2014
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time - 5 minutes
- Working Time - 2 hours
- Write using a blue or black pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11 - 14
- Begin each question on a new sheet of paper.

Total marks (70)

Section I

10 marks

- Attempt Questions 1-10
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section

Section II

60 marks

- Attempt questions 11 – 14
- Answer on the blank paper provided, unless otherwise instructed. Start a new page for each question.
- Allow about 1 hour 45 minutes for this section

Section I

Total marks (10)

Attempt Questions 1-10

Allow about 15 minutes for this section

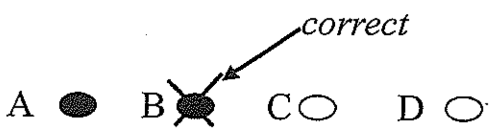
Use the multiple choice answer sheet.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample

$2+4=?$ (A) 2 (B) 6 (C) 8 (D) 9

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:



Question 1 Which of the following is an expression for $\int \cos^2 x \sin x dx$?

(A) $2\cos x \sin x + c$

(B) $\cos^3 x + c$

(C) $-\frac{1}{3}\cos^3 x + c$

(D) $\frac{1}{3}\cos^3 x + c$

Question 2 A particle is moving along the x-axis. Its velocity v at position x is given by $v = \sqrt{8x - x^2}$. What is the acceleration when $x = 3$?

(A) 1

(B) 2

(C) 3

(D) 4

Question 3 If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = e^{y-2}$

(B) $f^{-1}(x) = e^{y+2}$

(C) $f^{-1}(x) = \log_e x - 2$

(D) $f^{-1}(x) = \log_e x + 2$

Question 4 A particle is moving in a straight line with $v^2 = 36 - 4x^2$ and undergoing simple harmonic notion. If the particle is initially at the origin, which of the following is the correct equation for its displacement in terms of t?

(A) $x = 2\sin(3t)$

(B) $x = 3\sin(2t)$

(C) $x = 2\sin(9t)$

(D) $x = 3\sin(4t)$

Question 5 What is the solution to the equation $\dfrac{\cos^3\theta}{\sin\theta}+\sin\theta\cos\theta=1$ in the domain $0\le x\le 2\pi$?

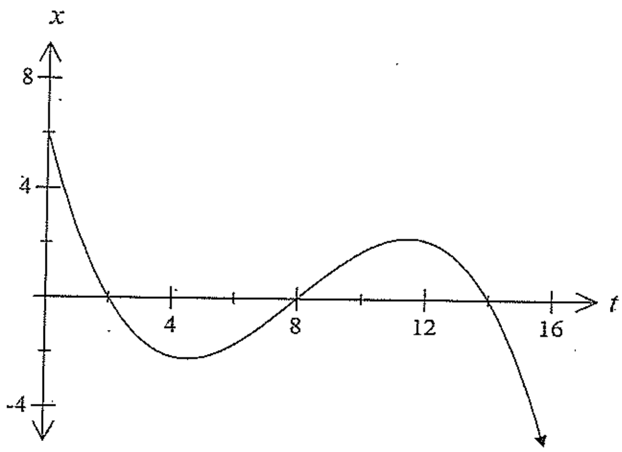
(A) $\theta=\dfrac{\pi}{4},\dfrac{3\pi}{4}$

(B) $\theta=\dfrac{\pi}{4},\dfrac{5\pi}{4}$

(C) $\theta=\dfrac{5\pi}{4},\dfrac{7\pi}{4}$

(D) $\theta=\dfrac{3\pi}{4},\dfrac{5\pi}{4}$

Question 6 The displacement, x metres, from the origin of a particle moving in a straight line at any time (t seconds) is shown in the graph.



When was the particle at rest?

(A) $t=4.5$ and $t=11.5$

(B) $t=0$

(C) $t=2$, $t=8$ and $t=14$

(D) $t=1.5$ and $t=8$

Question 7 The population (P) of a colony of bugs is increasing continuously at a rate proportional to the existing population. The present population is 20 000 and the population 3 months ago was 8000. If $P=Ae^{kt}$, what is the value of k?

(A) -0.916

(B) -0.305

(C) 0.305

(D) 0.916

Question 8 A particle moves along a straight line about a fixed point O so that its acceleration, a ms^{-2}, at time t seconds is given by $a = 4\cos\left(2t + \frac{\pi}{6}\right)$. Initially the particle is moving to the right with a velocity of 1 ms^{-1} from a position $\frac{\sqrt{3}}{2}$ metres to the left of O. Which of the following is the correct expression for the velocity of the particle after t seconds?

