

Name: Maths Class:

SYDNEY TECHNICAL HIGH SCHOOL



Year 12 Mathematics Extension 1

Trial HSC Examination

2015

Time allowed: 120 minutes

General Instructions:

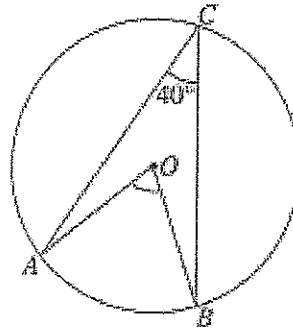
- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Standard Integrals is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1 Multiple Choice
Questions 1-10
10 Marks

Section II Questions 11-14
60 Marks

1.

The points A , B and C lie on a circle with centre O , as shown in the diagram. The size of $\angle ACB$ is 40° .



NOT TO SCALE

What is the size of $\angle AOB$?

- (A) 20°
- (B) 40°
- (C) 70°
- (D) 80°

2.

The angle θ satisfies $\sin \theta = \frac{5}{13}$ and $\frac{\pi}{2} < \theta < \pi$.

What is the value of $\sin 2\theta$?

- (A) $\frac{10}{13}$
- (B) $-\frac{10}{13}$

- (C) $\frac{120}{169}$
- (D) $-\frac{120}{169}$

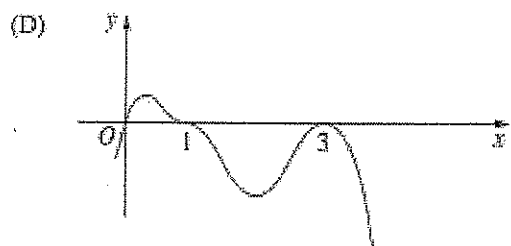
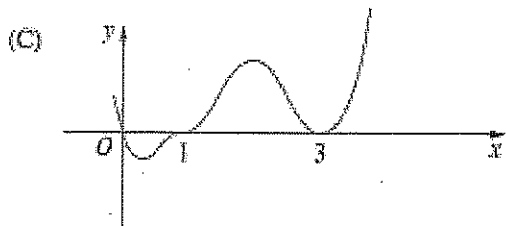
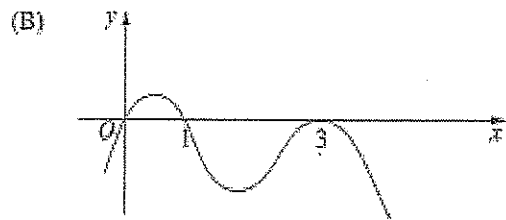
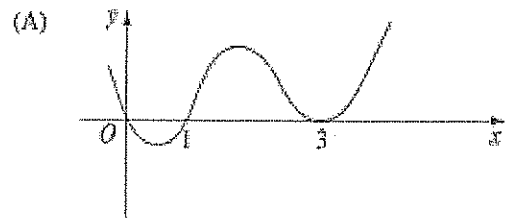
3.

Which of the following is the correct expression for $\int \frac{dx}{\sqrt{49-x^2}}$?

- (A) $-\cos^{-1} \frac{x}{7} + c$
- (B) $-\cos^{-1} 7x + c$
- (C) $-\sin^{-1} \frac{x}{7} + c$
- (D) $-\sin^{-1} 7x + c$

4.

Which diagram best represents the graph $y = x(1-x)^3(3-x)^2$?



5.

If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

- (A) $f^{-1}(x) = e^{y-2}$
- (B) $f^{-1}(x) = e^{y+2}$
- (C) $f^{-1}(x) = \log_e x - 2$
- (D) $f^{-1}(x) = \log_e x + 2$

6.

Which of the following is an expression for $\int \sin^2 6x dx$?

