Sydney Technical High School

**Name:** ..................................................

**Maths Class:** .....................

# Year 12

# Mathematics Extension 1

# Trial HSC

# August 2017

*Time allowed: 120 minutes (plus 5 minutes reading time)*

***General Instructions:***

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided at the rear of this Question Booklet, and may be removed at any time.

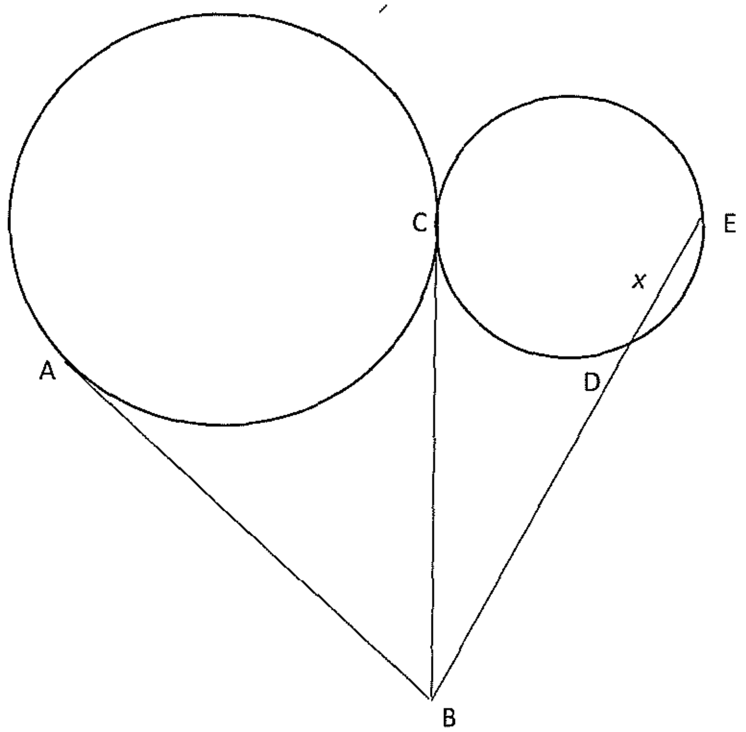
Section 1 Multiple Choice
Questions 1-10
10 Marks

Section II Questions 11-14
60 Marks

Total = 70 marks

## SECTION I – Multiple choice – 10 marks

Fill in the circle on your Multiple Choice answer sheet which corresponds to the correct answer

| | |
|---|---|
| 1. | <br>AB and BC are tangents to the larger circle, which touches the smaller circle at C.<br>DE = $x$<br>BD = 4DE<br><br>The length of AB is<br><br>A. $2\sqrt{5}x$ B. $3x$ C. $2x$ D. $x\sqrt{5}$ |
| 2 | The line joining A(2, 3) to B(5, -1) is divided **externally** by the point M in the ratio 2:3.<br>The point M has coordinates:<br><br>A. (-4, 11) B. (14, 11) C. $(\frac{16}{5}, \frac{23}{5})$ D. $(\frac{31}{2}, 5)$ |
| 3 | $\frac{d}{dx}\ln\left(\frac{2x+1}{3x+2}\right) =$<br><br>A. $\frac{2}{3}$ B. $\ln\left(\frac{2}{3}\right)$ C. $\frac{-1}{(2x+1)(3x+2)}$ D. $\frac{1}{(2x+1)(3x+2)}$ |

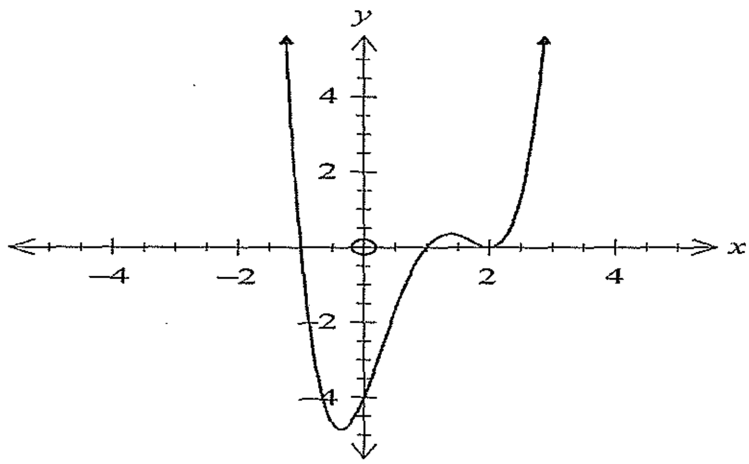
4 The acute angle between the line $x = 2$ and the line $\sqrt{3}y - x + 1 = 0$ is

