Abbotsleigh August 2002 (3 hours)

Total marks (120)
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{dx}{x^2+2x+5}$. **2**

(b) (i) Find real numbers A, B and C such that

$$\frac{3x+7}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$ **2**

(ii) Hence evaluate $\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx$ **2**

(c) Show $\int e^{ax}\sin 3x\,dx = \frac{e^{ax}}{a^2+9}\left[a\sin 3x - 3\cos 3x\right]$ and hence evaluate $\int_0^{2\pi} e^{2x}\sin 3x\,dx$. **4**

(d) $I_m = \int x^m e^x dx$

(i) Show that $I_m = x^m e^x - mI_{m-1}$ **2**

(ii) Find the value of $\int_1^2 x^2 e^x dx$ **3**

End of Question 1

Marks

QUESTION 2 (15 marks) Use a SEPARATE writing booklet.

(a) The complex number z is given by $z = 1 + \frac{1+i}{1-i}$. Find

(i) $\overline{z}$, giving your answer in the form $x+iy$ where x and y are real. **2**

(ii) iz **1**

(b) Find u and v if $(u-iv)^2 = -21-20i$. **4**

(c) If $z_1 = 1+3i$ and $z_2 = 1-3i$, find $\left|\frac{z_1^{10}}{z_2^9}\right|$. **3**

(d) Shade the region in the complex plane for which $2 \le z + \overline{z} \le 6$. **2**

(e) In the Argand diagram $P(z)$ is a point in the first quadrant of the circle $|z| = 2$. If $\arg z = \theta$ find, in terms of θ, expressions for

(i) $\arg z^2$ **1**

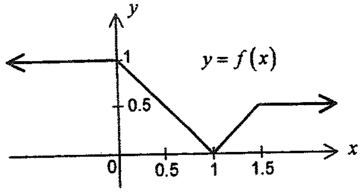
(ii) $\arg(z-2)$ **2**

End of Question 2

Marks

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below is a sketch of the function $y = f(x)$.



On separate diagrams sketch

(i) $y = f(-x)$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $|y| = f(x)$ **2**

(b) (i) Express $\dfrac{x^2-8}{x^2-4}$ in the form $c + \dfrac{d}{x^2-4}$ where c and d are integers. **1**

(ii) Draw a neat sketch of $y = \dfrac{x^2-8}{x^2-4}$. Clearly indicate the intercepts with the coordinate axes and the position and equation of all asymptotes. **4**

(iii) Shade the region R described by $0 \le y \le \dfrac{x^2-8}{x^2-4}$ and $3 \le |x| \le 4$. **1**

(iv) Show that the area of $R = 2\left(1 + \log_e \dfrac{3}{5}\right)$. **3**

End of Question 3

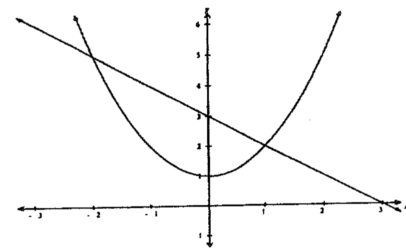
Marks

QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

(a) Show that $2 - \sqrt{3}$ is a zero of the polynomial $a(x) = x^3 - 15x + 4$.
Hence reduce $a(x)$ to irreducible factors over the real field. **3**

(b) Factorise $Q(x) = x^6 - 3x^2 + 2$ over the field of complex numbers, given that it has two double roots. **4**

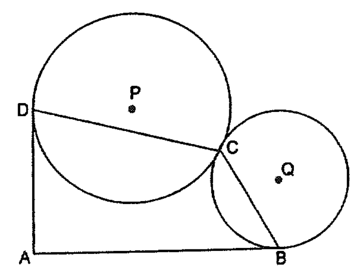
(c) The area bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x-axis to form a solid.