(A) $\frac{x}{2} - \frac{1}{24} \sin 6x + c$

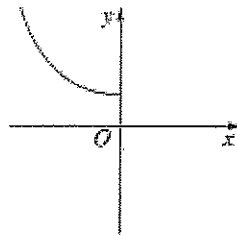
(B) $\frac{x}{2} + \frac{1}{24} \sin 6x + c$

(C) $\frac{x}{2} - \frac{1}{24} \sin 12x + c$

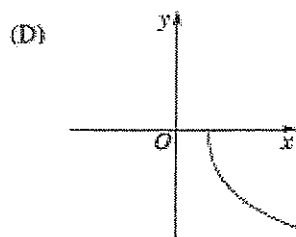
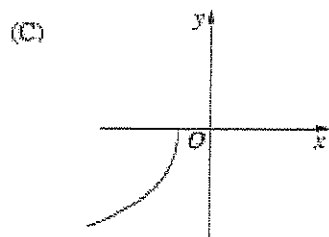
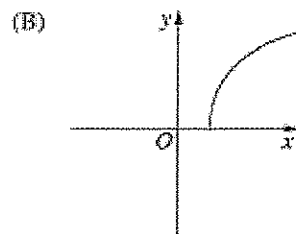
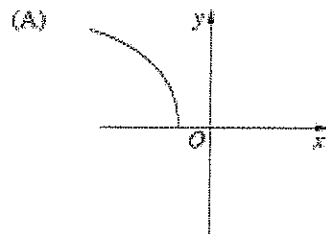
(D) $\frac{x}{2} + \frac{1}{24} \sin 12x + c$

7.

The diagram shows the graph $y = f(x)$.

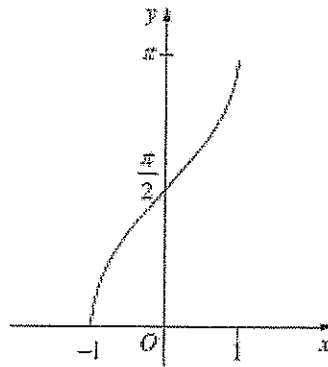


Which diagram shows the graph $y = f^{-1}(x)$?



8.

The diagram shows the graph of a function.



Which function does the graph represent?

- (A) $y = \cos^{-1} x$
- (B) $y = \frac{\pi}{2} + \sin^{-1} x$
- (C) $y = -\cos^{-1} x$
- (D) $y = -\frac{\pi}{2} - \sin^{-1} x$

9.

Which inequality has the same solution as $|x+2| + |x-3| = 5$?

- (A) $\frac{5}{3-x} \geq 1$
- (B) $\frac{1}{x-3} - \frac{1}{x+2} \leq 0$
- (C) $x^2 - x - 6 \leq 0$
- (D) $|2x-1| \geq 5$

10.

After t minutes the temperature (T) of a bottle of water placed in a refrigerator is given by $T = A + Be^{kt}$ where A is the temperature inside the refrigerator and B and k are constants.

A bottle of water with a temperature of 20°C is placed in a refrigerator with temperature of 2°C . If 20 minutes later the temperature of the bottle of water has decreased to 10°C then the value of k is?

(A) $k = -\frac{1}{20} \log_e \frac{9}{4}$

(B) $k = -\frac{1}{10} \log_e \frac{4}{9}$

(C) $k = \frac{1}{20} \log_e \frac{9}{4}$

(D) $k = \frac{1}{10} \log_e \frac{4}{9}$

Question 11 (15 Marks) Start a fresh sheet of paper.

Marks

- (a) Solve the inequality $\frac{4-2x}{x+5} \leq 2$ 2
- (b) Find $\lim_{x \rightarrow 0} \frac{\sin\left(\frac{\pi}{4}x\right)}{2x}$ 2
- (c) Find, to the nearest degree, the acute angle between the lines:
 $x - 2y + 1 = 0$ and $y = 5x - 4$ 2
- (d) Differentiate $\ln(\sin^{-1} 2x)$ 2
- (e) Find the Cartesian equation of the parabola given $x = t - 2$ and $y = 3t^2 - 1$. 1
- (f) i. Prove that $\sin \theta \sec \theta = \tan \theta$ 1
ii. Hence solve $\sin \theta \sec \theta = \sqrt{3}$. ($0 \leq \theta \leq 2\pi$) 1
- (g) Find the remainder when the polynomial $P(x) = x^3 - 4x$ is divided by $x + 3$. 1
- (h) Given that $\log_3 7 = m$, find an expression for $\log_3 21$ 1
- (i) Differentiate the following, expressing your answer as a single fraction: 2