A. $\frac{\pi}{6}$  B. $\frac{\pi}{3}$  C. $\pi$  D. $0^c$

5. If $\tan x = \frac{-1}{k}$ and $0 \leq x \leq \pi$ then $\sec x =$

A. $\frac{\sqrt{1+k^2}}{k}$  B. $\frac{-\sqrt{1+k^2}}{k}$  C. $\frac{k}{\sqrt{1+k^2}}$  D. $\frac{-k}{\sqrt{1+k^2}}$

6.




The Polynomial graphed above could be:

A. $P(x) = (2 - x)(x^2 - 1)$

B. $P(x) = (x - 2)(x^2 - 1)$

C. $P(x) = (2 - x)^2(x^2 - 1)$

D. $P(x) = (x^2 - 2)(x^2 - 1)$

| | |
|---|---|
| 7 | $\int \frac{dx}{\sqrt{9-25x^2}} =$ <br><br> A. $\frac{1}{3}sin^{-1}(5x)$ B. $\frac{1}{5}sin^{-1}(3x)$ C. $\frac{1}{5}sin^{-1}\left(\frac{5x}{3}\right)$ D. $\frac{1}{5}sin^{-1}\left(\frac{3x}{5}\right)$ |
| 8 | $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx =$ <br><br> A. 0 B. $\frac{-1}{24}$ C. $\frac{1}{24}$ D. $\frac{8-9\sqrt{3}}{24}$ |
| 9 | Simplify $\frac{1-\cos 2\theta}{1+cos2\theta}$ <br><br> A. $tan^2\theta$ B. $cot^2\theta$ C. $1 - tan^2\theta$ D. $1 - cot^2\theta$ |
| 10 | For what value of *x* is the ratio of its natural logarithm (*lnx*) to the number itself (*x*) a maximum? <br><br> A. 0 B. 1 C. *e* D. *ln x* |

***End of Section I***

## SECTION II - 60 Marks

Complete all answers in your answer booklets

Begin a new page for each new question.

## QUESTION 11: (15 Marks)

**Marks**

(a) Find **4**

(i) $\int \frac{1}{(2x+3)\sqrt{2x+3}}\,dx$ (ii) $\int sin^2 4x\,dx$

(b) Find the value(s) of $k$ for which the quadratic equation $x^2 - 3kx + (k+3) = 0$ has one root twice the other. **2**

(c) For $y = e^{x^3}$ find $\frac{d^2y}{dx^2}$, in factored form **2**

(d) Using the substitution $u = 1 - x^3$, or otherwise, find **2**

$$\int x^2(1 - x^3)^4\,dx$$

(e) If $\alpha$ and $\beta$ are the roots of the quadratic equation $3x^2 - x + 2 = 0$, **2**

find the quadratic equation with roots $1 - \alpha$ and $1 - \beta$

***QUESTION 11 continues overleaf.......***

***QUESTION 11 continued……***

(f) A body in a room with a constant temperature of $25^oC$ cools from $150^oC$ to $100^oC$ in 30 minutes.

(i) Given that the rate of cooling of the temperature (T) of this body is given by $\frac{dT}{dt} = -k(T - 25)$, show that $T = 25 + 125e^{-kt}$ is a solution to this expression, where $t$ is the time taken in minutes. **1**

