ABBOTSLEIGH

TRIAL HIGHER SCHOOL CERTIFICATE

1999

MATHEMATICS

4 UNIT

Time allowed: 3 hours

- **All questions may be attempted**
- **Answer each question in a separate booklet**
- **All questions are of equal value**
- **Approved calculators may be used**

Question One — **Marks**

(a) Evaluate $\displaystyle\int_0^1 \frac{2dx}{\sqrt{2-x^2}}$ — 2

(b) Find $\displaystyle\int xe^{-x}dx$ — 2

(c) Find $\displaystyle\int \sin^5 x\cos^2 x dx$ — 3

(d) Using the substitution $x=\frac{1}{u}$, where $u>0$ find $\displaystyle\int \frac{dx}{x\sqrt{x^2+1}}$ — 3

(e) Find $\displaystyle\int \sin(\log x)dx$ — 5

Question Two

Marks

(a) Let $z = \dfrac{-i}{1+i\sqrt{3}}$ — 2

(i) Sketch z on the Argand diagram.

(ii) Find the modulus and argument of z

(b) Let $A = 1 + 2i$ and $B = -3 + 4i$ — 3

Draw sketches to show the loci satisfied on the Argand diagram by

(i) $|z - A| = |B|$

(ii) $|z - A| = |z - B|$

(iii) $\arg(z - A) = \dfrac{\pi}{4}$

(c) (i) Solve the equation $z^4 = 1$ — 3

(ii) Hence find all solutions of the equation $z^4 = (z-1)^4$

(d) Use De Moivre's Theorem to express $\cos 4\theta$ in terms of $\cos\theta$ — 3

(e) Express the roots of the equation $z^2 + 2(1+2i)z - (11+2i) = 0$ in the form $a + ib$ where a and b are real. — 2

(f) Draw Argand diagrams to represent the following regions — 2

(i) $1 \le |z + 3 - 2i| \le 3$

(ii) $\dfrac{\pi}{6} \le \arg z \le \dfrac{\pi}{3}$

Question Three

(a) Make neat sketches of the following graphs, labelling any important features. — 4

(i) $y = \sin^2 2x$ for $-2\pi \le x \le 2\pi$

(ii) $|x| - |y| = 1$

(b) (i) Express $\dfrac{3x+1}{(x+1)(x^2+1)}$ in the form $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ — 4

(ii) Hence find $\displaystyle\int \frac{3x+1}{(x+1)(x^2+1)}\,dx$

Question Three (continued) — Marks

(c) Given $I_n = \int_0^{\pi/2} \cos^n x\,dx$ where n is a positive integer — 4

(i) Prove that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$

(ii) Hence evaluate I_4

(d) (i) Show that $1+i$ is a zero of the polynomial — 3
$P(x) = x^3 + x^2 - 4x + 6$

(ii) Express P(x) as a product of irreducible factors over the set of real numbers.

Question Four

(a) (i) Show that the tangent to the ellipse $\frac{x^2}{12} + \frac{y^2}{4} = 1$ at the point P(3, 1) has equation $x + y = 4$. — 8

(ii) If this tangent cuts the directrix in the fourth quadrant at the point T, and S is the corresponding focus, show that SP and ST are at right angles to each other.

(b) (i) Show that the tangent to the rectangular hyperbola $xy = c^2$ at the point $T(ct, \frac{c}{t})$ has equation $x + t^2y = 2ct$. — 7

(ii) The tangents to the rectangular hyperbola $xy = c^2$ at the points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$, where pq = 1, intersect at R.
Find the equation of the locus of R and state any restrictions on the values of x for this locus.