$$\frac{2x^2 - 4x}{x\sqrt{x}}$$

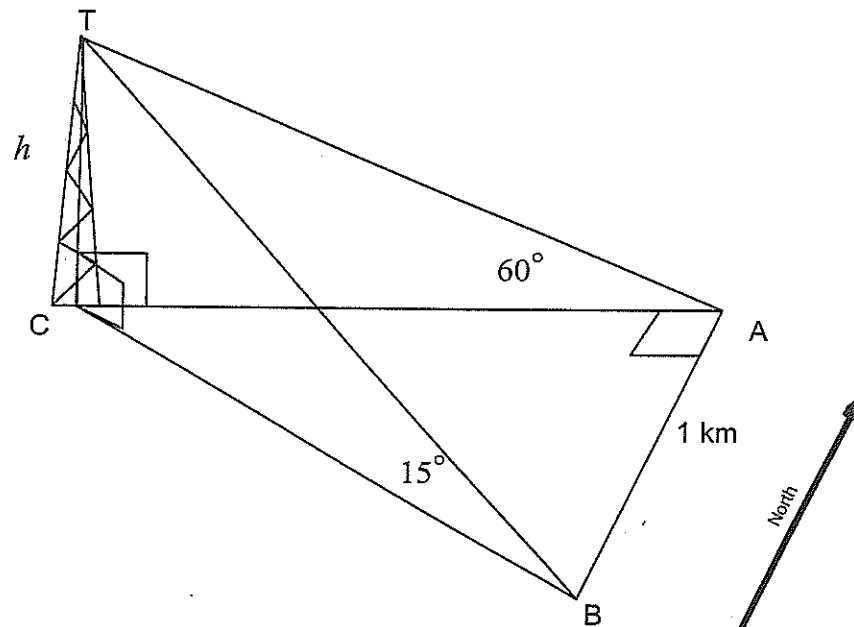
- (a) Use the process of mathematical induction to show that:

4

$$1 + 3 + 9 + \dots + 3^{n-1} = \frac{1}{2}(3^n - 1) \quad \text{for all positive integers } n$$

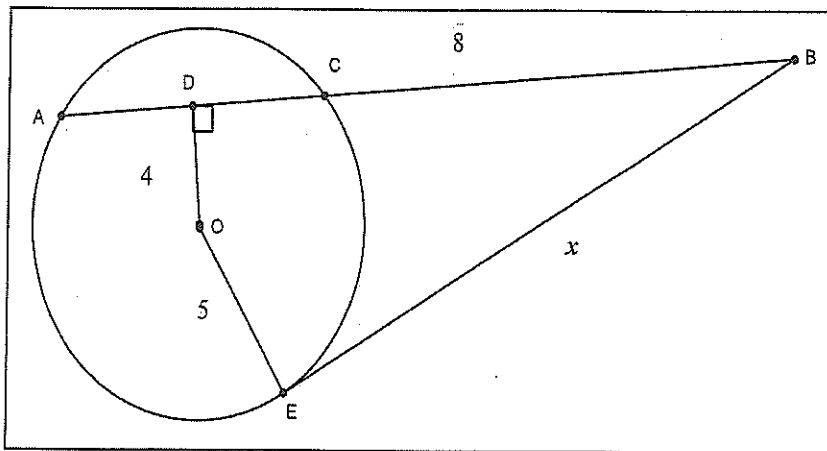
- (b) The angle of elevation of the top of a tower (T) from a point A due East of the tower is 60° . From a point B due South of A, the angle of elevation of T is 15° . A and B are at the same elevation as the base of the tower. If the distance $AB = 1$ km, find the height (h) of the tower to the nearest metre.

3



(c)

2



Given the length OD is 4cm, OE is 5cm, BC is 8cm, and BE is a tangent to the circle, centre O. OD is perpendicular to the secant AC. Find the value of x , correct to 2 decimal places.

(d) If $\alpha = \sin^{-1}\left(\frac{2}{3}\right)$ and $\beta = \sin^{-1}\left(\frac{3}{5}\right)$, find the value of $\sin(\alpha + \beta)$ 3

(e) Using the substitution $u = x - 2$, evaluate $\int_3^4 \frac{x^2}{(x-2)^2} dx$ 3

End of Question 12

Question 13 (15 Marks) Start a fresh sheet of paper.