(i) By considering slices perpendicular to the x-axis show that the area of one slice is given by $A = \pi\left(8 - 6x - x^2 - x^4\right)$. **2**

(ii) Hence find the volume of the solid formed. **2**

(d) Two circles touch each other at C. AD is a tangent to the circle with centre P and touches the circle at D. AB is a tangent to the circle with centre Q and touches the circle at B.



NOT TO SCALE

Copy the diagram into your answer booklet.
Prove that $\angle BCD = 180^\circ - \dfrac{1}{2}\angle BAD$. **4**

End of Question 4

Marks

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

(a) The velocity v cm/s of a particle moving in a straight line is given by $v^2 = 48 + 16x - 4x^2$ where x cm is the displacement of the particle from a fixed point.

(i) Show that the particle is moving with simple harmonic motion. Hence write down the centre of motion. **3**

(ii) What is the amplitude of the motion? **2**

(iii) If initially the particle is at one of the extreme points, how far will it travel in the first $\frac{3\pi}{4}$ seconds? **2**

(iv) Where is the particle when $t = \frac{3\pi}{4}$ seconds? **1**

(b) A torus is generated by revolving the region $x^2 + y^2 \le 4$ about the line $x = 5$.

(i) By using the method of cylindrical shells show that the volume of one shell is given by $\Delta V = 4\pi(5-x)\sqrt{4-x^2}\,\Delta x$. **2**

(ii) Hence find the volume of the torus. **2**

(c) Solve the equation $4x^3 - 12x^2 + 11x - 3 = 0$ given that the roots form an arithmetic series. **3**

End of Question 5

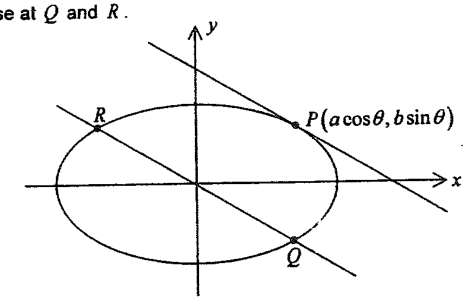
Marks

QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction $3^n - 1 \ge 2n$ where n is a positive integer. **3**

(b) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre O.

A line drawn through O, parallel to the tangent to the ellipse at P, meets the ellipse at Q and R.



(i) Show that Q and R are the points $(a\sin\theta, -b\cos\theta)$ and $(-a\sin\theta, b\cos\theta)$. **2**

(ii) Prove that the area of ΔPQR is independent of the position of P. **3**

(c) A solid is built on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ such that cross-sections perpendicular to the x-axis are squares with a side in the elliptical base.

Find the volume of the solid. **4**

(d) Z and W represent the complex numbers z and w respectively.

If $|z| = 2$ and $w = \frac{z+3}{z}$, find the locus of W. **3**

End of Question 6

QUESTION 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The tangent to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at $P(4\sqrt{2},3)$ meets the asymptotes of the hyperbola at A and B.

Show that P is the midpoint of AB. **4**

(b) LMN is an isosceles triangle with $LM=LN$ and P is a point on the bisector of $\angle MLN$. MP produced meets LN at Q and NP produced meets LM at R.

Prove that M, N, Q and R are concyclic. **4**

(c) (i) Show that the equation of the chord joining the points $P\left(2p,\frac{2}{p}\right)$ and $Q\left(2q,\frac{2}{q}\right)$ on the hyperbola $y=\frac{4}{x}$ is $x+pqy=2(p+q)$. **3**

(ii) By letting q approach p, or otherwise, write down the equation of the tangent to the curve at P. **1**

(iii) If the chord in (i) passes through the point $(2(p+q-1),2)$, and T is the point of intersection of the tangents at P and Q, show that the locus of T is $y=x$. **3**

End of Question 7

QUESTION 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Use this result to solve the equation $8x^3-6x=-1$ for x. **5**