(A) $v = 2\sin\left(2t + \frac{\pi}{6}\right)$

(B) $v = 2\sin\left(2t + \frac{\pi}{6}\right) + 1 - \sqrt{3}$

(C) $v = 4\sin\left(2t + \frac{\pi}{6}\right)$

(D) $v = 4\sin\left(2t + \frac{\pi}{6}\right) - 1$

Question 9 What is the exact value of the definite integral $\int_{\frac{2}{\sqrt{3}}}^{2\sqrt{3}} \frac{dx}{x^2 + 4}$?

(A) $\frac{\pi}{12}$

(B) $\frac{\pi}{6}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

Question 10 A particle moves in a straight line and its position at any time t is given by $x = 3\cos 2t + 4\sin 2t$. The motion is simple harmonic. What is the greatest speed?

(A) 6

(B) 10

(C) 12

(D) 20

End of Section 1

Section II

Total marks (60)
Attempt Questions 11 - 14
Allow about 1 hour 45 minutes for this section.

Answer all questions, starting each question on a new page and question number at the top of the page.

Question 11 (15 marks) Use a separate page/booklet **Marks**

(a) Find $\lim_{x \to \infty} \frac{3x^2 - 2x}{x^2 + 4}$ **1**

(b) (i) Show that $x - 2$ is a factor of $x^3 - 4x^2 + 7x - 6$ **1**

(ii) Explain why $x^3 - 4x^2 + 7x - 6 = 0$ has only 1 real root. **2**

(c) Find the value of k if the roots of the equation $x^3 - 3x^2 - 6x + k = 0$ are in arithmetic progression. **3**

(d) Differentiate with respect to x: $y = \log_7 x^2$ **2**

(e) Find the coordinates of the point P that divides the interval $(2, -6)$ and $(7, 9)$ internally in the ratio $2:3$. **2**

(f) The acceleration of a particle is given by $a = -e^{-x}$. Initially $v = \sqrt{2}, x = 0$. Find the velocity as a function of x. **2**

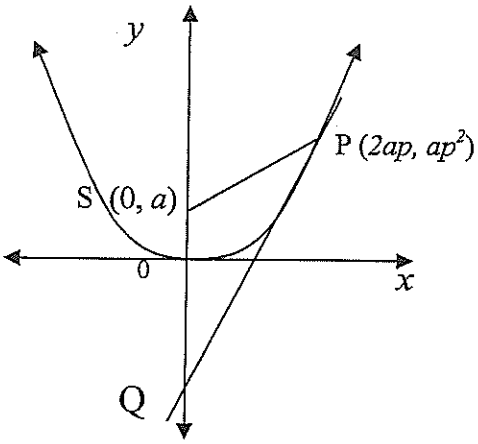
(g) (i) State the domain of $y = 4\cos^{-1}\frac{x}{3}$ **1**

(ii) Hence, sketch the curve $y = 4\cos^{-1}\frac{x}{3}$ **1**

Question 12 (15 marks) Use a separate page/booklet **Marks**

(a) Solve: $\dfrac{x}{x-2} \geq 4, \quad x \neq 2$ **2**

(b) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$ whose focus is at S.
The tangent at P meets the Y-axis at Q



(i) Derive and show that the equation of the tangent at P is $y = px - ap^2$. **2**

(ii) Find the coordinates of Q. **1**

(ii) Show that $\angle SPQ = \angle SQP$ **2**

(c) A particle moves in simple harmonic motion. It starts from rest at a point 6 cm to the right of the centre of motion O. The particle has a speed of 10 *cm/s*, when it passes through O.

(i) Write the expression for displacement in the form of $x = a\cos(nt + \alpha)$. **2**

(ii) Find the period of motion. **1**

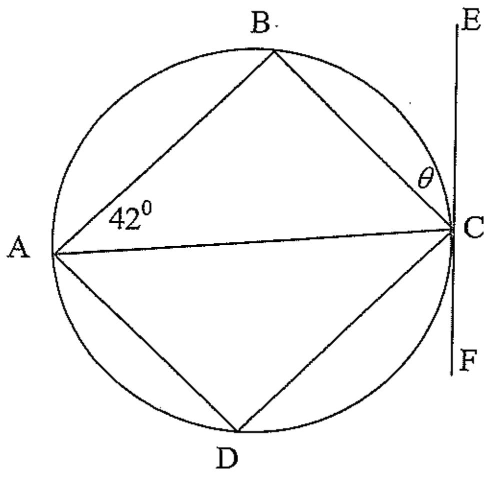
(iii) Find the acceleration after 3 seconds. **1**

