ABBOTSLEIGH

2001
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks **(120)**

- Attempt Questions 1-8.
- All questions are of equal value.

Question One (15 marks) (Start a new booklet)

(a) Let $Z = \dfrac{-i}{1+i\sqrt{3}}$

(i) Plot Z on the Argand diagram. **2**

(ii) Find the modulus and argument of Z. **2**

(b) Let A = 1 + 2i , B = -3 + 4i and Z = x + iy
Draw clearly labelled sketches to show the loci satisfied on the Argand diagram by:

(i) $|Z - A| = |B|$ **1**

(ii) $|Z - A| = |Z - B|$ **1**

(iii) $\arg(Z - A) = \dfrac{\pi}{4}$ **1**

(c) (i) Solve the equation $Z^4 = 1$ where Z is a complex number. **1**
(ii) Hence find all solutions of the equation $Z^4 = (Z-1)^4$. **3**

(d) (i) Show that the function $Q(x) = \dfrac{1}{x^2-1}$ is an even function. **1**

(ii) Show that $Q(x)$ has two vertical asymptotes and a horizontal asymptote. **1**

(iii) Find the coordinates of any stationary points. **1**

(iv) By considering the behaviour of the function as $x \to \pm\infty$ and $x \to \pm 1$, sketch the curve. **1**

Question Two (15 marks) (Start a new booklet)

(a) Find $\int \sin^3 2x\,dx$ **3**

(b) Use the substitution $t = \tan\frac{\theta}{2}$ to find the exact value of **4**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta + 2}$$

(c) (i) In the Cartesian plane indicate (by shading) the region R consisting of those points simultaneously satisfied by these five relations: **3**

$0 \le x \le \frac{\pi}{2}$, $\quad y \ge 0$, $\quad y \ge \sin x$, $\quad y \le \cos x$, $\quad y \le \tan x$

(ii) Show that y = cos x and y = tan x (from part (i)) intersect where $\sin x = \frac{\sqrt{5}-1}{2}$. **2**

(d) On separate diagrams carefully sketch the following graphs for the domain $-2\pi \le x \le 2\pi$. Each sketch should be about a third of a page in size.

(i) $y = \cos^2 x$ **1**

(ii) $y = |\cos^2 x|$ **1**

(iii) $y = \frac{1}{\cos^2 x}$ **1**

Question Three (15 marks) (Start a new booklet)

The ellipse, *E*, has equation $9x^2 + 16y^2 = 144$.
The points P(4 cos θ, 3 sin θ) and Q(-4 sin θ, 3 cos θ) lie on *E*.

(a) Find the equations of the tangents at the points P and Q. **4**

(b) Find the point of intersection, T, of these tangents. **3**

(c) Prove that, as θ varies, the locus of T is another ellipse, *F*, with equation $9x^2 + 16y^2 = 288$. **3**

(d) For the ellipse, *F*, find the coordinates of the foci, S and S' and the equations of the directrices. Sketch *F* showing all its features. **4**

(e) Show that both *E* and *F* have the same eccentricity. **1**

Question Four (15 marks) (Start a new booklet)

(a) (i) Given that the polynomial **4**

$P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

has a triple zero, find all roots of P(x) = 0.

(ii) Sketch the function y = P(x) (Make no attempt to evaluate the coordinates of stationary points.) **1**

(b) Show that (x + 1) is a factor of

$P(x) = x^3 + 2x^2 + 2x + 1$

and hence factorise P(x) over the complex numbers. **3**

(c) Find the two square roots of $-3 + 4i$ expressing each root in the form a + ib where a and b are real. **2**

(d) If α, β, γ are the roots of $x^3 - 3x^2 + 2x - 1 = 0$, find

(i) $\alpha + \beta + \gamma$ and $\alpha\beta + \beta\gamma + \alpha\gamma$. **1**

(ii) $\alpha^3 + \beta^3 + \gamma^3$. **2**

(iii) the equation whose roots are α^{-1}, β^{-1}, γ^{-1}. **2**

Question Five (15 marks) (Start a new booklet)