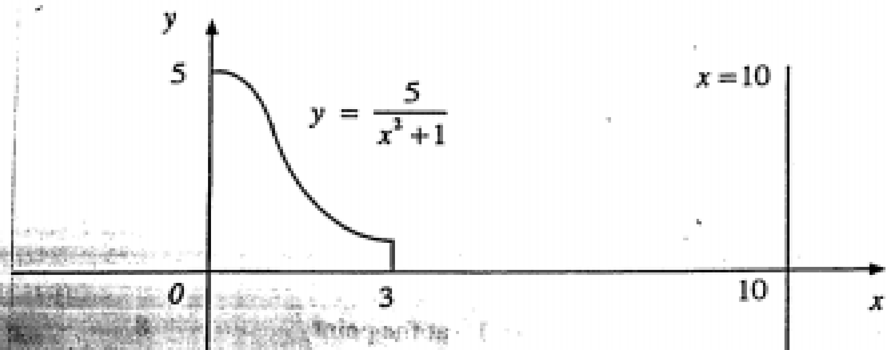
Question Five **Marks**

(a) (i) Sketch the curve $y = \sin^{-1}x$ 5

(ii) Find the volume of the solid generated by rotating the region bounded by the curve $y = \sin^{-1}x$, the x-axis and the ordinate $x = 1$ about the y-axis. Use the method of slices.

(b) The base of a solid is the circle $x^2 + y^2 = 25$. Find the volume of the solid if every section perpendicular to the x-axis is a semi-circle whose diameter lies in the base of the solid. 5

(c) 5



The region bounded by the curve $y = \dfrac{5}{x^2+1}$, the x-axis and the lines $x = 0$ and $x = 3$ is rotated about the line $x = 10$.

(i) Use the method of cylindrical shells to show that the volume V cm^3 is given by $V = \displaystyle\int_0^3 \frac{100\pi - 10\pi x}{x^2+1}\,dx$

(ii) Hence find the volume V to the nearest cm^3.

Question Six

Marks

(a) The equation $x^4 + 4x^3 - 3x^2 - 4x - 2 = 0$ has roots α, β, γ, δ. **3**
Find the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$.

(b) Solve $x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2 = 0$ if it has a root of multiplicity 4. **6**

(c) The chord of contact of the point $T(x_0, y_0)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the axes at M and N. If the mid-point of MN lies on the circle $x^2 + y^2 = 1$ what is the locus of T? **6**

Question Seven

(a) A railway line has been constructed around a circular curve of radius 500 m. The distance between the rails is 1.5 m and the outside rail is 0.1 m above the inside rail. Find the speed that eliminates a sideways force on the wheels for a train on this curve. (Take $g = 9.8\ ms^{-2}$.) **4**

(b) A particle of mass m is set in motion with speed u. Subsequently the only force acting upon the particle directly opposes its motion and is of magnitude $mk(1 + v^2)$ where k is a constant and v is its speed at time t. **6**

(i) Show that the particle is brought to rest after a time $\frac{1}{k}\tan^{-1} u$.

(ii) Find an expression for the distance travelled by the particle in this time.

(c) In Δ ABC, AB = AC. The bisector of ∠ ABC meets AC at M. The circle through A, B and M cuts BC at Q. Show, with reasons that AM = CQ. **5**

Question Eight

Marks

(a) The tangent at a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts the x-axis at M, while the normal cuts the x-axis at N. Prove that $OM.ON = a^2e^2$. **5**

(b) A particle is projected from the origin with initial velocity U to pass through a point (a,b). Prove that there are two possible trajectories if
$(U^2 - gb)^2 > g^2(a^2 + b^2)$ **5**

(c) A cone is placed with its vertex upward. A light string of length l metres is attached at one end to the vertex and the other end to a particle of mass m kg, which is made to describe a circle of uniform angular velocity ω in contact with the cone. Assume there is no friction on the cone's surface. Find the tension in the string, and the normal reaction of the surface. Hence, find the condition for this to happen. **5**

End of paper

ABBOTSLEIGH 4UNIT TRIAL HSC 1999 SOLUTIONS

Question One (15 marks)

(a) $\int_0^1 \frac{2\,dx}{\sqrt{2-x^2}} = 2\int_0^1 \frac{dx}{\sqrt{2-x^2}}$

$= 2\left[\sin^{-1}\frac{x}{\sqrt2}\right]_0^1$ (✓)

$= 2\left(\sin^{-1}\frac{1}{\sqrt2} - \sin^{-1}0\right)$ (½)

$= 2.\frac{\pi}{4} - 0$

$= \frac{\pi}{2}$ (½)

(b) $\int xe^{-x}\,dx$

$u = x$, $\frac{du}{dx} = 1$, $\frac{dv}{dx} = e^{-x}$, $v = -e^{-x}$

$= uv - \int v\frac{du}{dx}\,dx$

$= -xe^{-x} - \int -e^{-x}.1\,dx$ (✓)