(b) If $a>b$ and $c>d$, prove that $ac+bd>ad+bc$. **2**

(c) If a,b,c,d are positive numbers with $a\ge c+d$ and $b\ge c+d$, prove that $ab\ge ad+bc$. **3**

(d) (i) Show that $\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$ **3**

(ii) Hence prove that

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

where n is a positive integer. **2**

END OF PAPER

ion 1

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4}$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

$3x+7 = A(x+2)(x+3) + B(x+1)(x+3)$
$\quad + C(x+1)(x+2)$
$\quad = Ax^2 + 5Ax + 6A + Bx^2 + 4Bx + 3B$
$\quad + Cx^2 + 3Cx + 2C$

$$\left.\begin{array}{l} A+B+C=0 \\ 5A+4B+3C=3 \\ 6A+3B+2C=7 \end{array}\right\} \quad A=2,\ B=-1,\ C=-1$$

$$\frac{x+7}{(x+2)(x+3)} = \frac{2}{x+1} - \frac{1}{x+2} - \frac{1}{x+3}$$

$$\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx$$
$$= \Big[2\ln|x+1| - \ln|x+2| - \ln|x+3|\Big]_0^1$$
$$= 2\ln 2 - \ln 3 - \ln 4$$
$$\quad -(2\ln 1 - \ln 2 - \ln 3)$$
$$= \ln 2$$

$$\int e^{ax}\sin 3x\,dx$$
$$= -\frac{1}{3}e^{ax}\cos 3x + \frac{a}{3}\int e^{ax}\cos 3x\,dx$$
$$= -\frac{1}{3}e^{ax}\cos 3x + \frac{a}{3}\left(\frac{1}{3}e^{ax}\sin 3x - \frac{a}{3}\int e^{ax}\sin 3x\,dx\right)$$
$$\frac{a^2}{9}\Big)\int e^{ax}\sin 3x\,dx = \frac{e^{ax}}{9}(a\sin 3x - 3\cos 3x)$$
$$\int e^{ax}\sin 3x\,dx = \frac{e^{ax}}{9+a^2}(a\sin 3x - 3\cos 3x)$$
$$\int^{\pi} e^{2x}\sin 3x\,dx = \left[\frac{e^{2x}}{2^2+9}(2\sin 3x - 3\cos 3x)\right]_0^{2\pi}$$

$$\int_0 e^{\cdots}\sin 3x\,dx = \frac{\cdots}{13}(2\sin\cdots - 3\cos\cdots)$$
$$- \frac{e^0}{13}(2\sin 0 - 3\cos 0)$$
$$= \frac{3}{13}\left(1 - e^{4\pi}\right)$$

(d)(i) $\int x^m e^x\,dx = x^m e^x - \int m x^{m-1} e^x\,dx$
$\quad = x^m e^x - m\int x^{m-1}e^x\,dx$
$\quad = x^m e^x - m I_{m-1}$

(ii) $\int_1^2 x^2 e^x\,dx = \left[x^2 e^x\right]_1^2 - 2\int_1^2 x e^x\,dx$
$\quad = 4e^2 - e^1 - 2\left(\left[xe^x\right]_1^2 - \int_1^2 e^x\,dx\right)$
$\quad = 4e^2 - e - 4e^2 + 2e + 2\left[e^x\right]_1^2$
$\quad = e + 2(e^2 - e)$
$\quad = 2e^2 - e$

Question 2

(a) $z = 1 + \frac{1+i}{1-i} \times \frac{1+i}{1+i}$
$\quad = 1 + i$

(i) $\bar{z} = 1 - i$

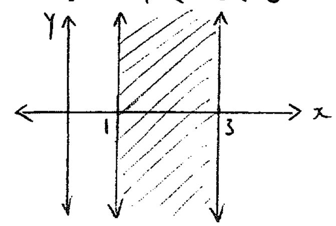
(ii) $iz = -1 + i$

(b) $(u - iv)^2 = u^2 - v^2 - 2iuv$
$u^2 - v^2 = -21$
$2uv = 20 \Rightarrow v = \frac{10}{u}$
$u^2 - \frac{100}{u^2} = -21$
$u^4 + 21u^2 - 100 = 0$
$(u^2+25)(u^2-4) = 0$
$u^2 = -25,\ u^2 = 4$
$u = \pm 5i,\ \pm 2$
$v = \mp 2i,\ \pm 5$