(d) (i) Express $\sqrt{3}\sin 2\theta - \cos 2\theta$ in the form $R\sin(2\theta - \alpha)$, where α acute. **2**

(ii) Hence solve $\sqrt{3}\sin 2\theta - \cos 2\theta = 1$; $0 \leq \theta \leq \pi$. Answer in exact form. **2**

Question 13 (15 marks) Use a separate page/booklet. **Marks**

(a) ABCD is a cyclic quadrilateral where AC bisects $\angle DAB$, $\angle BAC = 42^0$ and FE is a tangent to the circle at C.



NOT TO SCALE

(i) Find the size of θ ($\angle BCE$). Give reasons. **1**

(ii) Prove that FE is parallel to DB. **3**

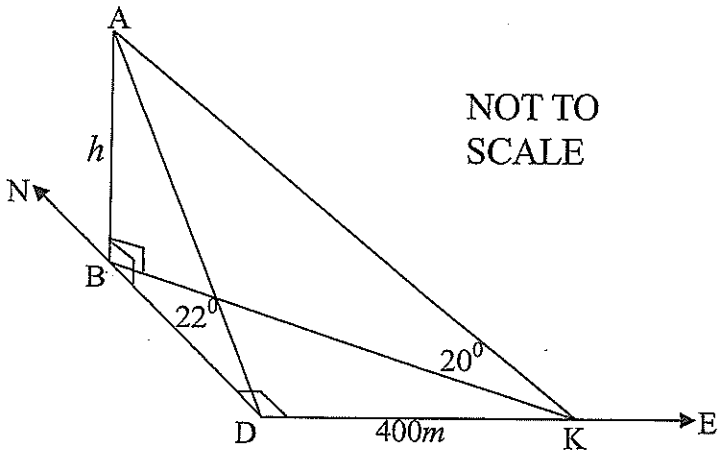
(b) (i) Differentiate: $y = x\sin^{-1}x + \sqrt{1-x^2}$ **2**

(ii) Hence evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$ **1**

(c) Prove that $\tan^{-1}x = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$ **2**

Question 13 continues on page 8

(d) Donna is standing at D and observes the angle of elevation of the tip of a flagpole A, on top of a building to be 22^0. Her friend Kate, who is standing at K, 400 *metres* due east of Donna, finds the angle of elevation of the tip of the flagpole to be 20^0. The building is due north of Donna and B is the base of the building. The points B, D and K are all on level ground.



(i) Show that the height (h) of the flagpole above the ground is given by: **3**

$$h = \frac{400}{\sqrt{\cot^2 20^0 - \cot^2 22^0}}$$

(ii) Find the value of h, correct to 3 significant figures. **1**

(e) Using the substitution $x = u^2 - 2$ or otherwise, find $\int \frac{x}{\sqrt{x+2}}\,dx$ **2**

Question 14 (15 marks) Use a separate page/booklet. **Marks**

(a) A particle is projected with speed v *m/s* at an angle of projection, θ to the horizontal

(i) Derive expressions for the horizontal and vertical displacements x and y at any time t seconds after projection. Let gravity = g m/s^2. **2**

(ii) Show that the equation of the path of the particle is given by

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$$ **2**

(iii) The particle has an initial speed of $2\sqrt{70}$ m/s and just clears a pole. The pole is 5m high and its base is 20m from the point of projection. Find two possible angles of projection to the nearest degree. (Take $g = 9{\cdot}8$ m/s^2) **2**

(b) Prove by Mathematical Induction that, **3**

$(n)^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all positive whole numbers n

(c) (i) Sketch the curve $y = \ln(x-2)$ **1**

(ii) The inner surface of a bowl is of the shape formed by rotating about the y axis, the curve $y = \ln(x-2)$ between $y = 0$ and $y = 2$

The bowl is placed with its axis vertical and water is poured in.