(ii) Show that $k = \frac{1}{30}\ln\left(\frac{5}{3}\right)$ **1**

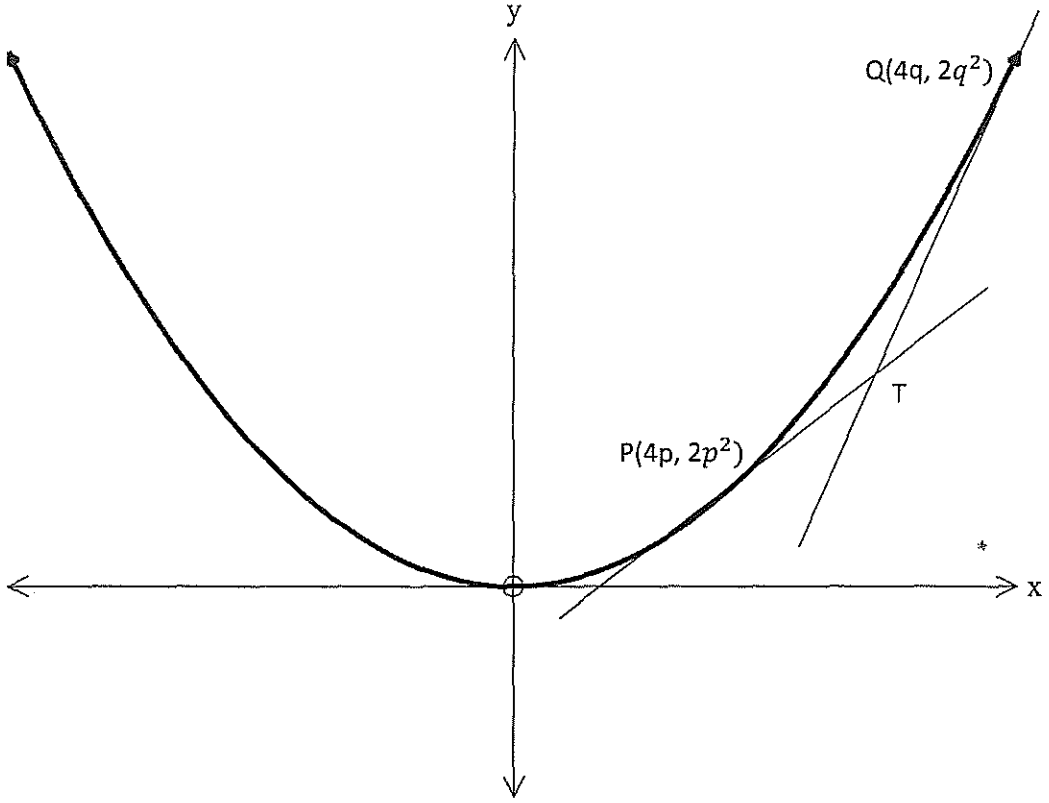
(iii) Find the temperature of the body after another 30 minutes. **1**

## *End of Question 11*

**QUESTION 12: (15 Marks)** ***Start a new page***

**Marks**

(a)




(i) Given the point P ($4p$, $2p^2$) lies on the parabola $8y = x^2$, prove that the equation of the tangent at the point P is given by **1**

$$y = px - 2p^2$$

(ii) If Q $(4q, 2q^2)$ also lies on the same arc of the parabola as P, find the equation of the chord PQ. **1**

(iii) PQ passes through the point (0, -4). **1**
Prove that $pq$ = 2.

(iv) T is the point of intersection of the tangents at P and Q. **2**
Find the equation of the locus of T.

(v) Find any restrictions on x in the locus of T. **1**

***QUESTION 12 continued overleaf.......***

*QUESTION 12 continued.......*

(b) A particle is moving with simple harmonic motion in a straight line. Its speed, $v$, when it is a distance $x$ from the centre of the oscillation is given by

$$v^2 = \pi^2(9 - x^2)$$

(i) What is the period of the motion? **1**

(ii) What is the maximum acceleration? **1**

(c) (i) Find $\frac{d}{dx}( sin^{-1}x \ + \ sin^{-1}\sqrt{1 - x^2} \ )$ **2**

(ii) Explain the meaning of the answer to part (i) above. **1**

(d) The Volume and the Surface Area of a sphere are given by the formulae:

$$V = \frac{4}{3}\pi r^3 \qquad \text{and } S = 4\pi r^2$$

(i) Show that $\frac{dV}{dt} = S\frac{dr}{dt}$ **1**

(ii) A spherical ball of radius 48 mm has its Volume changing at a rate equal to 6 times its Surface Area, while remaining spherical. (The rate is in $mm^3/sec$). Show that the rate of change of the radius is a constant. **1**

(iii) How many seconds does it take for the Volume to reduce to $\frac{1}{8}$ of its original? **2**