$|z_2| = |1 - 3i| = \sqrt{10}$

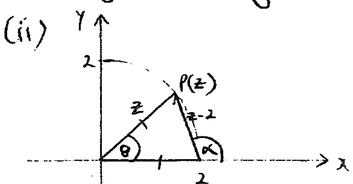
$\left|\frac{z_1^{10}}{z_2^{9}}\right| = \sqrt{10}$

(d) $z + \bar{z} = x + iy + x - iy = 2x$
$1 \leqslant x \leqslant 3$



(e) (i) $\arg z^2 = 2\arg z = 2\theta$

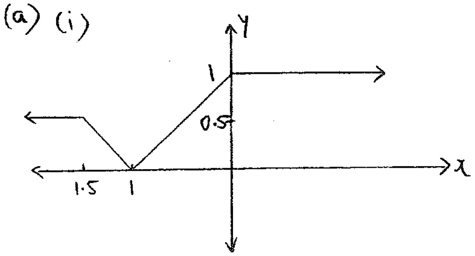
(ii)

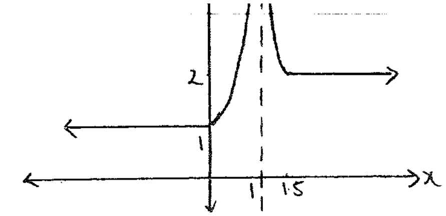


Isosceles Δ - base ∠ $= \frac{\pi - \theta}{2}$
$\arg(z-2) = \alpha$
$\quad = \frac{\pi}{2} - \frac{\theta}{2} + \theta$
$\quad = \frac{\pi}{2} + \frac{\theta}{2}$

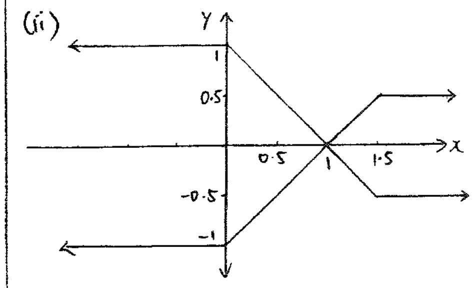
Question 3

(a) (i)



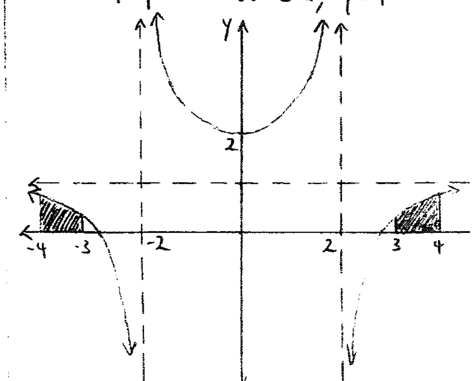


(ii)



(b) (i) $\frac{x^2-8}{x^2-4} = \frac{x^2-4}{x^2-4} - \frac{4}{x^2-4}$
$\quad = 1 - \frac{4}{x^2-4}$

(ii) intercepts $x = \pm 2\sqrt{2}$
$y = 2$ (min + pt.)
asymptotes $x = \pm 2,\ y = 1$