$= -xe^{-x} + -e^{-x} + C$

$= -xe^{-x} - e^{-x} + C$ (✓)

$= -e^{-x}(x+1) + C$

(c) $\int \sin^3 x\cos^2 x\,dx$

$= \int (1-\cos^2 x)^2 \sin x\cos^2 x\,dx$ (✓)

$= \int (1+\cos^4 x - 2\cos^2 x)\cos^2 x\sin x\,dx$

$= \int (\cos^2 x + \cos^6 x - 2\cos^4 x)\sin x\,dx$ (✓)

$= -\frac{\cos^3 x}{3} - \frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} + C$ (✓)

(d) $\int \frac{dx}{x\sqrt{x^2+1}}$

$x = \frac{1}{u}$, $\frac{dx}{du} = -\frac{1}{u^2}$

$= \int \frac{-\,du/u^2}{\frac{1}{u}\sqrt{\frac{1}{u^2}+1}}$ (✓) $u>0$

$= \int -\frac{du}{u^2} \times \frac{u.u}{\sqrt{1+u^2}}$

$= -\int \frac{du}{\sqrt{1+u^2}}$ (✓)

$= -\log\left(u + \sqrt{1+u^2}\right) + C$ (½)

$= -\log\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right) + C$ (½)

(e) $I = \int \sin(\log x)\,dx$

$= \int \sin(\log x)\cdot\frac{d}{dx}(x)\,dx$ (✓)

$= \sin(\log x).x - \int x\cos(\log x)\frac{1}{x}\,dx$ (✓)

$= x\sin(\log x) - \int \cos(\log x)\frac{d}{dx}(x).\,dx$

$= x\sin(\log x) - \left[\cos(\log x).x - \int x\left[-\sin(\log x).\frac{1}{x}\right]dx\right]$ (✓)

$= x\sin(\log x) - x\cos(\log x) - \int \sin(\log x)\,dx$ (½)

$\therefore 2I = x(\sin(\log x) - \cos(\log x))$ (✓)

$\therefore I = \frac{1}{2}x\left(\sin(\log x) - \cos(\log x)\right) + C$ (½)

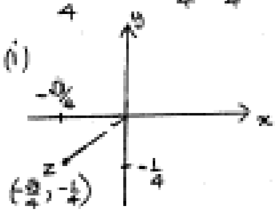
Question Two (15 marks)

(a) $z = \frac{-i}{1+i\sqrt3} \times \frac{1-i\sqrt3}{1-i\sqrt3}$

$= \frac{-i + i^2\sqrt3}{1 - i^2 3}$

$= \frac{-i-\sqrt3}{4} = -\frac{\sqrt3}{4} - \frac{1}{4}i$ (½)

(i)



(½)

(ii) modulus $= \sqrt{3/16 + 1/16} = \frac{1}{2}$ (½)

$\tan\theta = \frac{1/4}{\sqrt3/4} = \frac{1}{\sqrt3}$

$\therefore \theta = 30°$

$\therefore \arg z = -150°$ (½)

(b) $A = 1+2i$, $B = -3+4i$

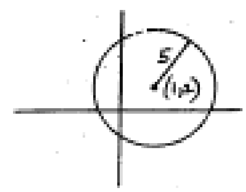
(i) $|z - A| = |B|$

$|z - (1+2i)| = \sqrt{9+16}$

$|z - (1+2i)| = 5$

represents a circle centre $(1,2)$, radius $= 5$ (½)

(b) (continued)



(½)

(ii) $|z - A| = |z - B|$

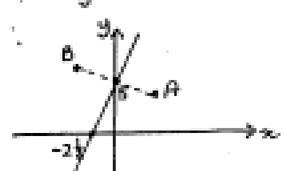
Locus is the perpendicular bisector of AB

OR Let $z = x+iy$ then

$(x-1)^2 + (y-2)^2 = (x+3)^2 + (y-4)^2$

Expand + collect like terms

$2x - y + 5 = 0$



(iii)

Question Two (continued)

(c)(i) $z^4-1=0$

$(z^2-1)(z^2+1)=0$

$(z-1)(z+1)(z-i)(z+i)=0$

$\therefore z=\pm1, \pm i$ (1)

