^3$ increases at a constant rate of $72mm^3$/second. What is the rate of increase of its surface area $A\ mm^2$ when the radius is 12mm. — 3

E. Use mathematical induction to prove that $n^3\ +\ (n + 1)^3 +\ (n + 2)^3$ is divisible by 9 for all positive integers n — 3

Question 13 (15 marks) (Start a new page)

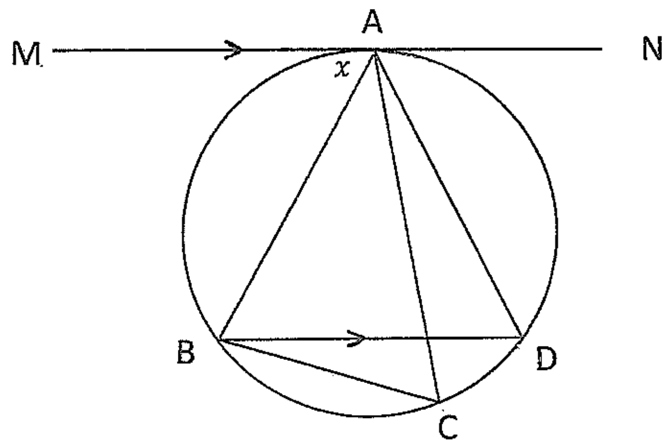
A. A particle moves in a straight line and at time t seconds, its distance xcm from a fixed point is given by $x = 1 + \frac{1}{2} \cos\ 2t$

(i) Show that the motion of the particle is simple harmonic by expressing $\ddot{x} = -n^2 (x - A)$ 1

(ii) State the period of its motion 1

(iii) Find the displacement of the particle from the origin when it is at rest, and determine its amplitude. 2

B.



ABC is a triangle inscribed in a circle. MAN is a tangent to the circle at A. BD is a chord of the circle such that BD||MN. Let $\angle$MAB= x 3
Copy diagram onto your answer sheet.
Show that CA bisects $\angle$BCD.

C. Newton's law of cooling states that the rate of change of the temperature T of a body at any time t is proportional to the difference in temperature T of the body and the temperature m of the surrounding medium ie: $\frac{dT}{dt} = k(T - m)$ where k is a constant.

(i) Show that T = m + Ae^{kt} where A is a constant, satisfies this equation — 1

(ii) If the temperature of the surrounding air is 40°C and the temperature of the body drops from 170°C to 105°C in 45 minutes, find the temperature of the body in another 90 minutes (nearest whole degree) — 3
[Find k correct to 3 decimal places]

(iii) Find the time taken for the temperature of the body to drop to 80°C (to the nearest minute) — 2

D. Find $\int cos^2 2x\, dx$ — 2

Question 14 (15 marks) (Start a new page)

A. (i) Prove $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ where v denotes velocity 2

(ii) The acceleration of a particle moving in a straight line is given by $\ddot{x} = -2e^{-x}$ where x is the displacement from the origin. Initially the particle is at the origin with velocity 2m/s.

α. Prove that $v^2 = 4e^{-x}$ 2

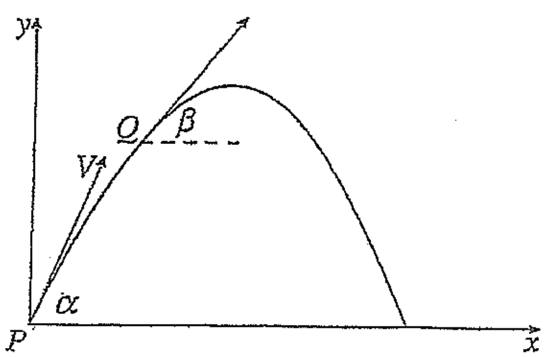
β. Describe the subsequent motion of the particle with reference to its speed and direction 2

B. P(x) is a monic polynomial of degree 3. P(x) has a quadratic factor of $x^2 - 1$ and when P(x) is divided by $x - 2$, the remainder is -9. Form an equation for P(x) and hence solve P(x) =0 2

C. A particle is projected from a point P on horizontal ground, with initial Speed $V\ m/s$ at an angle of elevation α to the horizontal. Its equations of motion are $\ddot{x} = 0\ and\ \ddot{y} = -g$. The horizontal and vertical components of velocity and displacement of the particle at any time t are given by

$$\frac{dx}{dt} = V\cos\alpha\ \ and\ \ \frac{dy}{dt} = V\sin\alpha - gt$$

$$x = Vt\ cos\ \alpha\ and\ \ y = Vt\sin\alpha - \frac{1}{2}gt^2\ (do\ not\ prove\ these)$$