Marks

- (a) Simplify $1 + \cos^2 x + \cos^4 x + \cos^6 x + \dots$ ($0 < x < \frac{\pi}{2}$) **2**
- (b) (i) Show that $\frac{d}{dx} \ln\left(\frac{x-1}{x+1}\right) = \frac{2}{(x-1)(x+1)}$ **2**
- (ii) Hence evaluate $\int_2^3 \frac{4dx}{(x-1)(x+1)}$ (answer in exact form) **2**
- (c) Let $P(2ap, 2ap^2)$ and $Q(2aq, 2aq^2)$ be points on the parabola $y = \frac{x^2}{2a}$.
- i. Find the equation of the chord PQ. **2**
- ii. If PQ is a focal chord, find the relationship between p and q . **2**
- iii. Show that the locus of the midpoint of PQ is a parabola. **2**
- (d) Find the area under the curve $y = \frac{1}{\sqrt{4-x^2}}$ from $x = 1$ to $x = 2$. **2**
- (e) Write $\sqrt{3} \cos x - \sin x$ in the form $2 \cos(x + \alpha)$
where $0 < \alpha < \frac{\pi}{2}$ **1**

End of Question 13

- (a) The polynomial $x^3 - 4x^2 + 5x - 1 = 0$ has 3 roots. If the roots are α , β and γ .
- i. Find the value of $\alpha + \beta + \gamma$ 1
 - ii. Find the value of $\alpha\beta\gamma$ 1
 - iii. Find the equation of the polynomial with roots 2α , 2β and 2γ . 2
- (b) The two equal sides of an isosceles triangle are of length 6cm. If the angle between them is increasing at the rate of 0.05 radians per second, find the rate at which the area of the triangle is increasing when the angle between the equal sides is $\frac{\pi}{6}$ radians. 2
- (c) A particle moves so that its distance x centimetres from a fixed point O at time t seconds is $x = 8\sin 3t$.
- i. Using the equation $\ddot{x} = -n^2 x$, show that the particle is displaying simple harmonic motion 1
 - ii. What is the period of the motion? 1
 - iii. Find the velocity of the particle when it first reaches 4 centimetres to the right of the origin. 2
- (d) The function $f(x) = \sec x$ for $0 \leq x < \frac{\pi}{2}$, and is not defined for other values of x .
- (i) State the domain of the inverse function $f^{-1}(x)$. 1
 - (ii) Show that $f^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$. 2
 - (iii) Hence find $\frac{d}{dx}f^{-1}(x)$. 2

End of Examination

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Student Name: Teacher Name:

1 D	(c) $x-2y+1=0$ $y=5x-4$
2 D	$m=1/2$ $m=5$
3 A	
4 D	$\tan \theta = \left \frac{m_1 - m_2}{1 + m_1 m_2} \right $
5 C	
6 C	$= \left \frac{5 - 1/2}{1 + (5)(1/2)} \right $
7 D	
8 B	$\tan \theta = 9/7$
9 C	$\theta = 52^\circ$
10 A	
	(d) $\ln(\sin^{-1} 2x)$
Q11 $\frac{4-2x}{x+5} \leq 2$	$y = \ln u$ $u = \sin^{-1} 2x$
$x \neq -5$	$\frac{dy}{dx} = \frac{1}{\sin^{-1} 2x} \times \frac{2}{\sqrt{1-4x^2}}$
$(x+5)(4-2x) - 2(x+5)^2 \leq 0$	$= \frac{2}{\sin^{-1} 2x \sqrt{1-4x^2}}$
$(x+5)(-4x-6) \leq 0$	
c.v. $x = -5, -3/2$	
$x < -5, x > -3/2$	(e) $x = t-2$ $y = 3t^2 - 1$
	$t = x+2$
	$y = 3(x+2)^2 - 1$
	$= 3x^2 + 12x + 11$
(b) $\lim_{x \rightarrow 0} \frac{\sin \pi/4 x}{2x}$	(f) $\sin \theta \sec \theta = \tan \theta$
$= \lim_{x \rightarrow 0} \left(\frac{1}{2}\right) \left(\frac{\pi}{4}\right) \frac{\sin \pi/4 x}{\pi/4}$	LHS $\frac{\sin \theta}{\cos \theta} = \tan \theta$
$= \pi/8$	(ii) $\sin \theta \sec \theta = \sqrt{3}$
	$\tan \theta = \sqrt{3}$
	$\theta = \pi/3, 4\pi/3$

Student Name: Teacher Name:

(g) $P(x) = x^3 - 4x$	Q12
$P(-3) = -27 + 12$	(a) let $n=1$
$= -15$	LHS: $3^0 = 1$
	RHS: $1/2(3^1 - 1) = 1$
	$= \text{LHS.}$
(h) $\log_3 7 = m$	\therefore true for $n=1$
$\log_3 21 = \log_3 7 + \log_3 3$	let $n=k$.
$= m+1$	$1+3+9+\dots+3^{k-1} = 1/2(3^k - 1)$
(i) $\frac{d}{dx} \frac{uv' - uv'}{v^2}$	prove true for $n=k+1$
$u = 2x-4$ $v = x^{1/2}$	$1+3+9+\dots+3^{k-1} + 3^k = 1/2(3^{k+1} - 1)$
$u' = 2$ $v' = 1/2 x^{-1/2}$	
$\frac{dy}{dx} = \frac{2\sqrt{x} - \frac{x-2}{\sqrt{x}}}{x}$	LHS
$= \frac{2x - x + 2}{x\sqrt{x}}$	$= 1/2(3^k - 1) + 3^k$
$= \frac{x+2}{x\sqrt{x}}$	$= 1/2 3^k - 1/2 + 3^k$
	$= \frac{3}{2} 3^k - 1/2$
	$= 1/2(3^{k+1} - 1)$
	$= \text{RHS}$
<u>OR</u>	which is of the form
$y = (2x-4)(x^{-1/2})$	$1/2(3^n - 1)$ where $n=k+1$
$= (2x-4)\left(\frac{1}{2}x^{-3/2}\right) + (x^{-1/2})(2)$	\therefore true for \dots
$= \frac{(1/2)(2x-4)}{x\sqrt{x}} + \frac{2}{\sqrt{x}}$	
$= \frac{-x+2}{x\sqrt{x}} + \frac{2x}{x\sqrt{x}}$	
$= \frac{x+2}{x\sqrt{x}}$	

Q12 (c)

$$(BC)(BH) - (BE)^2$$

$$(8)(14) - (BE)^2$$

$$BE = 10.583 \dots$$

$$= 10.58$$

Q12 (d)

$$\sin \alpha = \frac{2}{3} \quad \sin \beta = \frac{3}{5}$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \left(\frac{2}{3}\right) \left(\frac{4}{5}\right) + \left(\frac{\sqrt{5}}{3}\right) \left(\frac{3}{5}\right)$$

$$= \frac{8}{15} + \frac{3\sqrt{5}}{15}$$

$$= \frac{8 + 3\sqrt{5}}{15}$$

Q12 (e)

$$u = x - 2$$

$$du = dx$$

$$x = u + 2$$

$$x = 3 \quad u = 1$$

$$x = 4 \quad u = 2$$

$$\int_4^3 \frac{x^2}{(x-2)^2} dx$$

$$= \int_2^1 \frac{(u+2)^2}{u^2} du$$

$$= \int_2^1 \left(1 + \frac{4}{u} + 4u^2\right) du$$

$$= \left[u + 4 \ln u - 4u^{-1} \right]_2^1$$

$$= 4 \ln 2 + 3$$

Q12

$$(b) AC^2 + AB^2 = BC^2$$

$$AC^2 + 1 = BC^2$$

$$AC^2 = BC^2 - 1$$

$$h = \tan 60 AC$$

$$= \sqrt{3} AC$$

$$\tan 15 = \frac{h}{BC}$$

$$h = \tan 15 BC$$

$$\sqrt{3} AC = \tan 15 BC$$

$$\left[\sqrt{3}(BC^2 - 1) \right] = \left[\tan 15 BC \right]^2$$

$$3(BC^2 - 1)^2 = \tan^2 15 BC^2$$

$$3BC^2 - 3 = \tan^2 15 BC^2$$

$$3(BC^2)^2 - 3 - \tan^2 15 BC^2 = 0$$

$$3(BC^2)^2 - 3 - \tan^2 15 BC^2 = 0$$

$$\left[BC^2 \right]^2 - 3 - \tan^2 15 BC^2 = 0$$

$$3 - \tan^2 15$$

$$\left[BC^2 \right]^2 = 1.0121 \dots$$

$$h = \tan 15 (BC)$$

$$= 0.271 \dots \text{ km}$$

$$= 271.2 \text{ m}$$

$$= 271 \text{ m}$$

Q13

(a) $1 + \cos^2 x + \cos^4 x + \cos^6 x + \dots$ ($0 < x < \frac{\pi}{2}$)
 $a = 1$ $r = \cos^2 x$

$$S_{\infty} = \frac{a}{1-r}$$

$$= \frac{1}{1 - \cos^2 x}$$

$$= \frac{1}{\sin^2 x}$$

$$= \operatorname{cosec}^2 x$$

(b) $\frac{d}{dx} \ln \left(\frac{x-1}{x+1} \right) = \frac{2}{(x-1)(x+1)}$

LHS $\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$ $u = \frac{x-1}{x+1}$
 $= \frac{-(x-1)}{(x+1)^2} + \frac{1}{x+1}$