(a) Evaluate $\int_0^1 \frac{x}{\sqrt{x+1}}dx$. **3**

(b) (i) Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ intersect at right angles. **4**

(ii) Find the equation of the circle through the points of intersection of these two conics. **2**

(c) (i) Show that the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > b > 0$ at the point $P(a \sec\theta, b\tan\theta)$ has equation **2**

$$bx\sec\theta - ay\tan\theta = ab$$

(ii) If this tangent passes through a focus of the ellipse **4**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{where } a > b > 0$$

show that it is parallel to one of the lines $y = x$ or $y = -x$ and that its point of contact with the hyperbola lies on a directrix of the ellipse.

Question Six (15 marks) (Start a new booklet)

(a) The three roots of the equation **3**

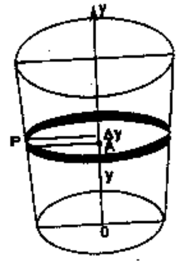
$$8x^3 - 36x^2 + 38x - 3 = 0$$

are in arithmetic sequence. Find the roots of the equation.

(b) Express $\frac{1}{(x+1)(x^2+4)}$ as partial fractions and use the result to evaluate **5**

$$\int_0^2 \frac{1}{(x+1)(x^2+4)}\,dx.$$

(c) A bucket has an internal radius of 10 cm at the bottom and 18 cm at the top. If the depth is 24 cm, find the volume of the bucket in cm^3. **3**



(d) (i) If $I_n = \int_1^e x(\ln x)^n\,dx$, $n = 0, 1, 2, 3, \ldots$ **2**

show that $I_n = \frac{e^2}{2} - \frac{n}{2}I_{n-1}$, $n = 1, 2, 3, \ldots$

(ii) Evaluate $\int_1^e x(\ln x)^3\,dx$. **2**

Question Seven (15 marks) (Start a new booklet)

(a) (i) If $z = \cos\theta + i\sin\theta$ use de Moivre's Theorem to show that **3**

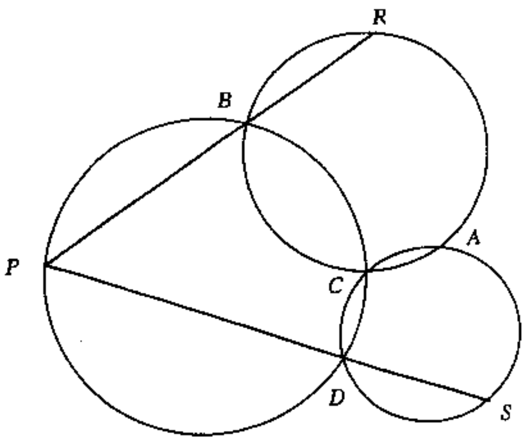
$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(ii) By expanding $\left(z + \frac{1}{z}\right)^4$ show that **3**

$$\cos^4\theta = \tfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

(iii) Evaluate $\int_0^{\frac{\pi}{4}} \cos^4\theta \, d\theta$. **2**

(b)



In the diagram above, three circles intersect at a common point C. PBR and PDS are straight lines.

(i) Copy the diagram into your booklet.

(ii) Show that R, A, S are collinear points. **4**

(iii) If CA is perpendicular to RAS explain where the centre of the circle through P, B, C, D is located relative to the line PC. **3**

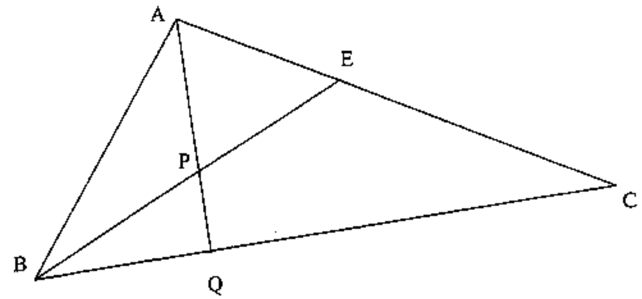
Question Eight (15 marks) (Start a new booklet)

(a) Solve the following pair of equations for z and w where z and w are complex numbers. Express your answers in the form a + ib. **3**

$$2z + 3iw = 0$$

$$(1 - i)z + 2w = i - 7$$

(b) In ΔABC, BE bisects $\angle ABC$, and APQ is a straight line such that AP = AE.