(i) Show that the time of the flight of the of the particle is given by $t = \frac{2Vsin\alpha}{g}$ 2

(ii) The particle reaches a point Q, as shown, where the direction of the flight makes an angle β with the horizontal. Show that 1

$$tan\beta = \frac{Vsin\alpha - gt}{Vcos\alpha}$$

(iii) Hence show that the time taken to travel from $P\ to\ Q$ is 2

$$\frac{Vsin(\alpha-\beta)}{gcos\,\beta} \text{ seconds}$$

(iv) Consider the case where $\beta = \frac{\alpha}{2}$. If the time taken to travel 2

from $P\ to\ Q$ is one third of the total time of the flight, find the value of α.

Student Name: ____________ Teacher Name: ____________

2013 STHS Extension 1 Trial Solutions

Section 1

1. B 2. C 3. A 4. D 5. B

6. A 7. B 8. C 9. C 10. A

Section II

11. a) $A(-3, 6)$ $B(12, -4)$

$k : l$
$-2 : 3$ $x = \dfrac{mx_2 + nx_1}{m+n}$

$x = \dfrac{-2\times 12 + 3\times -3}{-2+3} = -33$

$y = \dfrac{-2\times -4 + 3\times 6}{1} = 26$

$(-33, 26)$ (2)

b) $\cos 105$

$\cos(60+45)$

$\cos 60 \cos 45 - \sin 60 \sin 45$ (1)

$\frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$ (1)

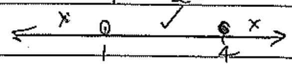
$\dfrac{1-\sqrt{3}}{2\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}-\sqrt{6}}{4}$ (1)

c) $\dfrac{2x+1}{x-1} \geq 3$

$x \neq 1$

$2x+1 = 3x-3$

$4 = x$



(1)

$1 < x \leq 4$ (2)

d) $\displaystyle\lim_{x\to 0} \frac{\sin 6x}{6x} \times \frac{1}{\frac{7}{6}}$

$\displaystyle\frac{6}{7}\times \lim_{x\to 0}\frac{\sin 6x}{6x}$

$\frac{6}{7}\times 1$

$= \frac{6}{7}$ (1)

e) $\displaystyle\int_0^1 \frac{t}{\sqrt{t+1}}\,dt$

$u = t+1$

$du = dt$

$t=0,\ u=1,\ t=1,\ u=2$

$\Rightarrow \displaystyle\int_1^2 \frac{u-1}{\sqrt{u}}\,du$ (1)

$\displaystyle\int_1^2 u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$

$\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^2$ (1)

$\frac{2}{3}\times 2^{\frac{3}{2}} - 2\sqrt{2} - \left(\frac{2}{3} - 2\right)$

$\frac{2}{3}\times 2^{\frac{3}{2}} - 2\sqrt{2} + 1\frac{1}{3}$ (1)

Student Name: ____________ Teacher Name: ____________

f) $y = 3x+1$ $y = -x+6$

$m_1 = 3$ $m_2 = -1$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$ (1)

$= \left|\dfrac{3 - -1}{1 + -3}\right|$ (1)

$= \left|\dfrac{4}{-2}\right| = 2$

$\therefore \theta = 63^\circ$ (1)

12. $y = \frac{1}{4}x^2$

$\frac{dy}{dx} = \frac{1}{2}x$

when $x = -2t$, $\frac{dy}{dx} = m = -t$

$y - t^2 = -t(x - -2t)$ (1)

$y - t^2 = -tx - 2t^2$

$y + tx + t^2 = 0$ (1)

(ii) A is when $y = 0$

$tx + t^2 = 0$

$x = -t$

$A(-t, 0)$ Now M (1) is midpoint of $A(-t, 0)$ and $T(-2t, t^2)$

$\Rightarrow M\left(\frac{-3t}{2}, \frac{t^2}{2}\right)$ (1)

$x = \frac{-3t}{2}$, $y = \frac{t^2}{2}$

$2x = -3t$ $y = \frac{t^2}{2}$

$t = \frac{2x}{-3}$ $y = \frac{4x^2}{9}\times\frac{1}{2}$

$2x^2 = 9y$ (1)

b) $\displaystyle\int \frac{dx}{4+x^2}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$ (1)