***End of Question 12***

## QUESTION 13: (15 Marks) *Start a new page*

**Marks**

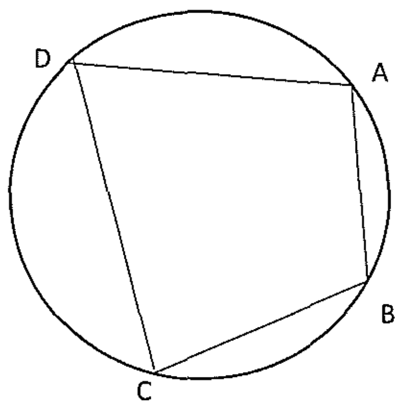
(a) A man walks at a speed of $\frac{16}{4+t}$ km per hour, after walking for t hours.

(i) How long does he walk for his speed to reduce to 3 kph? **1**

(ii) How far has he walked in that time? (give to 2 decimal places) **2**

(b) Prove that $\frac{d}{dx}\ln(secx) = tanx$ **2**

(c) The points A, B, C and D lie on the circumference of the circle below. **3**




Prove (giving all reasons) that tan A + tan B + tan C + tan D = 0

(d) Find $\frac{d}{dx}\ln(x+\sqrt{x^2+1})$ and hence find the value, in exact form, of **3**

$$\int_0^1 \frac{dx}{\sqrt{x^2+1}}$$

(e) Prove, by the process of Mathematical Induction, that $3^{2n}+7$ is divisible by 8 for all positive integral *n*. **4**

***End of Question 13***

**QUESTION 14: (15 Marks)** ***Start a new page***

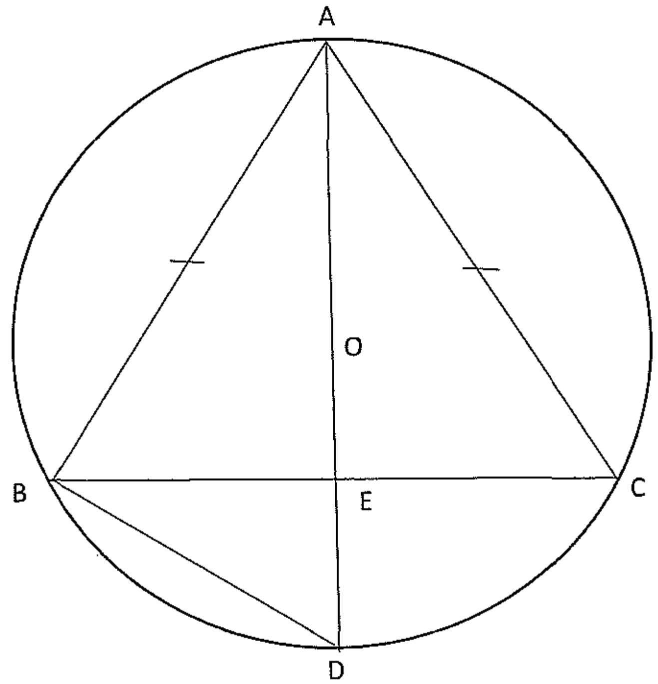
**Marks**

(a) If $\frac{dy}{dx} = 1 + y^2$ and at $x = \frac{\pi}{2}$, $y = 1$, show that $y = \tan(x - \frac{\pi}{4})$ **2**

(b) (i) If $f(x) = cosec\ x$ show that $f^{-1}(x) = sin^{-1}\left(\frac{1}{x}\right)$ **2**

(ii) Find the Domain of y =$f^{-1}(x)$ **1**

(c)




AD is a diameter of the circle, centre O
ΔABC is isosceles, with AC = AB

(i) Redraw the diagram onto your answer booklet (at least one third of a page)