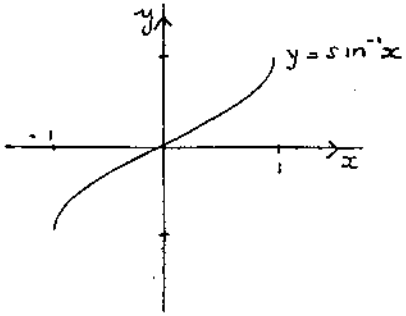
(i) Copy the diagram into your own booklet.

(ii) Prove that AB is a tangent to the circle which passes through the points A, Q and C. **5**

Question Eight (continued)

(c) A solid shaped like an egg timer is made by rotating the curve

$y = \sin^{-1} x$ around the y-axis.



(i) Show, by summing horizontal slices, that the volume so obtained is $\frac{\pi^2}{2}$ cubic units. **3**

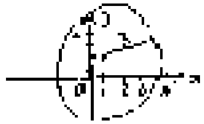
(ii) Confirm this answer by using the method of cylindrical shells to find the volume obtained by rotating the region bounded by the curve, the x-axis and the line $x = 1$ about the y-axis. **4**

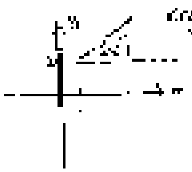
End of paper

Extension 1 Mathematics — [illegible]

Complex Nos

(ii) $|z| = \sqrt{\frac{3}{16} + \frac{1}{16}} = \frac{1}{2}$



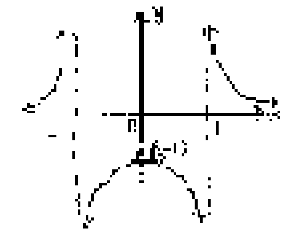


(c) (i) $z^4 = 1$

(ii) $z^4 = (z+1)^4$

$\left(\frac{z}{z+1}\right)^4 = 1$

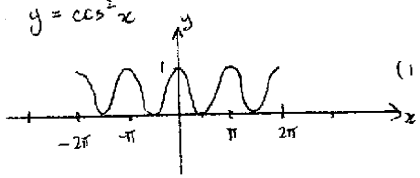
Extension 1 Mathematics (continued)



Integration

Question 8 (continued)

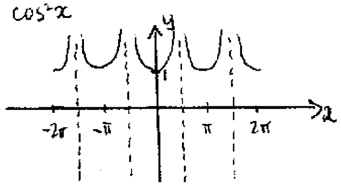
$y = \cos^2 x$



(1)

$y = |\cos^2 x|$ same as (1) (1)

$y = \dfrac{1}{\cos^2 x}$



(1)

stion 3

$9x^2 + 16y^2 = 144$

$18x + 32y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-18x}{32y} = \dfrac{-9x}{16y}$ (1)

ngent at $P(4\cos\theta, 3\sin\theta)$

$m_P = \dfrac{-36}{48}\dfrac{\cos\theta}{\sin\theta} = -\dfrac{3}{4}\dfrac{\cos\theta}{\sin\theta}$

$y - 3\sin\theta = \dfrac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$ (1)

$y\sin\theta - 12\sin^2\theta = -3\cos\theta\, x + 12\cos^2\theta$

$4y\sin\theta + 3x\cos\theta = 12$ (1)

ngent at $Q(-4\sin\theta, 3\cos\theta)$

$m_Q = \dfrac{36\sin\theta}{48\cos\theta} = \dfrac{3\sin\theta}{4\cos\theta}$

$- 3\cos\theta = \dfrac{3\sin\theta}{4\cos\theta}(x + 4\sin\theta)$

$y\cos\theta - 12\cos^2\theta = 3x\sin\theta + 12\sin^2\theta$

$4y\cos\theta - 3x\sin\theta = 12$ (1)