Show that the volume of water in the bowl when it is filled to a depth h, where $h < 2$, is given by $\pi(4h - 4\frac{1}{2} + 4e^h + \frac{1}{2}e^{2h})$ $unit^3$. **3**

(iii) If the bowl is filled at the rate of 60 $unit^3$ / s, find the rate at which the water level is rising when the depth of water is $1{\cdot}25$ *units*. Give your answer correct to 2 decimal places. **2**

MC: C A C B B A C A A B

Q11

(a) $\lim_{x\to\infty} \dfrac{\frac{3x^2}{x^2} - \frac{2x}{x}}{\frac{x^2}{x^2} - \frac{4}{x^2}}$

$= \lim_{x\to\infty} \dfrac{3 - \frac{2}{x}}{1 - \frac{4}{x^2}}$

$= \lim_{x\to\infty} 3 = 3$

(b)(i) $f(x) = x^3 - 4x^2 + 7x - 6$

$f(2) = 2^3 - 4(2)^2 + 7(2) - 6$

$= 0$

ii) $x - 2 \overline{\big)\, x^3 - 4x^2 + 7x - 6} = 0$ quotient $x^2 - 2x + 3$

$x^3 - 2x^2$

$-2x^2 + 7x - 6$

$-2x^2 + 4x$

$3x - 6$.

$\therefore (x-2)(x^2 - 2x + 3) = 0$.

$x^2 - 2x + 3 = 0$ is a factor and has no solution

$x = \dfrac{2 \pm \sqrt{4 - 4\times1\times3}}{2\times1}$

$= \dfrac{2 \pm \sqrt{-8}}{2}$ $\therefore$ no solution.

(c) let the roots be.

$\alpha - d, \alpha, \alpha + d$

$\alpha + \alpha - d + d + d = -\frac{-3}{1}$

$3\alpha = 3$

$\alpha = 1$.

$\alpha(\alpha - d) + \alpha(\alpha + d) + (\alpha - d)(\alpha + d) = -6$

$\alpha^2 - \alpha d + \alpha^2 + \alpha d + \alpha^2 - d^2 = -6$.

$\therefore 3\alpha^2 - d^2 = -6$.

$3 - d^2 = -6$

$d^2 = 9$

$\alpha(\alpha - d)(\alpha - d) = -k$.

$\alpha(\alpha^2 - d^2) = -k$.

$1(1^2 - 9) = -k$.

$-8 = -k$

$\therefore k = 8$.

(d) $y = \dfrac{\ln x^2}{\ln 7}$

$= \dfrac{2}{\ln 7} \cdot \ln x$

$y' = \dfrac{2}{\ln 7} \cdot \dfrac{1}{x}$

$= \dfrac{2}{x \ln 7}$

(e) $\left(\dfrac{2\times7 + 3\times2}{2+3}, \dfrac{2\times9 + 3\times-6}{2+3} \right)$

$= (4, 0)$

(f) $a = -e^{-x}$

$\dfrac{d(\frac{1}{2}v^2)}{dx} = -e^{-x}$

$\frac{1}{2}v^2 = \int -e^{-x}$

$= e^{-x} + c$

$\frac{1}{2}(\sqrt{2})^2 = e^{-0} + c$

$1 = 1 + c$

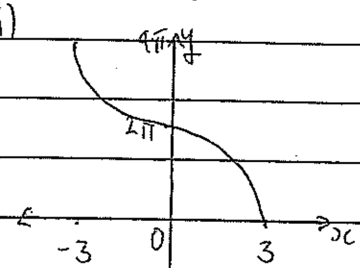
$c = 0$

$\therefore \frac{1}{2}v^2 = e^{-x}$

$v^2 = 2e^{-x}$

$v = \sqrt{2}\, e^{\frac{-x}{2}}$

(g)(i)



ii) ii) $-3 \le x \le 3$

Q12

(a) $\dfrac{x}{x-2} \cdot (x-2)^2 \ge 4(x-2)^2$

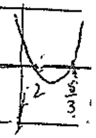
$x(x-2) \ge 4(x-2)^2$

$4(x-2)^2 - x(x-2) \le 0$

$(x-2)[4x - 8 - x] \le 0$

$(x-2)(3x-8) \le 0$

$\therefore 2 < x \le \frac{8}{3}$.



(b)(i) $y = \dfrac{x^2}{4a}$

$\dfrac{dy}{dx} = \dfrac{2x}{4a} = \dfrac{x}{2a}$ $\therefore m = \dfrac{2ap}{2a} = p$

$y - ap^2 = p(x - 2ap)$

(ii) Q at $x = 0$.