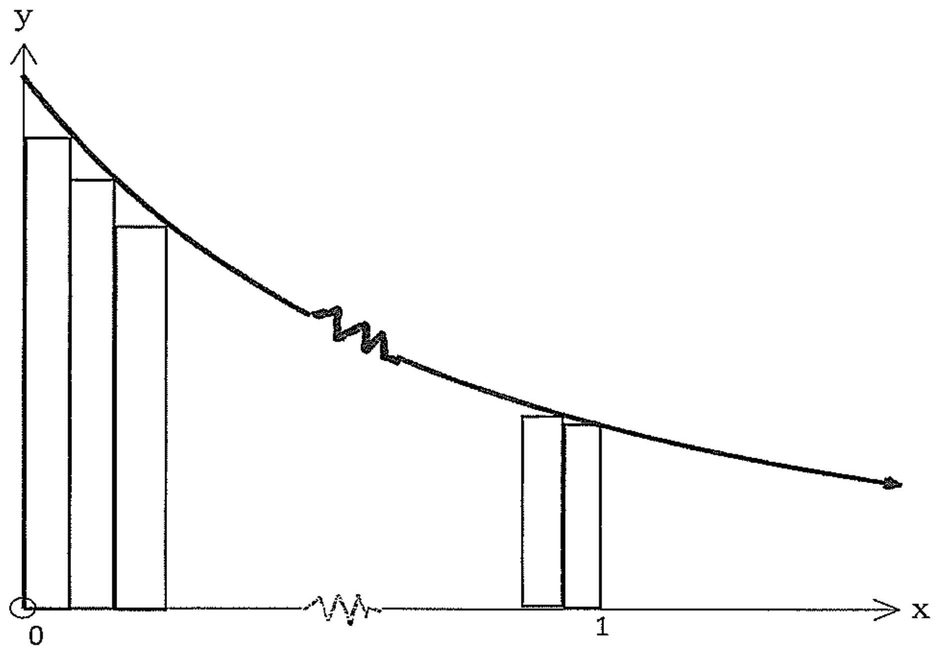
(ii) Let ∠ABC = $x$. Prove that ∠BDE = $x$ (give all reasons) **1**

(iii) Prove that BC is perpendicular to AD (give all reasons)
***(DO NOT use congruency or the properties of an isosceles triangle)*)** **2**

***QUESTION 14 continued overleaf.......***

***QUESTION 14 continued.......***

(d) The curve $y = \frac{1}{(x+1)^2}$ shown below, has *n* rectangles of equal width inscribed on it, between the values x = 0 and *x* = 1.
The height of the rectangles is determined by the *x* – value on the right of the rectangle, forming what are called *Lower Rectangles*.




(i) Give the area of the first rectangle on the left. **2**

(ii) Find a simplified expression for the area of the *kth* rectangle from the left **2**

(iii) Deduce that **3**

$$\lim_{n\to\infty} n\left\{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \;\dots\dots\; + \frac{1}{(2n)^2}\right\} = \frac{1}{2}$$

***End of Examination***

SOLUTIONS [EXT 1 2017]

Multiple Choice

(1) $EB \times BD = CB^2$

$5x \times 4x = CB^2$

$CB = 2x\sqrt{5}$

and $AB = CB = 2\sqrt{5}x$

$\therefore$ $\boxed{A}$

(2) $(2,3)$ $(5,-1)$

$-2 : 3$

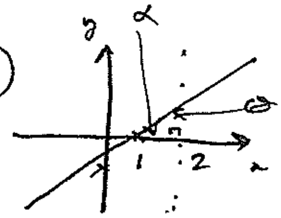
$\therefore M$ is $\left(\frac{-2\times 5 + 3\times 2}{1}, \frac{-2\times -1+9}{1}\right)$

$= (-4, 11)$ $\boxed{A}$

(3) $\frac{2}{2x+1} - \frac{3}{3x+2}$

$= \frac{1}{(2x+1)(3x+2)}$ $\boxed{D}$

(4)



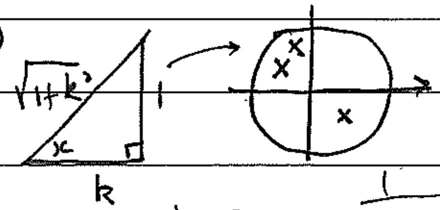

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \pi/6$

$\therefore \theta = \pi/3$

$\boxed{B}$

(5)




$\therefore \sec x = \frac{1}{\cos x}$

$= -\frac{\sqrt{1+k^2}}{k}$

$\boxed{B}$

(6) $\boxed{C}$

(7) $\int \frac{1/5}{\sqrt{9/25 - x^2}}\,dx = \frac{1}{5}\sin^{-1}\frac{x}{3/5}$

$= \frac{1}{5}\sin^{-1}\frac{5x}{3}$

$\boxed{C}$

(8) $\int_{\pi/3}^{\pi/2} \sin x \cos^2 x\,dx$

$= -\frac{1}{3}\cos^3 x\Big]_{\pi/3}^{\pi/2}$

$= \frac{1}{3}\left(\frac{1}{8}\right)$

$= \frac{1}{24}.$ $\boxed{C}$

(9) $\frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x - 1)} = \frac{2\sin^2 x}{2\cos^2 x}$