Question 3 (cont)

(b) Solve tangent equations simultaneously

$4y\sin\theta + 3x\cos\theta = 12$ —①

$4y\cos\theta - 3x\sin\theta = 12$ —②

① × cos θ, ② × sin θ $(\frac{1}{2})$

$4y\cos\theta\sin\theta + 3x\cos^2\theta = 12\cos\theta$ —③

$4y\cos\theta\sin\theta - 3x\sin^2\theta = 12\sin\theta$ —④

③ − ④ $3x(\cos^2\theta + \sin^2\theta) = 12(\cos\theta - \sin\theta)$

$3x = 12(\cos\theta - \sin\theta)$ (1)

$x = 4(\cos\theta - \sin\theta)$

① × sin θ, ② × cos θ

$4y\sin^2\theta + 3x\cos\theta\sin\theta = 12\sin\theta$ —⑤ $(\frac{1}{2})$

$4y\cos^2\theta - 3x\sin\theta\cos\theta = 12\cos\theta$ —⑥

⑤ + ⑥

$4y(\sin^2\theta + \cos^2\theta) = 12(\sin\theta + \cos\theta)$

$\therefore\ y = 3(\sin\theta + \cos\theta)$ (1)

$\therefore T\ \big(4(\cos\theta - \sin\theta),\ 3(\sin\theta + \cos\theta)\big)$

(c) $x = 4(\cos\theta - \sin\theta)$

$y = 3(\sin\theta + \cos\theta)$ (1)

$\therefore \left(\dfrac{x}{4}\right)^2 = \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta$

$\left(\dfrac{y}{3}\right)^2 = \sin^2\theta + 2\cos\theta\sin\theta + \cos^2\theta$ (1)

$\therefore \left(\dfrac{x}{4}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 2\cos^2\theta + 2\sin^2\theta$

$= 2$

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 2$ (1)

$9x^2 + 16y^2 = 288$

(d) $\dfrac{x^2}{32} + \dfrac{y^2}{18} = 1$

$a = \sqrt{32} = 4\sqrt{2}$

$b = \sqrt{18} = 3\sqrt{2}$

$b^2 = a^2(1 - e^2)$

$18 = 32(1 - e^2)$

$\dfrac{18}{32} = 1 - e^2 \quad \therefore \quad e^2 = \dfrac{14}{32} \quad = \dfrac{\sqrt{7}}{4}$ (1)

$\therefore$ Focus $= S(ae, 0) = (\sqrt{14}, 0)$

$S'(-ae, 0) = (-\sqrt{14}, 0)$ (1)

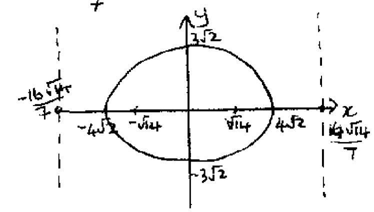
Directrices $x = \pm\dfrac{a}{e}$

$= \pm\dfrac{4\sqrt{2}}{\sqrt{7}} \cdot 4$

$= \pm 16\dfrac{\sqrt{2}}{\sqrt{7}} \cdot \dfrac{\sqrt{7}}{\sqrt{7}}$ (1)

Question 3 (continued)

$x = \pm\dfrac{16\sqrt{14}}{7}$ (1)



(1)

(e) eccentricity of E

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

$e = \sqrt{1 - \dfrac{9}{16}} = \dfrac{\sqrt{7}}{4}$

eccentricity of F is $\dfrac{\sqrt{7}}{4}$

$\therefore$ E and F have the same "e".