(iv) $R = 2\int_3^4 \left(1 - \frac{1}{x-2} + \frac{1}{x+2}\right)dx$
$\quad = 2\left[x - \ln|x-2| + \ln|x+2|\right]$
$\quad = 2(4 - \ln 2 + \ln 6 - (3 - \ln 1 + \ln \cdots$
$\quad = 2\left(1 + \ln\frac{3}{5}\right)$

tion 4

$a(2-\sqrt3) = (2-\sqrt3)^3 - 15(2-\sqrt3) + 4$

$= 8 - 12\sqrt3 + 18 - 3\sqrt3 - 30 + 15\sqrt3 + 4$

$= 0$

∴ $2-\sqrt3$ is a zero

conjugate surd $2+\sqrt3$ is a zero

$x(x) = (x-(2-\sqrt3))(x-(2+\sqrt3))(x-b)$

$-b(2-\sqrt3)(2+\sqrt3) = 4 \Rightarrow b = -4$

$(x) = (x-2+\sqrt3)(x-2-\sqrt3)(x+4)$

$Q(x) = x^6 - 3x^2 + 2$

$Q'(x) = 6x^5 - 6x$

double root $\Rightarrow Q'(x) = 0$

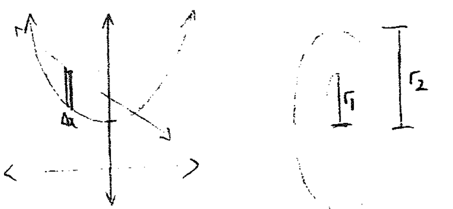
$6x(x^4-1) = 0$

$x = 0, \pm1, \pm i$

test these $\Rightarrow Q(1) = Q(-1) = 0$

∴ $Q(x) = (x-1)^2(x+1)^2(x^2+2)$

$Q(x) = (x-1)^2(x+1)^2(x-\sqrt2 i)(x+\sqrt2 i)$



i) $A(x) = \pi(r_2^2 - r_1^2)$

$= \pi((3-x)^2 - (x^2+1)^2)$

$= \pi(9 - 6x + x^2 - x^4 - 2x^2 - 1)$

$= \pi(8 - 6x - x^2 - x^4)$

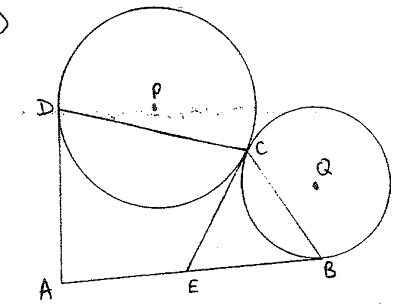
ii) $\Delta V = \pi(8 - 6x - x^2 - x^4)\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=-2}^{1} \pi(8 - 6x - x^2 - x^4)\Delta x$

$= \pi\int_{-2}^{1}(8 - 6x - x^2 - x^4)\,dx$

$= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5}u^3$

(d)



construct tangent FC

EC = EB tangents from external pt

∴ $\angle ECB = \angle CBE$

Similarly FD = FC

∴ $\angle ADC = \angle FCD$

F

$\angle BCD = 360° - (\angle ABC + \angle ADC + \angle BAD)$

$= 360° - (\angle ECB + \angle ECD) - \angle BAD$

$= 360° - \angle BCD - \angle BAD$

$2\angle BCD = 360° - \angle BAD$

∴ $\angle BCD = 180° - \frac12\angle BAD$

Question 5

(a) (i) $v^2 = 48 + 16x - 4x^2$

$\ddot x = \frac{d}{dx}(\frac12 \dot x^2)$

$= \frac{d}{dx}(24 + 8x - 2x^2)$

$= 8 - 4x$

$= -4(x-2)$

which is the form of SHM with centre $x=2$

(ii) endpts when $v=0$

$x^2 - 4x - 12 = 0$

$(x-6)(x+2) = 0$

oscillates between $x=-2$ & $x=6$

∴ amplitude 4 cm

(iii) $n^2 = 4 \Rightarrow n = 2$ ∴ period $= \pi$ s

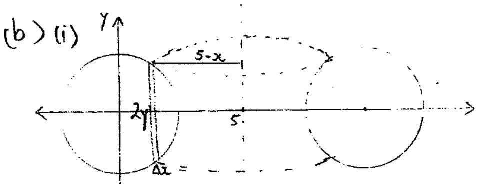
In π s travels 16cm

$\frac{3\pi}{4}$ is $\frac34$ of period

∴ travels 12cm

(iv) starts at endpoint ∴ in $\frac{3\pi}{4}$ s is at centre of motion $x=2$.