(ii) $z^4=(z-1)^4$

$\left(\frac{z}{z-1}\right)^4=1$

From (i) $\frac{z}{z-1}=\pm1, \pm i$

Consider all solutions

If $\frac{z}{z-1}=1$, $z=z-1$ $\therefore$ no solutions (½)

If $\frac{z}{z-1}=-1$, $z=-z+1$ $\therefore z=\frac{1}{2}$ (½)

If $\frac{z}{z-1}=i$ $\therefore z=\frac{-i}{1-i}$ (½)

If $\frac{z}{z-1}=-i$ then $z=\frac{i}{1+i}$ (½)

(d) $\cos4\theta+i\sin4\theta=(\cos\theta+i\sin\theta)^4$ by De Moivre's Theorem. (½)

$=(c+is)^4$

$=c^4+4c^3is+6c^2i^2s^2+4ci^3s^3+i^4s^4$ (✓)

$=c^4-6c^2s^2+s^4+i(4c^3s-4cs^3)$

Equate real parts

$\cos4\theta=c^4-6c^2s^2+s^4$ (½)

$=c^4-6c^2(1-c^2)+(1-c^2)^2$

$=8\cos^4\theta-8\cos^2\theta+1$ (✓)

(e) $z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{(2-4i)\pm\sqrt{4(1+2i)^2+4(11+2i)}}{2}$

$=\frac{2(-1-2i)\pm2\sqrt{1-4+4i+11+2i}}{2}$

$=(-1-2i)\pm\sqrt{8+6i}$ (✓)

If $(a+ib)^2=8+6i$

then $a^2-b^2=8$

and $2iab=6i$

$\therefore ab=3$

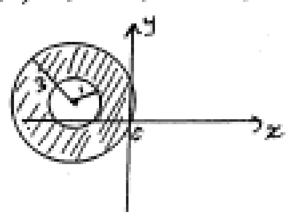
$\therefore a=3, b=1$

$\therefore \sqrt{8+6i}=3+i$ (✓)

$\therefore z=(-1-2i)\pm3+i$

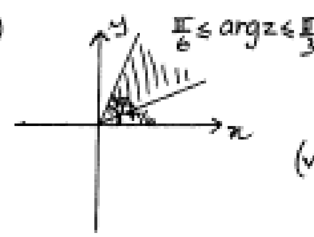
$=(2-i)$ or $(-4-3i)$

(f) (i) $1\le|z+3-2i|\le3$



(✓)

(ii)

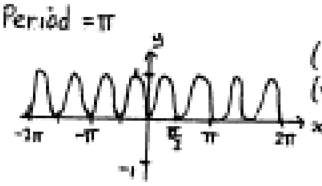


(✓)

Question Three (15 marks)

(a) (i) $y=\sin^2 2x$ for $-2\pi\le x\le2\pi$

Period $=\pi$

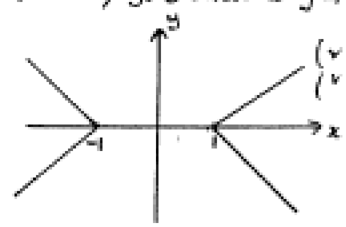


(✓) (✓)

(ii) $|x|-|y|=1$

If (x,y) lies on the curve then so does $(-x,y)$, $(x,-y)$ and $(-x,-y)$

If $x>0$, $y>0$ then $x-y=1$



(✓) (✓)

(b)(i) $\frac{3x+1}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$

$A(x^2+1)+(Bx+C)(x+1)=3x+1$

When $x=-1$ $A=-1$

When $x=0$ $A+C=1$

$\therefore C=2$

Equate coefficients of x^2

$\therefore A+B=0$ $\therefore B=1$

$\therefore \frac{3x+1}{(x+1)(x^2+1)}=\frac{-1}{x+1}+\frac{x+2}{x^2+1}$ (✓)

(ii) $I=-\int\frac{1}{x+1}+\int\frac{x+2}{x^2+1}$

$=-\ln(x+1)+\frac{1}{2}\ln(x^2+1)+2\tan^{-1}x$ (✓)

Question Three (continued)

(c) $I_n = \int_0^{\pi/2} \cos^n x \, dx$

(i)

$I_n = \int_0^{\pi/2} \cos^{n-1} x \frac{d}{dx}(\sin x)\, dx$ (½)