Question Four

(a) $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

(i) $P'(x) = 4x^3 + 3x^2 - 6x - 5$ $(\frac{1}{2})$

$P''(x) = 12x^2 + 6x - 6$ $(\frac{1}{2})$

$P''(x) = 0$ when $6(x+1)(2x-1) = 0$ $(\frac{1}{2})$

$\therefore$ root is $x = -1, \frac{1}{2}$ $(\frac{1}{2})$

Test $P(-1) = 0$ $(\frac{1}{2})$

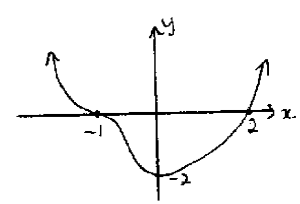
$\therefore (x+1)^3$ is a factor of $P(x)$ $(\frac{1}{2})$

Since constant term is −2, then $(\frac{1}{2})$

$P(x) = (x+1)^3(x-2)$ $(\frac{1}{2})$

$\therefore$ roots are −1, −1, −1, 2

(ii)



(1)

(b) $P(-1) = 0$ $(\frac{1}{2})$

$\therefore (x+1)$ is a factor of $P(x)$

$$\begin{array}{r} x^2 + x + 1 \\ x+1\,\overline{)\,x^3 + 2x^2 + 2x + 1} \\ \underline{x^3 + x^2} \\ x^2 + 2x \\ \underline{x^2 + x} \\ x + 1 \\ x + 1 \end{array}$$

(1)

Question Four (cont)

$x^2 + x + 1 = (x + \frac{1}{2})^2 - \frac{1}{4} + 1$ (1)

$= (x + \frac{1}{2})^2 + \frac{3}{4}$

$= (x + \frac{1}{2})^2 - \left(\frac{\sqrt{3}}{2}i\right)^2$

$\therefore P(x) = (x+1)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

(c) $(a + ib)^2 = -3 + 4i$

$a^2 - b^2 + 2abi = -3 + 4i$

$a^2 - b^2 = -3$ $(\frac{1}{2})$

$ab = 2$

$a^2 - \left(\dfrac{2}{a}\right)^2 = -3$ $(\frac{1}{2})$

$a^4 + 3a^2 - 4 = 0$

$(a^2 + 4)(a^2 - 1) = 0$ $(\frac{1}{2})$

$a^2 = 1$ since a is real

$a = \pm 1$

$\therefore b = \pm 2$ $(\frac{1}{2})$

$\therefore \sqrt{-3 + 4i} = \pm(1 + 2i)$

(d) $x^3 - 3x^2 + 2x - 1 = 0$

(i) $\alpha + \beta + \gamma = 3$ $(\frac{1}{2})$

$\alpha\beta + \beta\gamma + \alpha\gamma = 2$ $(\frac{1}{2})$

(ii) $\alpha^3 - 3\alpha^2 + 2\alpha - 1 = 0$

$\beta^3 - 3\beta^2 + 2\beta - 1 = 0$

$\gamma^3 - 3\gamma^2 + 2\gamma - 1 = 0$

Add

$\alpha^3 + \beta^3 + \gamma^3 = 3(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha + \beta + \gamma) + 3$

$= 3(\alpha^2 + \beta^2 + \gamma^2) - 2 \times 3 + 3$

$= 3\left[(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)\right] - 3$

$= 3[9 - 2 \times 2] - 3$ (2)

$= 3 \times 5 - 3 = 12$

(iii) $\left(\dfrac{1}{x}\right)^3 - 3\left(\dfrac{1}{x}\right)^2 + 2\left(\dfrac{1}{x}\right) - 1 = 0$

$\dfrac{1}{x^3} - \dfrac{3}{x^2} + \dfrac{2}{x} - 1 = 0$ (2)

$1 - 3x + 2x^2 - x^3 = 0$

$x^3 - 2x^2 + 3x - 1 = 0$

uestion Five

(a) $\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$

Let $u = x+1$ (1)

$du = dx$

$I = \int_1^2 \frac{u-1}{\sqrt{u}}\,du$

$= \int_1^2 \sqrt{u} - \frac{1}{\sqrt{u}}\,du$

$= \left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^2$ (1)