(b) (i)



$x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4-x^2}$

$\Delta V = 2\pi(5-x)2y\,\Delta x$

$= 4\pi(5-x)\sqrt{4-x^2}\,\Delta x$

(ii) $V = \lim_{\Delta x \to 0}\sum_{x=-2}^{2} 4\pi(5-x)\sqrt{4-x^2}\,\Delta x$

$= 4\pi\int_{-2}^{2}(5\underbrace{\sqrt{4-x^2}}_{\text{semi-circle}} - x\sqrt{4-x^2})\,dx$

$= 4\pi\left(5\times\frac12\times\pi\times2^2 + \frac13\left[(4-x^2)^{3/2}\right]_{-2}^{2}\right)$

$= 40\pi^2 u^3$

(c) $4x^3 - 12x^2 + 11x - 3 = 0$

Let roots be $\alpha-d, \alpha, \alpha+d$

sum : $3\alpha = 3 \Rightarrow \alpha = 1$

$\Sigma\alpha\beta$: $3\alpha^2 - d^2 = \frac{11}{4}$

$d^2 = \frac14$

$d = \pm\frac12$

∴ roots are $\frac12, 1, 1\frac12$

ie solution $x = \frac12, 1, 1\frac12$

Question 6

(a) $3^n - 1 \geq 2n$

Show true for $n=1$

LHS $= 3^1 - 1 = 2$, RHS $= 2\times1 = 2$

Assume true for $n=k$

ie $3^k - 1 \geq 2k$

Show true for $n=k+1$

$3^{k+1} - 1 = 3\times3^k - 1$

$\geq 3\times(2k+1) - 1$

$= 3\times2k + 2$

$= 2(k+1) + 4k$

$\geq 2(k+1)$ since 4k

∴ $3^{k+1} - 1 \geq 2(k+1)$

ie true for $n=k+1$ if true for n

Since true for $n=1$, also true for $n=1+1=2$ and thus true for $n=2+$ and so on for all positive integral

(b) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$

at P $m_T = \frac{-b\cos\theta}{a\sin\theta}$

∴ eq QR $y = \frac{-b\cos\theta}{a\sin\theta}x$

find Q & R

$\frac{x^2}{a^2} + \frac{b^2\cos^2\theta\, x^2}{b^2 a^2\sin^2\theta} = 1$

$\frac{x^2(\sin^2\theta + \cos^2\theta)}{a^2\sin^2\theta} = 1$

$x^2 = a^2\sin^2\theta$

$x = \pm a\sin\theta$

$x = \pm a\sin\theta$, $y = \mp b\cos\theta$

∴ Q & R $(\pm a\sin\theta, \mp b\cos$

(ii) $QR = \sqrt{(2a\sin\theta)^2 + (2b\cos\theta)^2}$

$= 2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

⊥ dist P to QR

$$d = \frac{|bx\cos\theta + ay\sin\theta|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

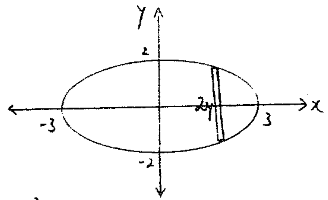
$$= \frac{ab(\cos^2\theta + \sin^2\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$A = \tfrac{1}{2}\times 2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} \times \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$= ab$ which is independent of θ

$\therefore$ Area ΔPQR is independent of position P.