(provided $n \geq 1$)

$= \left[\sin x \cos^{n-1} x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x (n-1)\cos^{n-2} x(-\sin x)\, dx$

(provided $n \geq 2$) (✓)

$= (n-1)\int_0^{\pi/2} \cos^{n-2} x \sin^2 x \, dx$

$= (n-1)\int_0^{\pi/2} \cos^{n-2}(1-\cos^2 x)\, dx$ (½)

$= (n-1) I_{n-2} - (n-1) I_n$ (✓)

$\therefore n I_n = (n-1) I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$

(ii) $I_4 = \frac{3}{4} I_2$ (½)

$I_2 = \frac{1}{2} I_0$

$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$

$\therefore I_2 = \frac{\pi}{4}$

$I_4 = \frac{3}{4} \cdot \frac{\pi}{4} = \frac{3\pi}{16}$ (½)

(d) (i)

$P(1+i) = (1+i)^3 + (1+i)^2 - 4(1+i) + 6$

$= 1 + 3i - 3 - i + 1 + 2i - 1 - 4 - 4i + 6$

$= 0$

$\therefore (1+i)$ is a zero of $P(x)$ (✓)

(ii) Since $P(x)$ has real coefficients, the conjugate of $(1+i)$ i.e. $(1-i)$ is also a zero of $P(x)$ (✓)

Let the other zero be α

Sum of zeros

$(1+i) + (1-i) + \alpha = -1$

$\alpha = -3$

Question Three (d) (continued)

$\therefore P(x) = [x-(1+i)][x-(1-i)](x+3)$

$= (x^2 - 2x + 2)(x+3)$ (✓)

Question Four (15 marks)

(a) (i) $\frac{x^2}{12} + \frac{y^2}{4} = 1$

$\frac{2x}{12} + \frac{2y}{4}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-x}{3y}$ (✓)

At $P(3,1)$, $\frac{dy}{dx} = \frac{-3}{3} = -1$ (✓)

$\therefore y - 1 = -1(x-3)$ (✓)

$\therefore x + y = 4$ (✓)

(ii) $e = \sqrt{1 - \frac{4}{12}} = \frac{\sqrt{2}}{\sqrt{3}}$ (✓)

$\therefore$ focus is $\left(2\sqrt{3} \cdot \frac{\sqrt{2}}{\sqrt{3}}, 0\right) = (2\sqrt{2}, 0)$

and the directrix is

$x = \frac{2\sqrt{3}}{\sqrt{2}/\sqrt{3}} = 3\sqrt{2}$ (✓)

$\therefore$ at T, $x + y = 4$

and $x = 3\sqrt{2}$

$\therefore y = 4 - 3\sqrt{2}$ (✓)

$\therefore P(3,1)$, $S(2\sqrt{2}, 0)$, $T(3\sqrt{2}, 4-3\sqrt{2})$ (✓)

$\therefore$ gradient SP $\times$ gradient ST

$= \frac{1}{3-2\sqrt{2}} \times \frac{4-3\sqrt{2}}{\sqrt{2}} = \frac{4-3\sqrt{2}}{3\sqrt{2}-4}$ (✓)

$= -1$

(5)

$\therefore$ SP and ST are at $90°$ to each other.

(b) (i) $xy = c^2$

$\therefore y = \frac{c^2}{x}$

$\therefore \frac{dy}{dx} = \frac{-c^2}{x^2}$ (✓)

$\therefore$ at $T\left(ct, \frac{c}{t}\right)$, $\frac{dy}{dx} = \frac{-c^2}{(ct)^2}$

$= -\frac{1}{t^2}$ (✓)

$\therefore$ tangent at $T(ct, c/t)$ has gradient $-\frac{1}{t^2}$ and equation

$y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$

$t^2 y - ct = -x + ct$ (✓)

$x + t^2 y = 2ct$

(3)

(ii) Tangent at $P\left(cp, \frac{c}{p}\right)$ and at $Q\left(cq, \frac{c}{q}\right)$ are

$x + p^2 y = 2cp$ — (1) (✓)

$x + q^2 y = 2cq$ — (2)

Solve (1) and (2) simultaneously

(1) − (2)

$(p^2 - q^2) y = 2c(p-q)$

$\therefore y = \frac{2c}{p+q}$ (✓)