$= \left(\frac{2}{3}\cdot 2^{3/2} - 2\cdot 2^{1/2}\right) - \left(\frac{2}{3} - 2\right)$

$= \frac{4}{3} - \frac{2}{3}\sqrt{2}$ (1)

(b)(i) $4x^2 + 9y^2 = 36$ ellipse

$8x + 18y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-8x}{18y} = \frac{-4x}{9y}$ (½)

$4x^2 - y^2 = 4$ hyperbola

$8x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{8x}{2y} = \frac{4x}{y}$ (½)

To find point of intersection

$4x^2 + 9y^2 = 36$ —①

$4x^2 - y^2 = 4$ —②

① − ② $10y^2 = 32$, $y^2 = 3.2$, $y = \pm\sqrt{3.2}$

$4x^2 - 3.2 = 4$, $4x^2 = 7.2$, $x^2 = 1.8$ (1), $x = \pm\sqrt{1.8}$

$m_{ellipse} = \frac{-4\sqrt{1.8}}{9\sqrt{3.2}} = -\frac{1}{3}$ (1)

$m_{hyperbola} = \frac{4\sqrt{1.8}}{\sqrt{3.2}} = 3$

$\therefore m_e \cdot m_h = -\frac{1}{3}\cdot 3 = -1$ (1)

∴ the conics intersect at 90°.

(ii) Midpoint of intersection points (1) is (0,0) is centre

$\therefore x^2 + y^2 = x_1^2 + y_1^2 = 5$ (1)

(c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $P(a\sec\theta, b\tan\theta)$

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2x}{a^2}\cdot\frac{b^2}{2y}$ (½)

At P, $\frac{dy}{dx} = \frac{2a\sec\theta\, b^2}{a^2 \cdot 2 \cdot b\tan\theta}$ (½)

∴ Equation of tangent is

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$ (½)

$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$

$bx\sec\theta - ay\tan\theta = ab$ (½)

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Focus $(ae, 0)$

$bae\sec\theta - 0 = ab$

$\sec\theta = \frac{1}{e}$ (1)

$m_{tangent} = \frac{b\sec\theta}{a\tan\theta}$

$\therefore m^2 = \frac{b^2\sec^2\theta}{a^2\tan^2\theta}$ $b^2 = a^2(1-e^2)$ (1), $\tan^2\theta = \sec^2\theta - 1$

$= \frac{a^2(1-e^2)\frac{1}{e^2}}{a^2\left(\frac{1}{e^2} - 1\right)}$ (1)

$= 1$

∴ parallel to $y = x$ or $y = -x$

$P(a\sec\theta, b\tan\theta)$

Since $\sec\theta = \frac{1}{e}$, $P\left(\frac{a}{e}, b\tan\theta\right)$

∴ P is on directrix of ellipse.

Question Six

(a) $8x^3 - 36x^2 + 38x - 3 = 0$

Let the roots be $a-d$, a, $a+d$ (½)

Sum of roots $= 3a = \frac{36}{8}$

$a = \frac{3}{2}$ (1)

Product of roots

$a(a^2 - d^2) = \frac{3}{8}$ $\frac{3}{2}\left(\frac{9}{4} - d^2\right) = \frac{3}{8}$

$d^2 = 2$ $\frac{9}{4} - d^2 = \frac{6}{24}$ (1)

$d = \pm\sqrt{2}$ (½) $d^2 = \frac{9}{4} - \frac{6}{24}$

Roots are $\frac{3}{2} - \sqrt{2}$, $\frac{3}{2}$, $\frac{3}{2} + \sqrt{2}$

(b) $\frac{1}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$

$1 = a(x^2+4) + (bx+c)(x+1)$

$1 = ax^2 + 4a + bx^2 + c + cx + bx$ (1)

$a + b = 0$

$b + c = 0$

$4a + c = 1$ (1)

$\therefore c = a$, $\therefore a = \frac{1}{5}$, $b = -\frac{1}{5}$, $c = \frac{1}{5}$