$= \tan^2 x$

$\boxed{A}$

(10) For max ratio $(r)$ $\frac{dr}{dx} = 0$

$\therefore \frac{d}{dx}\left(\frac{\ln x}{x}\right) = 0$

$\therefore \frac{x \cdot 1/x - \ln x}{x^2} = 0$

$\therefore \ln x = 1$

$\therefore x = e$ $\boxed{C}$

SECTION II

QUESTION 11:

(a) (i) $\int (2x+3)^{-3/2}\,dx$

$= -(2x+3)^{-1/2} + k$

$= \dfrac{-1}{\sqrt{2x+3}} + k$

(ii) $\cos 2A = 1 - 2\sin^2 A$

$\therefore \sin^2 A = \dfrac{1-\cos 2A}{2}$

$\therefore \int \sin^2 4x\,dx = \int \dfrac{1-\cos 8x}{2}\,dx$

$= \dfrac{x}{2} - \dfrac{1}{16}\sin 8x + k$

(b) Let the roots be $\alpha$ and $2\alpha$

Sum $3\alpha = 3k \Rightarrow \alpha = k$

Product $2\alpha^2 = k+3$

$\therefore 2k^2 - k - 3 = 0$

$(2k-3)(k+1) = 0 \Rightarrow k = -1$ or $k = 3/2$

(c) $\frac{d}{dx} e^{x^3} = 3x^2 e^{x^3}$

$\dfrac{d^2y}{dx^2} = e^{x^3} 6x + 3x^2 . 3x^2 e^{x^3}$

$= 3xe^{x^3}(2 + 3x^3 e^{x^3})$

(d) $u = 1 - x^3 \Rightarrow \dfrac{du}{dx} = -3x^2$

$\therefore dx = -du/3x^2$

$\therefore$ Integral $= \int x^2 . u^4 (-du/3x^2)$

$= -1/3 \int u^4\,du$

$= -1/15\, u^5 + k$

$= -1/15\,(1-x^3)^5 + k$

(e) Sum $= \alpha + \beta = 1/3$

Product $= \alpha\beta = 2/3$

Sum new $= 2 - (\alpha+\beta) = 5/3$

Product new $= 1 - (\alpha+\beta) + \alpha\beta$

$= 1 - 1/3 + 2/3$

$= 4/3$

$\therefore$ Q.E is

$x^2 - 5/3\,x + 4/3 = 0$

$\therefore 3x^2 - 5x + 4 = 0$

11.(f) $T = 25 + 125e^{-kt}$ (i). $\frac{dT}{dt} = -125ke^{-kt}$

$= -k(25 + 125e^{-kt} - 25)$

$= -k(T - 25)$

(ii) At $t = 30$, $T = 100$

$\therefore 75 = 125e^{-30k}$

$\therefore e^{-30k} = 3/5$

$\therefore -30k = \ln(3/5)$

$\therefore k = \frac{1}{30}\ln(5/3)$

(iii) At $t = 60$

$T = 25 + 125e^{-2\ln(5/3)}$

$= 25 + 125e^{\ln(9/25)}$

$= 25 + 125 \times 9/25$

$= 70°$.

QUESTION 12:

(i) $\frac{dy}{dx} = 2x/8 = x/4$

At P, $m_T = p$.

$\therefore$ Equation is $y - 2p^2 = p(x - 4p)$

$\therefore y - 2p^2 = px - 4p^2$

$y = px - 2p^2$

(ii) $m_{PQ} = \frac{2q^2 - 2p^2}{4q - 4p}$

$= \frac{p+q}{2}$

Equation PQ: $y - 2q^2 = \frac{p+q}{2}(x - 4q)$

$\therefore 2y - 4q^2 = (p+q)x - 4pq - 4q^2$

$\therefore 2y = (p+q)x - 4pq$

(iii) If PQ passes through $(0, -4)$

then $-8 = -4pq$

$\therefore pq = 2$.