Sub in (1)

$x + p^2\left(\frac{2c}{p+q}\right) = 2cp$

$\therefore x = \frac{2cpq}{p+q}$ (✓)

$\therefore R(x,y)$ is the point

$R\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

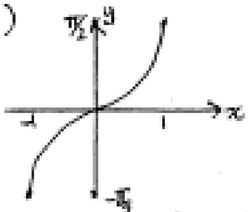
$= R\left(\frac{2c}{p+q}, \frac{2c}{p+q}\right)$

Since $pq = 1$ $\therefore$ Locus of R lies on the line $y = x$ (✓)

where $-c < x < 0$, $0 < x < c$ since $pq = 1$. $x \neq 0$.

Question Five (15 marks)

(a)(i)



(✓)

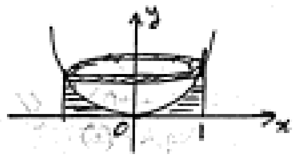
(ii) Consider a hollow disc perpendicular to the y-axis, of thickness δy radii x and 1

The volume of the disc is (✓)

$\delta V = \pi(1-x^2)\delta y$

$= \pi(1-\sin^2 y)\delta y$

$= \pi\cos^2 y\,\delta y$ (✓)



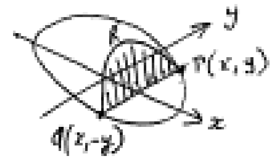
The volume of the solid

$V = \lim_{\delta y\to 0}\sum_{y=0}^{\pi/2}\pi\cos^2 y\,\delta y = \int_0^{\pi/2}\pi\cos^2 y\,dy$

$= \int_0^{\pi/2}\pi\left(\frac{1+\cos 2y}{2}\right)dy$ (✓)

$= \pi\left[\frac{y}{2}+\frac{\sin 2y}{4}\right]_0^{\pi/2} = \frac{\pi^2}{4}$ (✓)

(b) The base of the solid lies in the plane of the paper. The boundary of the solid is along the circle $x^2+y^2=25$



The cross section (is a semi-circle) PRQP at a distance x from the origin, at right angles to the x-axis is shown above. (½)

$PQ = 2y$

Radius of circle PRQ is y

$\therefore A(x) =$ area cross section $= \frac{1}{2}\pi y^2$ (✓)

Now $x^2+y^2=25$

$\therefore y^2 = 25-x^2$

$\therefore A(x) = \frac{\pi}{2}(25-x^2)$ (✓)

The volume of the solid is

$V = \int_{-5}^{5} A(x)\,dx$

$= \int_{-5}^{5}\frac{\pi}{2}(25-x^2)\,dx$ (✓)

$= \int_0^5 \pi(25-x^2)\,dx$

$= \pi\left[25x-\frac{x^3}{3}\right]_0^5$ (✓)

$= \pi\left(125-\frac{125}{3}\right)$

$= \frac{250}{3}\pi$ cubic units (½)

(5)

(c)(i) Take strips of thickness δx parallel to the y-axis

Volume of shell is

$\delta V \doteqdot 2\pi(10-x)\,y\,\delta x$ (✓)

$\therefore V = \lim_{\delta x\to 0}\sum 2\pi(10-x)\,y\,\delta x$ (✓)

$= \int_0^3 2\pi(10-x)\frac{5}{x^2+1}\,dx$ (✓)

$= \int_0^3 \frac{100\pi-10\pi x}{x^2+1}\,dx$

(ii) $V = 100\pi\int_0^3\frac{dx}{x^2+1} - 5\pi\int_0^3\frac{2x\,dx}{x^2+1}$ (✓)

$= 100\pi\left[\tan^{-1}x\right]_0^3 - 5\pi\left[\ln(x^2+1)\right]_0^3$

$= 100\pi\tan^{-1}3 - 5\pi\ln 10$ (✓)

$= 356\text{ cm}^3$

Question Six (15 marks)

(a) $x^4+4x^3-3x^2-4x-2=0$

$\alpha, \beta, \gamma, \delta$.