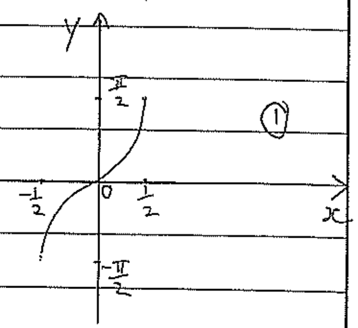
c)(i) $f(x) = \sin^{-1}2x$

Domain: $-1 \leq 2x \leq 1$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$ (1)

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ (1)

(ii)



(1)

Student Name: ______ Teacher Name: ______

d) $\frac{dV}{dt} = 72$, find $\frac{dA}{dt}$ when $r = 12$

$V = \frac{4}{3}\pi r^3$ $\quad A = 4\pi r^2$

$\frac{dV}{dr} = 4\pi r^2$ $\quad \frac{dA}{dr} = 8\pi r$ ①

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ ①

$= \frac{dA}{dr} \times \left(\frac{dr}{dV} \times \frac{dV}{dt}\right)$

$= 8\pi r \times \left(\frac{1}{4\pi r^2} \times 72\right)$

$= 2 \times 12 \times \frac{1}{12^2} \times 72$

$= 12 \text{ mm}^2/\text{sec}$ ①

e) Step 1

Show result is true for $n = 1$

ie: $n^3 + (n+1)^3 + (n+2)^3$ becomes

$1^3 + (1+1)^3 + (1+2)^3$

$1 + 8 + 27$

$= 36$ which divides by ✓

Step 2

Assume result is true for $n = k$

ie: $k^3 + (k+1)^3 + (k+2)^3 = M$ (M integral)

Step 3

Show result is true for $n = k+1$

$(k+1)^3 + (k+2)^3 + (k+3)^3$

$= (k+3)^3 + 9M - k^3$ from step 2 ②

$= k^3 + 9k^2 + 9k + 27 + 9M - k^3$

$= 9k^2 + 27k + 27 + 9M$

$= 9[k^2 + 3k + 3 + M]$ is integral since

Student Name: ______ Teacher Name: ______

M and k are integral ① for correct setting out

Step 4

Since result is true for $n = 1$, it must also be true for $n = 1 + 1 = 2$, $n = 2 + 1 = 3$ ... and hence for all positive integral values of n.

13. (i) $x = 1 + \frac{1}{2}\cos 2t$

SHM if $\ddot{x} = -n^2(x - A)$

$\dot{x} = -\sin 2t$

$\ddot{x} = -2\cos 2t$ but from x,

$\cos 2t = 2(x-1)$

$\therefore \ddot{x} = -4(x-1)$ is of required form ①

(ii) Period is $\frac{2\pi}{n}$

$n = 2$ ①

$\therefore$ period $= \pi$ seconds

(iii) At rest when

$\dot{x} = 0$

ie: $-\sin 2t = 0$ when

$2t = 0, \pi, 2\pi, \ldots$

$t = 0, \frac{\pi}{2}, \pi, \ldots$

sub. any into x

$x = 1 + \frac{1}{2}\cos 0 = 1\frac{1}{2}$ at rest ①

Amplitude $= \frac{1}{2}$ ①

b) Join CD

$\angle MAB = \angle ABD = x$

① (Alternate angles $MN \parallel BD$)

$\angle ABD = \angle ACD = x$

(Angles in same segment)

$\angle MAB = \angle ACB = x$

① (Alternate Segment Theorem)