$\therefore \int_0^2 \frac{1}{(x+1)(x^2+4)}\,dx$

$= \int_0^2 \frac{1}{5(x+1)} + \frac{-x}{5(x^2+4)} + \frac{1}{5(x^2+4)}\,dx$ (1)

$= \left[\frac{1}{5}\ln(x+1)\right]_0^2 - \left[\frac{1}{5}\cdot\frac{1}{2}\ln(x^2+4)\right]_0^2 + \left[\frac{1}{5}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$= \frac{1}{5}\left[\left(\ln 3 - \frac{1}{2}\ln 8 + \frac{1}{2}\cdot\frac{\pi}{4}\right) - \left(\ln 1 - \frac{1}{2}\ln 4 + 0\right)\right]$ (1)

$= \frac{1}{10}\left(\ln\frac{9}{2} + \frac{\pi}{4}\right)$ (1)

Question Six (continued)

(c) Consider the cross section at a distance y from the bottom. The radius of the cross section is a linear function of the height $\therefore r = ay + b$

When $y = 0$, $r = 10$ $\therefore b = 10$

$y = 24$, $r = 18$ $\therefore 18 = 24a + b$

$\therefore a = \frac{1}{3}$

and $r = \frac{y}{3} + 10$

$\therefore \delta V \doteq \pi r^2 \delta y$ (1)

$= \pi\left(\frac{y}{3} + 10\right)^2 \delta y$

$\therefore V = \int_0^{24} \pi\left(\frac{y}{3} + 10\right)^2 dy$ (1)

$= \pi\left[\left(\frac{y}{3} + 10\right)^3\right]_0^{24}$

$= \pi(18^3 - 10^3)$ (1)

$= 4832\pi\ \text{cm}^3$

(d)(i) $I_n = \int_1^e x(\ln x)^n\,dx$

$= \left[\frac{x^2}{2}(\ln x)^n\right]_1^e - \int_1^e \frac{x^2}{2}\cdot n(\ln x)^{n-1}\cdot\frac{1}{x}\,dx$ (1)

$= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\,dx$ (1)

$= \frac{e^2}{2} - \frac{n}{2}I_{n-1}$

(ii) $\int_1^e x(\ln x)^3\,dx = I_3$

$= \frac{e^2}{2} - \frac{3}{2}I_2$

$= \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2}{2} - I_1\right)$ (½)

$= -\frac{e^2}{4} + \frac{3}{2}I_1$ (½)

$= -\frac{e^2}{4} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2}I_0\right)$ (½)

$= \frac{e^2}{2} - \frac{3}{4}\left(\frac{e^2}{2} - \frac{1}{2}\right)$ (½)

$= \frac{1}{8}(e^2 + 3)$

stion Seven

) $z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$ (1)

$= \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$ (1)

$= \cos n\theta - i\sin n\theta$ (½)

$z^n + \frac{1}{z^n} = 2\cos n\theta$ (½)

$(z + \frac{1}{z})^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$

$(z^4 + \frac{1}{z^4}) + 4(z^2 + \frac{1}{z^2}) + 6$ (1)

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ (1)

$\ldots\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$ (1)

$\ldots\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

$\int_0^{\pi/2} \cos^4\theta \, d\theta$

$\int_0^{\pi/2} \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)\, d\theta$ (½)

$\left[\frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta\right]_0^{\pi/2}$ (1)

$\frac{3\pi}{16}$ (½)

See diagram

) $R\hat{A}C = P\hat{B}C$ (exterior angle of cylic quad ARBC) (1)

$S\hat{A}C = P\hat{D}C$ (exterior angle of cyclic quad ASDC) (1)

$P\hat{B}C + P\hat{D}C = 180^\circ$ (opp angles of cyclic quad are supplementary)

∴ $R\hat{A}C + S\hat{A}C = 180^\circ$ (1)

∴ RAS is a straight line (a pair of adjacent angles are supplementary) (1)

) $R\hat{A}C = 90^\circ$ (given)

$P\hat{B}C = R\hat{A}C = 90^\circ$ (as above) (1)