New equation has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$

New equation is (✓)

$\left(\frac{1}{x}\right)^4 + 4\left(\frac{1}{x}\right)^3 - 3\left(\frac{1}{x}\right)^2 - 4\left(\frac{1}{x}\right) - 2 =$ (✓)

$1+4x-3x^2-4x^3-2x^4=0$ (✓)

i.e. $2x^4+4x^3+3x^2-4x-1=$

(b) $P(x) = x^5+2x^4-2x^3-8x^2-7x-2$

$P'(x) = 5x^4+8x^3-6x^2-16x-7$

$P''(x) = 20x^3+24x^2-12x-16$

$P'''(x) = 60x^2+48x-12$ (✓)

$12(5x^2+4x-1)=0$ (✓)

$12(5x-1)(x+1)=0$

$x = \frac{1}{5}, -1$ (✓)

estion Six (continued)

$P(-1)=0$ ∴ $x=-1$ is a (✓) root of multiplicity 4 (✓)

∴ $P(x)=(x+1)^4(x-2)=0$

∴ $x=-1$ or 2 (✓)

) The equation of the chord of contact is

$\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$ (✓)

This chord meets the x-axis at $M(\frac{a^2}{x_0},0)$ and the y-axis at $N(0,\frac{-b^2}{y_0})$ (✓)

The co-ordinates of the midpoint of MN are

$x=\frac{a^2}{2x_0}$, $y=\frac{-b^2}{2y_0}$ (✓)

iven that $x^2+y^2=1$

ence $\left(\frac{a^2}{2x_0}\right)^2+\left(\frac{-b^2}{2y_0}\right)^2=1$ (✓)

he locus of T is

$\frac{a^4}{x_0^2}+\frac{b^4}{4y_0^2}=1$ (✓)

$a^4y_0^2+b^4x_0^2=4x_0^2y_0^2$

$a^4y^2+b^4x^2=4x^2y^2$ (✓)

stion Seven (15 marks)

1, $d=1.5$, $r=500$

y $T\cos\theta=mg$ —(1) (✓)

lly $T\sin\theta=\frac{mv^2}{r}$ —(2)

(2)÷(1) $\frac{v^2}{rg}=\tan\theta$ (✓)

But $\tan\theta=\frac{h}{\sqrt{d^2-h^2}}$

∴ $v^2=\frac{rgh}{\sqrt{d^2-h^2}}$ (✓)

∴ $v^2=\frac{500\times 9.8\times 0.1}{\sqrt{1.5^2-0.1^2}}$ (✓)

∴ $v=18.1\ ms^{-1}$

(b) $\ddot{x}=-k(1+v^2)$

When $t=0$, $x=0$, $v=u$

$\frac{dv}{dt}=-k(1+v^2)$ (✓)

$\frac{dv}{1+v^2}=-k\,dt$

$\tan^{-1}v=-kt+C$ (✓)

When $t=0$, $v=u$ ∴ $C=\tan^{-1}u$

∴ $t=\frac{1}{k}(\tan^{-1}u-\tan^{-1}v)$ (✓)

As the particle is brought to rest, its velocity is zero.

∴ $v=0$, ∴ $t=\frac{1}{k}\tan^{-1}u$

(ii) $\ddot{x}=-k(1+v^2)$

$v\frac{dv}{dx}=-k(1+v^2)$ (✓)

$\frac{v\,dv}{1+v^2}=-k\,dx$

$\frac{1}{2}\ln(1+v^2)=-kx+C$ (✓)

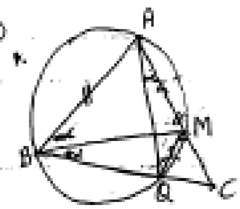
When $x=0$, $v=u$ ∴ $C=\frac{1}{2}\ln(1+u^2)$

$x=\frac{1}{2k}\left(\ln(1+u^2)-\ln(1+v^2)\right)$

$=\frac{1}{2k}\ln\left(\frac{1+u^2}{1+v^2}\right)$ (✓)

When $v=0$, $x=\frac{1}{2k}\ln(1+u^2)$ is the distance travelled

(c)



In △ABC, AB = AC. The bisector of $A\hat{B}C$ meets AC at M. The circle through A, B and M cuts BC at Q. Let $A\hat{B}C=2\alpha$. Then $A\hat{B}M=M\hat{B}C=\alpha$

∴ AM = MQ (Since arc AM = arc MQ) (✓)

Now $M\hat{B}Q=Q\hat{A}M$ (angles standing on same arc QM. ∴ $Q\hat{A}M=\alpha$. (✓)