$\therefore y - ap^2 = p(0 - 2ap)$

$y - ap^2 = -2ap^2$

$y = -ap^2$

$Q(0, -ap^2)$

(iii) $d^2_{SP} = (2ap)^2 + (ap^2 - a)^2$

$= 4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2$

$= a^2p^4 + 2a^2p^2 + a^2$

$= a^2(p^4 + 2p^2 + 1)$

$= a^2(p^2 + 1)^2$

$d_{SP} = \sqrt{a^2(p^2+1)^2}$

$= a(p^2+1) = ap^2 + a$

$d_{SQ} = a + ap^2$

$= d_{SP}$

$\therefore \angle SPQ = \angle SQP$ (equal angles opposite equal sides)

(c)(i) $x = a\cos(nt+\alpha)$

when $t=0$, $x=6$.

$6 = a\cos\alpha$.

$\dot{x} = -an\sin(nt+\alpha)$

when $t=0$, $\dot{x}=0$

$0 = -an\sin\alpha$.

$\sin\alpha = 0 \quad \therefore \alpha = 0$

$\therefore \cos\alpha = 1$

$\therefore a = 6 \quad x = 6\cos nt$

when $x=0$, $s=10$; $6\cos nt = 0$

$\therefore \cos nt = 0$

$nt = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$t = \frac{\pi}{2n}, \frac{3\pi}{2n}, \ldots$

$|-6n\sin nt| = 10$

$6n\sin n \cdot \frac{\pi}{2n} = 10$

$6n\sin\frac{\pi}{2} = 10$

$n = \frac{10}{6} = \frac{5}{3}$

$\therefore x = 6\cos\frac{5t}{3}$

(ii) Period $= \frac{2\pi}{n} = \frac{2\pi}{5/3}$

$= \frac{6\pi}{5}$ secs

(iii) $\dot{x} = -6 \cdot \frac{5}{3}\sin\frac{5t}{3}$

$= -10\sin\frac{5t}{3}$

$\ddot{x} = -\frac{5}{3} \cdot 10\cos\frac{5t}{3}$

$= -\frac{50}{3}\cos\frac{5t}{3}$

when $t=3$

$a = -\frac{50}{3}\cos 5$

$= -4.73 \text{ cm s}^{-2}$

d) i) $R = \sqrt{(\sqrt{3})^2 + (1)^2} = 2 \qquad \tan\alpha = -\frac{1}{\sqrt{3}}, \quad \alpha = -\frac{\pi}{6}$

$\therefore 2\sin(2\theta - \frac{\pi}{6})$

ii) $\sin(2\theta - \frac{\pi}{6}) = \frac{1}{2} \qquad -\frac{\pi}{6} \le 2\theta - \frac{\pi}{6} \le \frac{11\pi}{6}$

$2\theta - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$

$2\theta = \frac{\pi}{3}, \pi$

$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{2}$.

Q13

a) i) $\theta = 42°$ (angle in alt seg.)

ii) $\angle CAD = 42$ ($\angle BAD$ is bisected)

$\therefore \angle DCF = 42$ ($\angle$ in alt. seg.)

join DB

$\therefore \angle DBC = 42°$ ($\angle$ in alt seg.)

$\therefore BD \parallel FE$ (alt. $\angle$'s are equal)

b) i) $y' = x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{-2x(1-x^2)^{-\frac{1}{2}}}{2}$

$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}}$

$= \sin^{-1}x$.

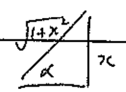
ii) $\left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$= \left[\frac{1}{2}\cdot\sin^{-1}\frac{1}{2} + \sqrt{1-(\frac{1}{2})^2} - (\sqrt{1})\right]$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{4} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

c) let $\tan^{-1}x = \alpha$

$\therefore x = \tan\alpha$



$\sin\alpha = \frac{x}{\sqrt{1+x^2}}$

$\sin(\tan^{-1}x) = \frac{x}{\sqrt{1+x^2}}$

$\therefore \tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$

d) i) $\frac{h}{BD} = \tan 22, \quad \frac{h}{BK} = \tan 20$

$BD = \frac{h}{\tan 22} = h\cot 22 \qquad BK = \frac{h}{\tan 20} = h\cot 20$

$BD^2 + 400^2 = BK^2$

$h^2\cot^2 22 + 400^2 = h^2\cot^2 20$

$h^2\cot^2 20 - h^2\cot^2 22 = 400^2$

$h^2(\cot^2 20 - \cot^2 22) = 400^2$

$h^2 = \frac{400^2}{\cot^2 20 - \cot^2 22}$

$h = \frac{400}{\sqrt{\cot^2 20 - \cot^2 22}}$

ii)