Question Seven (continued)

∴ PC is a diameter of the circle through PBCD (1)

∴ midpoint of PC is centre of circle. (1)

Question Eight

(a) $2z + 3iw = 0$ —①

$(1-i)z + 2w = i - 7$ —② (½)

② × $(1+i)$

$(1-i)(1+i)z + 2(1+i)w = (1+i)(i-7)$

$2z + (2+2i)w = i - 7i + i^2 + i^2 - 7$

$2z + (2+2i)w = -8 - 6i$ —③

① − ③

$(3i - 2 - 2i)w = 8 + 6i$ (½)

$w = \frac{8+6i}{-2+i} \times \frac{-2-i}{-2-i}$

$= -2 - 4i$ (1)

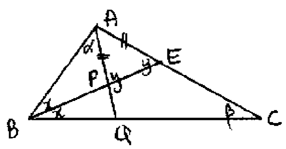
sub into ①

$2z + 3i(-2-4i) = 0$

$2z = -3i(-2-4i)$ (1)

$z = -6 + 3i$

(b)



Let $B\hat{A}P = \alpha$

$B\hat{C}E = \beta$ (1)

$A\hat{B}E = E\hat{B}Q = x$ (given)

$A\hat{P}E = A\hat{E}P = y$ (isosceles)

$E\hat{C}B = y - x = \beta$ (exterior angle of ΔEBC) (1)

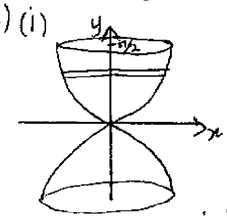
$B\hat{A}P = y - x = \alpha$ (exterior angle of ΔBAP) (1)

∴ $\alpha = \beta$ (angle b/w tangent + chord equals angle in alternate segment) (1)

∴ AB is a tangent to the circle passing through A, Q and C (1)

Question Eight (continued)

(c) (i)



$y = \sin^{-1}x$

$x = \sin y$

$\delta V \doteqdot \pi x^2 \delta y$

$= \pi \sin^2 y \, \delta y$

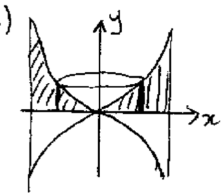
∴ $V = 2\int_0^{\pi/2} \pi \sin^2 y \, dy$ (1)

$= \pi \int_0^{\pi/2} 1 - \cos 2y \, dy$ (1)

$= \pi\left[y - \frac{1}{2}\sin 2y\right]_0^{\pi/2}$ (1)

$= \pi\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{2}$

(ii)



$\delta V \doteqdot 2\pi x y \, \delta x$

$V = 2\int_0^1 2\pi x y \, dx$

$= 4\pi \int_0^1 x \sin^{-1}x \, dx$ (1)

I

$I = \int_0^1 \sin^{-1}x \frac{d}{dx}\left(\frac{1}{2}x^2\right) dx$

$= \left[\frac{1}{2}x^2 \sin^{-1}x\right]_0^1 - \int_0^1 \frac{x^2}{2\sqrt{1-x^2}}\, dx$ (½)

$= \left(\frac{\pi}{4} - 0\right) + \frac{1}{2}\int_0^1 \left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right) dx$ (½)

$= \frac{\pi}{4} + \frac{1}{2}\int_0^1 \sqrt{1-x^2}\, dx - \left[\frac{1}{2}\sin^{-1}x\right]_0^1$ (½)

$= \frac{\pi}{4} + \frac{1}{2}\cdot\frac{\pi}{4} - \frac{\pi}{4} = \frac{\pi}{8}$

∴ $V = 4\pi \cdot \frac{\pi}{8} = \frac{\pi^2}{2}$ (½)

∴ Original volume = cylinder $- \frac{\pi^2}{2}$

$= \pi r^2 h - \frac{\pi^2}{2}$

$= \pi \cdot 1^2 \cdot \pi - \frac{\pi^2}{2}$ (1)

$= \frac{\pi^2}{2}$ as required

The end