$\therefore CA$ bisects $\angle BCD$

as $\angle ACB = \angle ACD = x$

①

Student Name: ______ Teacher Name: ______

c) (i) $T = m + Ae^{kt}$

$\frac{dT}{dt} = kAe^{kt}$ ① but

from $T = m + Ae^{kt}$

$T - m = Ae^{kt}$ ②

substituting ② into ①

$\frac{dT}{dt} = k(T - m)$ as req'd ①

(ii) $m = 40°$

when $t = 0$, $T = 170$

$t = 45$, $T = 105$

$170 = 40 + Ae^{k \times 0}$

$\therefore A = 130$ so ①

$T = 40 + 130e^{kt}$

$105 = 40 + 130e^{k \times 45}$

$65 = 130e^{45k}$

$\frac{1}{2} = e^{45k}$

$\ln\frac{1}{2} = 45k$ ①

$k = -0.015$ Now

when $t = 135$

$T = 40 + 130e^{-0.015 \times 135}$

$T = 57°$ ①

(iii) $T = 40 + 130e^{-0.015t}$

Find t when $T = 80$

$80 = 40 + 130e^{-0.015t}$

$40 = 130e^{-0.015t}$ ①

$\ln\frac{4}{13} = -0.015t$

$t = 79$ minutes ①

d) $\int \cos^2 2x \, dx$

$\int \frac{\cos 4x + 1}{2} \, dx$ ①

$= \frac{1}{8}\sin 4x + \frac{x}{2} + C$ ①

14. (i) $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$

$= \frac{d}{dt}(v)$

$= \frac{dv}{dt}$

$= \frac{dv}{dx} \times \frac{dx}{dt}$

$= v\frac{dv}{dx}$ since $v = \frac{dx}{dt}$ ①

$\frac{d}{dv}\left(\frac{1}{2}v^2\right) \cdot \frac{dv}{dx}$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ ①

Student Name: ______ Teacher Name: ______

(ii) α) From (i), sub $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ for $\ddot{x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$

$\int dx \quad \int dx$

$\frac{1}{2}v^2 = 2e^{-x} + c$ ①

when $x = 0$, $v = 2$

$\therefore \frac{1}{2} \times 2^2 = 2e^0 + c$

$2 = 2 + c$

$\therefore c = 0$

$\therefore v^2 = 4e^{-x}$ ①

β) As x increases from 0 it moves to the right or in a positive direction. ①

As $x \to \infty$, $v \to 0$ so it is slowing down but never reaches 0. ①

b) Let $P(x) = (x^2 - 1)(x - \alpha)$

$\Rightarrow P(2) = (4 - 1)(2 - \alpha) = -9$ ①

$6 - 3\alpha = -9$

$-3\alpha = -15$

$\alpha = 5$

$\therefore P(x) = (x - 1)(x + 1)(x - 5)$

$x = 1, -1, 5$ ①

c) (i) Solve $y = 0$

$Vt\sin\alpha - \frac{1}{2}gt^2 = 0$

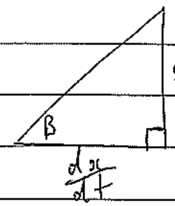
$t(V\sin\alpha - \frac{1}{2}gt) = 0$ ①

$V\sin\alpha = \frac{1}{2}gt$

$\frac{2V\sin\alpha}{g}$ or $0 = t$

$\Rightarrow$ Time of flight is $\frac{2V\sin\alpha}{g}$ ①

(ii)



$\tan\beta = \frac{dy/dt}{dx/dt}$

$= \frac{V\sin\alpha - gt}{V\cos\alpha}$ ①

Student Name: ______________________ Teacher Name: ______________________

(iii) $\tan\beta = \dfrac{V\sin\alpha - gt}{V\cos\alpha}$

$\dfrac{\sin\beta}{\cos\beta} = \dfrac{V\sin\alpha - gt}{V\cos\alpha}$

Now solve for t

$V\sin\beta\cos\alpha = V\sin\alpha\cos\beta - \cos\beta\, gt$

$\cos\beta\, gt = V(\sin\alpha\cos\beta - \sin\beta\cos\alpha)$ ①

$= V\sin(\alpha-\beta)$

so $t = \dfrac{V\sin(\alpha-\beta)}{g\cos\beta}$ ①

(iv) If $\beta = \frac{\alpha}{2}$ then time from P to Q is

$t = \dfrac{V\sin\frac{\alpha}{2}}{g\cos\frac{\alpha}{2}} = \dfrac{V\tan\frac{\alpha}{2}}{g}$ and this is $\frac{1}{3}$ of total flight time

$\Rightarrow \dfrac{V\tan\frac{\alpha}{2}}{g} = \dfrac{1}{3} \times \dfrac{2V\sin\alpha}{g}$

$3V\tan\frac{\alpha}{2} = 2V \times \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$ (t results) ①

$3 + 3\tan^2\frac{\alpha}{2} = 4$

$3\tan^2\frac{\alpha}{2} = 1$

$\tan^2\frac{\alpha}{2} = \frac{1}{3}$

$\tan\frac{\alpha}{2} = \frac{1}{\sqrt{3}}$ (α is acute)

$\frac{\alpha}{2} = \frac{\pi}{6}$

$\therefore \alpha = \frac{\pi}{3}$ ①