$h \doteq 335$ metres

e) $x = u^2 - 2 \Rightarrow u^2 = x + 2$

$\frac{dx}{du} = 2u, \quad dx = 2u\,du$

$\int \frac{x}{\sqrt{x+2}}\,dx = \int \frac{u^2-2}{\sqrt{u^2}} \cdot 2u\,du$

$= 2\int u^2 - 2\,du$

$= \frac{2u^3}{3} - 4u + C$

$= \frac{2(x+2)^{\frac{3}{2}}}{3} - 4(x+2)^{\frac{1}{2}} + C$

Q14 a) i)

$\ddot{x} = 0 \qquad \ddot{y} = -g$

$\dot{x} = c \qquad \dot{y} = -gt + c$

$\dot{x} = V\cos\theta \qquad \dot{y} = -gt + V\sin\theta$

$x = Vt\cos\theta + c \qquad y = -\frac{1}{2}gt^2 + Vt\sin\theta + c$

when $t=0$, $x=0$, $y=0$

$\therefore x = Vt\cos\theta$ & $y = -\frac{1}{2}gt^2 + Vt\sin\theta$

ii)

$t = \frac{x}{V\cos\theta}$

$y = \frac{-gx^2}{2V^2\cos^2\theta} + \frac{Vx\sin\theta}{V\cos\theta}$

$= \frac{-gx^2}{2V^2\cos^2\theta} + x\tan\theta$

$= x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$

iii) $V = 2\sqrt{70}$, $x = 20$, $y = 5$, $g = 9.8$

$20\tan\theta - \frac{7}{\cos^2\theta} = 5$

$20\tan\theta - 7\sec^2\theta = 5$

$20\tan\theta - 7(1+\tan^2\theta) = 5$

$20\tan\theta - 7 - 7\tan^2\theta = 5$

$7\tan^2\theta - 20\tan\theta + 12 = 0$

(5)

$(7\tan\theta - 6)(\tan\theta - 2) = 0$

$\tan\theta = \frac{6}{7}$ & $\tan\theta = 2$

$\therefore \theta \doteqdot 41^\circ$ & $\theta \doteqdot 63^\circ$

(b) If $n=1$, $(1)^3 + (1+1)^3 + (1+2)^3 = 36$

$\therefore$ true for $n=1$

Assume true for $n=k$

$(k)^3 + (k+1)^3 + (k+2)^3 = 9m$

m is a positive integer.

Prove true for $n=k+1$

$(k+1)^3 + (k+2)^3 + (k+3)^3 = 9p$

p is a positive integer.

$9m - k^3 = (k+1)^3 + (k+2)^3$

$\therefore LHS = 9m - k^3 + (k+3)^3$

$= 9m - k^3 + k^3 + 9k^2 + 27k + 27$

$= 9m + 9k^2 + 27k + 27$

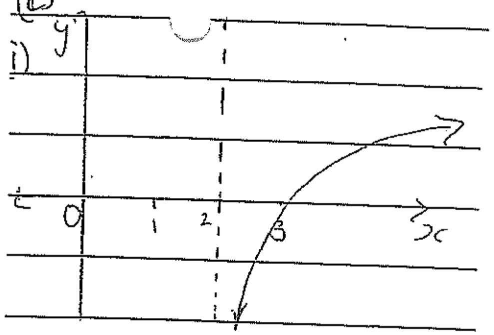
$= 9(m + k^2 + 3k + 3)$

$= 9q$

Hence, since true for $n=1$, and since true for $n=k$ & also true for $n=k+1$ then true for all positive integer n.

(c)

i)



(ii) $V = \pi\int_0^2 x^2 \, dy$

$\ln(x-2) = y$

$x - 2 = e^y$

$x = e^y + 2$

$x^2 = (e^y+2)^2$

$= e^{2y} + 4e^y + 4$

$\therefore V = \pi\int_0^h e^{2y} + 4e^y + 4 \, dy$

$= \pi\left[\frac{1}{2}e^{2y} + 4e^y + 4y\right]_0^h$

$= \pi\left[\left(\frac{1}{2}e^{2h} + 4e^h + 4h\right) - \left(\frac{1}{2} + 4\right)\right]$

$= \pi\left(\frac{1}{2}e^{2h} + 4e^h + 4h - 4\frac{1}{2}\right)$

iii) $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$60 = \pi(e^{2h} + 4e^h + 4) \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{60}{\pi(e^{2\times1.25} + 4e^{1.25} + 4)}$

$\doteqdot 0.63$

(6)