(c)



$\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow y^2 = \frac{4}{9}(9 - x^2)$

Area cross-section $= 4y^2$

$= \frac{16}{9}(9 - x^2)$

Volume slice $\Delta V = \frac{16}{9}(9 - x^2)\Delta x$

Volume $V = \lim_{\Delta x \to 0} \sum_{x=-3}^{3} \frac{16}{9}(9 - x^2)\Delta x$

$= \frac{16}{9}\int_{-3}^{3}(9 - x^2)\,dx$

$= \frac{32}{9}\left[9x - \frac{x^3}{3}\right]_0^3$

$= \frac{32}{9}(27 - 9)$

$= 64\ u^3$

(d) $w = \frac{z+3}{z} \Rightarrow zw - z = 3$

$z(w-1) = 3$

$z = \frac{3}{w-1}$

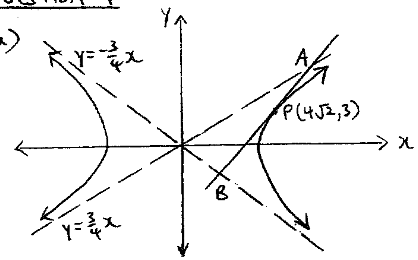
$|z| = 2 \quad \therefore 2 = \frac{3}{|w-1|}$

$|w-1| = \frac{3}{2}$

$\therefore$ W is a circle, centre $(1,0)$ radius $\sqrt{\tfrac{3}{2}}$

Question 7

(a)



$\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow \frac{x}{8} - \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{9x}{16y} \qquad m_T = \frac{9\times 4\sqrt{2}}{16\times 3} = \frac{3\sqrt{2}}{4}$

eq. tangent $y - 3 = \frac{3\sqrt{2}}{4}(x - 4\sqrt{2})$

$4y - 3\sqrt{2}x + 12 = 0$

when $y = \frac{3}{4}x \qquad 3x - 3\sqrt{2}x + 12 = 0$

$x = \frac{4}{\sqrt{2}-1}$

pt. A $\left(\frac{4}{\sqrt{2}-1}, \frac{3}{\sqrt{2}-1}\right)$

when $y = -\frac{3}{4}x \qquad -3x - 3\sqrt{2}x + 12 = 0$

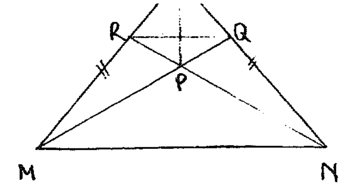
$x = \frac{4}{1+\sqrt{2}}$

pt. B $\left(\frac{4}{1+\sqrt{2}}, \frac{-3}{1+\sqrt{2}}\right)$

midpt AB $= \left(\frac{1}{2}\left(\frac{4}{\sqrt{2}-1} + \frac{4}{1+\sqrt{2}}\right), \frac{1}{2}\left(\frac{3}{\sqrt{2}-1} - \frac{3}{1+\sqrt{2}}\right)\right)$

$= \left(\frac{1}{2}\times 8\sqrt{2}, \frac{1}{2}\times 6\right)$

$= (4\sqrt{2}, 3)$ which is P



In ΔMPL and ΔNPL

$LM = LN$ given

LP is common

$\angle PLM = \angle PLN$ P on bisector of $\angle MLN$

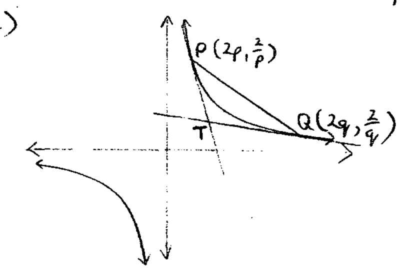
$\therefore \Delta MPL \equiv \Delta NPL$ SAS

$\therefore \angle LMP = \angle LNP$ corresp $\angle$s $\equiv \Delta$s

Interval RQ subtends equal angles $\angle LMP$ and $\angle LNP$ on the same side of it

$\therefore$ M, N, Q and R are concyclic

(c)



(i) $m_{PQ} = \frac{\frac{2}{p} - \frac{2}{q}}{2p - 2q}$

$= \frac{2(q-p)}{pq} \Big/ 2(p-q)$

$= \frac{-1}{pq}$

eq. PQ $\quad y - \frac{2}{p} = \frac{-1}{pq}(x - 2p)$

$pqy - 2q = -x + 2p$

$x + pqy = 2(p+q)$

tangent at P $\quad x + p^2y = 4p$

(iii) $(2(p+q-1), 2)$ satisfies eq.