Q 13 (a) (iv) $y = px - 2p^2$ (1)

$y = qx - 2q^2$ (2)

(1)−(2) $x(p-q) = 2(p^2-q^2)$

$\therefore x = 2(p+q)$

$\therefore y = 2p^2 + 2pq - 2p^2$

$\therefore y = 2pq$

$\therefore y = 4$ since $pq = 2$ from part (iii)

(v) $y = 4$ intersects the parabola at $(\sqrt{32}, 0)$ and $(-\sqrt{32}, 0)$

Since $T$ must lie OUTSIDE the parabola,

then $x > \sqrt{32}$ or $x < -\sqrt{32}$.

Question 12(b) $n = \pi$

(i) $\therefore T = 2\pi/\pi$

$= 2$

(ii) At max acceleration, $v = 0$

$\therefore x = \pm 3$ (positive for min, negative for max)

$a = \frac{d}{dx} \tfrac{1}{2} v^2$

$= -\pi^2 x$

$\therefore$ max acceleration is $3\pi^2$

(c) (i) $\frac{1}{\sqrt{1-x^2}} + \frac{\frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x)}{\sqrt{1-(1-x^2)}}$

$= \frac{1}{\sqrt{1-x^2}} - \frac{x}{x\sqrt{1-x^2}}$

$= 0$

(ii) this means that $\sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$ is a constant.

(d) (i) $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$= 4\pi r^2 \frac{dr}{dt}$

$= 8 \, dr/dt$

(ii) $\frac{dv}{dt} = 6S = S\,\frac{dr}{dt}$

$\therefore \frac{dr}{dt} = 6$

(iii) If the volume is $\frac{1}{8}$ of its original, the radius is $\frac{1}{2}$ of its original. (Since $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$)

Which means 48 mm becomes 24 mm, and since the rate of change of the radius is 6mm/sec (see part (ii)) then 24mm takes 4 seconds.

QUESTION 13:

(a) (i) $\frac{16}{4+t} = 3$

$\therefore 12+3t = 16$

$t = \frac{4}{3}$ hrs.

(ii) $\frac{dx}{dt} = \frac{16}{4+t}$

$\therefore x = 16\ln(4+t) + k$

At $x=0$, $t=0$

$\therefore -16\ln(4) = k$

$\therefore x = 16\ln(4+t) - 16\ln(4)$

At $t = \frac{4}{3}$

$x = 16\ln\left(\frac{16/3}{4}\right)$

$= 16\ln\left(\frac{16}{12}\right)$

$= 16\ln\left(\frac{4}{3}\right)$ km

$\approx 4.60$ km.

(b) $\sec x = \cos^{-1} x$

$\therefore \frac{d}{dx}\ln\sec x = \frac{d}{dx}(-\ln\cos x)$

$= -\frac{-\sin}{\cos x}$

$= \tan x$

OR $\frac{d}{dx}\sec x = -(\cos x)^{-2}(-\sin x)$

$= \frac{\sin x}{\cos^2 x}$

$\therefore \frac{d}{dx}\ln\sec x = \frac{\sin x/\cos^2 x}{\sec x} = \tan x$

(c) Since ABCD is cyclic,

$$\angle D + \angle B = 180^\circ$$

$$\therefore \tan(D+B) = \tan 180^\circ = \frac{\tan D + \tan B}{1 - \tan D \tan B}$$

$$\Rightarrow \tan D + \tan B = 0$$

Similarly $\tan A + \tan C = 0$

$$\therefore \tan A + \tan B + \tan C + \tan D = 0$$

(d) $$\frac{d}{dx}\left(x + \sqrt{x^2+1}\right) = 1 + \tfrac{1}{2}\cdot 2x\left(x^2+1\right)^{-1/2}$$

$$= 1 + \frac{x}{\sqrt{x^2+1}}$$

$$\therefore \frac{d}{dx} \ln\left(x + \sqrt{x^2+1}\right) = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

$$= \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}\left[x + \sqrt{x^2+1}\right]}$$

$$= \frac{1}{\sqrt{x^2+1}}$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{x^2+1}} = \left.\ln\left(x + \sqrt{x^2+1}\right)\right]_0^1$$