$\frac{d}{dx} \ln \left(\frac{x-1}{x+1} \right) =$
 $= \frac{-x+1+x+1}{(x+1)^2}$
 $= \frac{2}{(x+1)^2}$

$\frac{d}{dx} \ln \left(\frac{x-1}{x+1} \right) = \frac{x+1}{x-1} \cdot \frac{2}{(x+1)^2}$
 $= \frac{2}{(x-1)(x+1)}$

= RHS.

(ii) $\int_2^3 \frac{4 dx}{(x-1)(x+1)} = 2 \left[\ln \frac{(x-1)}{(x+1)} \right]_2^3$
 $= 2 (\ln \frac{1}{2} - \ln \frac{1}{3})$
 $= 2 \ln \left(\frac{1/2}{1/3} \right) = 2 \ln \frac{3}{2}$

Q13

(c) $P(2ap, 2ap^2) \vee Q(2aq, 2aq^2)$ $y = x^2/2a$

(i) $m = p+q$

(ii) $y - 2ap^2 = (p+q)(x - 2ap)$ $x^2 = 2ay$
 $y = (p+q)x - 2ap(p+q) + 2ap^2$ focus = $(0, \frac{a}{2})$
 $y = (p+q)x - 2apq$
 $\frac{a}{2} = 0 - 2apq$
 $a = -4apq$
 $pq = -1/4$

(iii) mid pt $\left(\frac{2ap+2aq}{2}, \frac{2ap^2+2aq^2}{2} \right)$

$x = a(p+q)$

$y = a(p^2+q^2)$
 $y = a((p+q)^2 - 2pq)$
 $= a \left(\frac{x^2}{a^2} + \frac{1}{2} \right)$
 $y = \frac{x^2}{a} + \frac{a}{2}$

$y = \frac{2x^2 + a^2}{2a}$

$2ay = 2x^2 + a^2$

$2x^2 = 2ay - a^2$

which is of the form of a parabola.

$x^2 = a \left(y - \frac{a}{2} \right)$

$V = (0, \frac{a}{2})$ focal length = $\frac{a}{4}$

Q14 (iii) alternative answer

$$4 = 8 \sin 3t$$

$$\sin 3t = \frac{1}{2}$$

$$3t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \dots$$

$$t = \frac{\pi}{18}, \frac{5\pi}{18}, \dots$$

$$\dot{x} = 24 \cos 3t$$

$$\text{at } t = \frac{\pi}{18}$$

$$\dot{x} = 24 \cos 3\left(\frac{\pi}{18}\right)$$

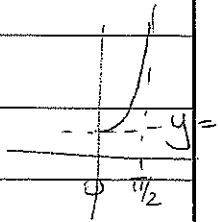
$$= 24 \cos \frac{\pi}{6}$$

$$= 12\sqrt{3} \text{ cm/s.}$$

Q14 (d)

(i) $f(x) = \sec x$ $0 \leq x \leq \frac{\pi}{2}$ range: $y \geq 1$

$f^{-1}(y)$ domain $y \geq 1$



(ii) $y = \frac{1}{\cos x}$

$$x = \frac{1}{\cos y}$$

$$\cos y = \frac{1}{x}$$

$$y = \cos^{-1} \frac{1}{x}$$

$$f^{-1}(x) = \cos^{-1} \frac{1}{x}$$

(iii) $f^{-1} x = \cos^{-1} \frac{1}{x}$

$$y = \cos^{-1} u$$

$$u = \frac{1}{x}$$

$$y' = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{-1}{x^2}$$

$$= \frac{1}{x^2 \sqrt{x^2-1}}$$

$$= \frac{1}{x^2 \sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{1}{x^2 \sqrt{\frac{x^2-1}{x^2}}}$$