Now △AMQ is isosceles since AM = QM

∴ $A\hat{M}Q=180-2\alpha$ (angle sum of △) (✓)

∴ $Q\hat{M}C=2\alpha=A\hat{C}B$

Hence △CMQ is isosceles and QM = CQ (✓)

∴ AM = CQ

Question Eight (15 marks)

(a) Let P be $(a\cos\theta, b\sin\theta)$

The equation of the tangent to the ellipse at P is $\frac{\cos\theta\, x}{a} + \frac{\sin\theta\, y}{b} = 1$ (✓)

The tangent meets the x-axis at $y=0$, $x = \frac{a}{\cos\theta}$

∴ The coordinates of M are $\left(\frac{a}{\cos\theta}, 0\right)$ (✓)

The equation of the normal to the ellipse at P is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad (✓)$$

The co-ordinates of N are $\left(\frac{(a^2-b^2)\cos\theta}{a}, 0\right)$ (✓)

$$\therefore OM.ON = \frac{a}{\cos\theta} \times \frac{(a^2-b^2)\cos\theta}{a} \quad (✓)$$

$$= |a^2-b^2| = a^2 - a^2(1-e^2)$$

$$= a^2e^2$$

b) Equations of motion are

Horizontal	Vertical
$\ddot{x} = 0$	$\ddot{y} = -g$
$\dot{x} = U\cos\alpha$	$\dot{y} = -gt + U\sin\alpha$
$x = U\cos\alpha\, t$	$y = -\frac{gt^2}{2} + Ut\sin\alpha$

(✓)

The point (a,b) lies on the trajectory, (a,b) satisfies the particle's Cartesian equation

$$y = \frac{-gx^2}{2U^2\cos^2\alpha} + x\tan\alpha \quad (✓)$$

hence

$$b = \frac{-ga^2}{2U^2\cos^2\alpha} + a\tan\alpha$$

$$= -\frac{ga^2\sec^2\alpha}{2U^2} + a\tan\alpha \quad (✓)$$

$$= \frac{-ga^2(1+\tan^2\alpha) + 2Ua\tan\alpha}{2U^2}$$

$$\therefore ga^2\tan^2\alpha - 2aU^2\tan\alpha + ga^2 + 2bU^2 = 0$$

This is a quadratic in terms of $\tan\alpha$. There are two solutions if $\Delta > 0$

$$\Delta = (2aU^2)^2 - 4(ga^2)(ga^2 + 2bU^2) \quad (✓)$$

$$= 4a^2(U^4 - 2bgU^2 - g^2a^2)$$

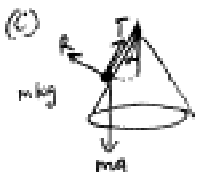
$\Delta > 0$ if $U^4 - 2bgU^2 - g^2a^2 > 0$

By completing the square

$$(U^2 - bg)^2 - g^2a^2 - b^2g^2 > 0 \quad (✓)$$

$$\therefore (U^2 - bg)^2 > g^2(a^2+b^2)$$

(c)



(✓)

The particle experiences three forces: the weight mg, the tension T in the string and the normal reaction R, perpendicular to the surface of the cone.

Resolving these forces

vertically, $T\cos\alpha + R\sin\alpha = mg$

horizontally, (✓)

$T\sin\alpha - R\cos\alpha = ml\sin\alpha\,\omega^2$

(note $\sin\alpha = \frac{r}{l}$

$\therefore r = l\sin\alpha$)

(1) by $\cos\alpha$ + (2) by $\sin\alpha$ gives

$T = mg\cos\alpha + ml\sin^2\alpha\,\omega^2$ (✓)

(1) by $\sin\alpha$ − (2) by $\cos\alpha$ give

$R = mg\sin\alpha - ml\sin\alpha\cos\alpha\,\omega^2$ (✓)

For the particle not to lose contact with the cone

$R \geq 0$, $m\sin\alpha(g - l\cos\alpha\,\omega^2) \geq 0$

$$\omega^2 \leq \frac{g}{l\cos\alpha}$$

$$\therefore \omega \leq \sqrt{\frac{g}{l\cos\alpha}} \quad (✓)$$

End of Paper

Total marks = 120