$$= \ln\left(1+\sqrt{2}\right) - \ln\left(\sqrt{1}\right)$$

$$= \ln\left(1+\sqrt{2}\right)$$

(e) For $n=1$, $3^2 + 7 = 16$ which is divisible by 8

$\therefore$ The formula is true for $n=1$

Assume it is true for $n=k$

i.e., $3^{2k} + 7 = 8m$ where $m \in \mathbb{J}$

For $n=k+1$

$$3^{2k+2} + 7 = 3^2\left(3^{2k}+7\right) + 7 - 63$$

$$= 9 \cdot 8m - 56$$

$$= 8(9m - 7)$$

$$= 8M \quad \text{where} \quad M \in \mathbb{J}$$

$\therefore$ If the statement is true for $n = k$,

it is also true for $n = k+1$

AND it is true for $n = 1$

$\therefore$ it is true for $n = 2$ and so on

ie true $\forall\ n \in \mathbb{Z}$.

QUESTION 14:

(a) $dy/dx = 1 + y^2$

$dx/dy = \dfrac{1}{1+y^2}$

$\therefore\ x = \tan^{-1} y + k$

At $x = \pi/2,\ y = 1$

$\therefore\ \pi/2 = \tan^{-1}(1) + k \Rightarrow k = \pi/4$

$\therefore\ x = \tan^{-1} y + \pi/4$

$\therefore\ y = \tan(x - \pi/4)$

(b) (i) $y = \operatorname{cosec} x \longrightarrow$ for Inverse $x = \operatorname{cosec} y$

$\therefore\ 1/x = \sin y$

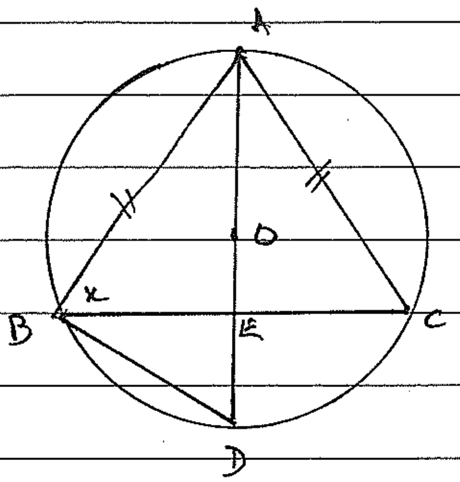
$\therefore\ y = \sin^{-1}(1/x)$

(ii) for Inverse sine normally $\mathcal{D}: -1 \leq x \leq 1$

In this case $-1 \leq 1/x \leq 1$

$\therefore\ x \leq -1$ or $x \geq 1$ for $\mathcal{D}_{f^{-1}}$

(c)




(ii) Since $\triangle ABC$ is isosceles $\angle ACB = x$

$\therefore \begin{cases} \angle ADB = \angle ACB = x \\ \angle BDE \end{cases}$ (angles at the circumference standing on arc AB)

(iii) Since AD is a diameter, $\angle ABD = 90°$ (angle in a semi-circle)

$\therefore \angle EBD = (90 - x)°$

$\therefore \angle BED = 90°$ (angle sum of $\triangle BED$)

(d) (i) width of each rectangle is $1/n$

Height of first rectangle is $f(1/n) = \dfrac{1}{(1/n + 1)^2}$

$\therefore$ Area $= 1/n \left(\dfrac{1}{(1/n + 1)^2}\right)$

$= \dfrac{1}{n} \dfrac{n^2}{(1+n)^2}$

$= n/(1+n)^2$

(ii) The height of the $k^{th}$ rectangle is $\dfrac{1}{(k/n + 1)^2}$

$\therefore$ Area $= 1/n \left[\dfrac{1}{(k/n + 1)^2}\right] = \dfrac{n}{(k+n)^2}$

(iii) The sum of the area of all rectangle is

$\dfrac{n}{(n+1)^2} + \dfrac{n}{(n+2)^2} + \dots + \dfrac{n}{(n+k)^2} + \dots + \dfrac{1}{(2n)^2}$

As $n \to \infty$, this area becomes the exact area under $y = 1/(x+1)^2$

$\therefore \lim\limits_{n \to \infty} n \left\{ \dfrac{1}{(n+1)^2} + \dots + \dfrac{1}{(2n)^2} \right\} = \int_0 \dfrac{1}{(x+1)^2} \, dx$

$= 1/2$