$2(p+q-1) + 2pq = 2(p+q)$

$-2 + 2pq = 0$

$pq = 1$

tangent at P $\quad x + p^2y = 4p$

tangent at Q $\quad x + q^2y = 4q$

solve simult. $\quad (p^2 - q^2)y = 4(p-q)$

$y = \frac{4(p-q)}{(p-q)(p+q)}$

$y = \frac{4}{p+q}$

$x + p^2 \times \frac{4}{p+q} = 4p$

$x = 4p - \frac{4p^2}{p+q}$

$= \frac{4p(p+q) - 4p^2}{p+q}$

$= \frac{4pq}{p+q}$

$= \frac{4}{p+q}$ since pq

$\therefore y = x$ is locus of T

Question 8

(a) $(\cos\theta + i\sin\theta)^3$ expand + de M

$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i$

$(\text{cis}\,\theta)^3 = \cos 3\theta + i\sin 3\theta$

equating real parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos$

$= 4\cos^3\theta - 3\cos\theta$

[or expand $\cos(2\theta + \theta)$ ...]

$\text{Let } x = \cos\theta$

$8x^3 - 6x = -1$

$2(4\cos^3\theta - 3\cos\theta) = -1$

$2\cos 3\theta = -1$

$\cos 3\theta = -\frac{1}{2}$

$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

$\therefore$ solution is $x = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$

$a > b \;, c > d$

$\therefore a-b > 0 \;, c-d > 0$

$(a-b)(c-d) > 0$

$ac - ad - bc + bd > 0$

$ac + bd > ad + bc$

) $a \geqslant c+d \;, b \geqslant c+d$

$(a-c) \geqslant d \;, (b-d) \geqslant c$

$(a-c)(b-d) \geqslant dc$

$ab - ad - bc + dc \geqslant dc$

$ab \geqslant ad + bc$

) (i)

$LHS = \frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} \times \frac{1+\sin\theta + i\cos\theta}{1+\sin\theta + i\cos\theta}$

$= \frac{(1+\sin\theta + i\cos\theta)^2}{(1+\sin\theta)^2 - i^2\cos^2\theta}$

$= \frac{(1+\sin\theta)^2 + 2i(1+\sin\theta)\cos\theta - \cos^2\theta}{(1+\sin\theta)^2 + \cos^2\theta}$

$= \frac{(1+\sin\theta)^2 + 2i(1+\sin\theta)\cos\theta - (1-\sin^2\theta)}{(1+\sin\theta)^2 + (1-\sin^2\theta)}$

$= \frac{(1+\sin\theta)^{\cancel{2}} + 2i\cancel{(1+\sin\theta)}\cos\theta - (1-\sin\theta)\cancel{(1+\sin\theta)}}{(1+\sin\theta)^{\cancel{2}} + (1-\sin\theta)\cancel{(1+\sin\theta)}}$

$= \frac{1+\sin\theta + 2i\cos\theta - 1 + \sin\theta}{1+\sin\theta + 1 - \sin\theta}$

$= \frac{2\sin\theta + 2i\cos\theta}{2}$

$= \sin\theta + i\cos\theta$

(ii) $RHS = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$

$= \sin(n\theta) + i\cos(n\theta)$

$= (\sin\theta + i\cos\theta)^n$

$= \left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n$

$= LHS$