ABBOTSLEIGH

AUGUST 2003

YEAR 12
ASSESSMENT 4
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks) Use a SEPARATE writing booklet.

(a) By completing the square, find $\int \frac{dx}{x^2-4x+8}$ **2**

(b) Use the substitution $x=\sin\theta$ to evaluate

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2\,dx}{\sqrt{1-x^2}}$$ **3**

(c) Use integration by parts to find $\int_1^e \frac{\ln x}{\sqrt{x}}\,dx$ **3**

(d) (i) Find real numbers a, b and c such that

$$\frac{x+7}{(1+x^2)(1+x)} \equiv \frac{ax+b}{1+x^2}+\frac{c}{1+x}$$ **2**

(ii) Find $\int \frac{x+7}{(1+x^2)(1+x)}\,dx$ **2**

(e) Use the substitution $t=\tan\frac{x}{2}$ to find $\int \frac{\tan x}{1+\cos x}\,dx$ **3**

QUESTION 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Express $z = 1+\sqrt{3}\,i$ in modulus-argument form. **2**

(ii) Show that $z^7 - 64z = 0$ **3**

(b) Let $z = x+iy$, where x and y are real numbers.

(i) Solve $z\bar{z} + 2z = \frac{1}{4} + i$ **4**

(ii) Draw a neat sketch of the locus of $\mathrm{Re}(z) = |z-2|$ **3**

(c) The points A, B, C, D on an Argand diagram represent the complex numbers a, b, c, d respectively. **3**

If $a+c = b+d$ and $a-c = i(b-d)$ find what type of quadrilateral is defined by $ABCD$. Clearly justify your answer.

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

(a) Given the function $f(x) = x\sqrt{4-x^2}$.

(i) State its natural domain and show that it is an odd function. **2**

(ii) Show that on the curve $y = f(x)$, stationary points occur at $x = \pm\sqrt{2}$. Find the coordinates of the stationary points and determine their nature. **3**

(iii) Draw a neat sketch of the curve $y = f(x)$, indicating the above features, and given that there is a point of inflexion at the origin. **2**

(iv) On separate diagrams, sketch the curves

1. $y^2 = x^2(4-x^2)$ **2**
2. $y = \dfrac{1}{f(x)}$ **2**

(b) Given that the sum of two of the roots of the equation $x^4 - x^3 - x^2 - x - 2 = 0$ is zero, find all four roots. **4**

Marks

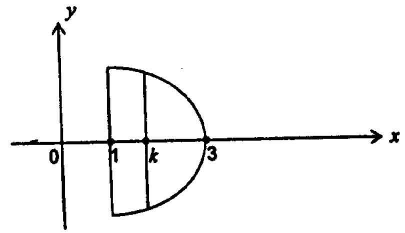
QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

(a) The ellipse E has equation $\frac{x^2}{8} + \frac{y^2}{4} = 1$

(i) Write down its eccentricity, the coordinates of its foci, S and S', and the equation of each directrix. Sketch the ellipse E. **4**

(ii) If $P(x_1, y_1)$ is an arbitrary point on E, prove that the sum of the distances SP and $S'P$ is independent of the position of P. **2**

(b)



The base of a particular solid is the region enclosed by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $x = 1$. Each cross-section of the solid perpendicular to the x-axis is an equilateral triangle.

(i) Show that the area of the triangle at $x = k$ is $\frac{\sqrt{3}}{9}(36 - 4k^2)$ **2**

(ii) Find the volume of the solid. **3**

(iii) Consider a second solid which is obtained by rotating the region enclosed by the ellipse and the line $x = 1$ about the y-axis. Find the volume of the solid formed. **4**

Marks

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

(a) Factorise x^2+4x+3 and hence, or otherwise, show that the coefficient of x^4 in the expansion of $(x^2+4x+3)^6$ is 61 695. **4**

(b) (i) Prove that the equation of the tangent to the hyperbola $x^2-y^2=c^2$ at the point $P(x_1,y_1)$ is $xx_1-yy_1=c^2$. **2**

(ii) This tangent meets the lines $y=x$ and $y=-x$ at Q and R respectively and O is the origin. Prove that the area of triangle OQR is constant. **4**

(c) A particle moves in a straight line and its position x at any time t is given by
$x=\sqrt{3}\cos 3t-\sin 3t$

(i) Show that the motion is simple harmonic. **2**

(ii) Determine the period and amplitude of the motion. **3**

Marks

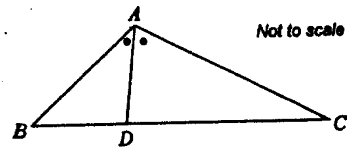
QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

(a) If α,β,γ are the roots of $2x^3-4x^2-3x-1=0$, find the value of $(\alpha-1)(\beta-1)(\gamma-1)$. **3**

(b) Solve for x, y, z over the complex numbers: **4**

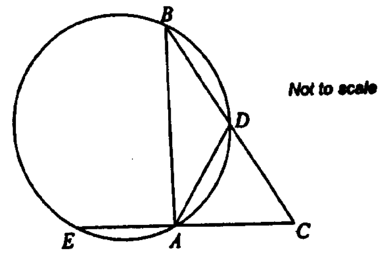
$x+y+z=1$
$xy+yz+zx=9$
$xyz=9$

(c) (i) In the triangle ABC, AD bisects angle BAC.



Prove that $\dfrac{BD}{DC}=\dfrac{BA}{AC}$ **4**

(ii)



In the diagram $AB=BC$ and AD bisects angle BAC.

Prove that $BD=CE$. **4**

QUESTION 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The minute hand OP and the hour hand OQ of a clock are 4cm and 3cm long respectively. Let $PQ =$ the distance between the tips of the hands of the clock.

(i) Show that $\dfrac{dPQ}{d\theta} = \dfrac{12\sin\theta}{\sqrt{25-24\cos\theta}}$ where θ is the acute angle between the hands of the clock. **2**

(ii) Hence show that the rate of increase (in cm per hour) of the length of PQ at 9 o'clock is $\dfrac{22\pi}{5}$ cm/h. **3**

(b) (i) If $f(x)$, $g(x)$ and $h(x)$ are distinct non-negative continuous functions of x in the interval $a \le x \le b$ and $f(x) < g(x) < h(x)$, explain why

$$\int_a^b f(x)\,dx < \int_a^b g(x)\,dx < \int_a^b h(x)\,dx$$ **2**

(ii) By considering the interval $0 < x < 1$ as an inequality, use algebra to show that

$$\frac{1}{2}x(1-x)^3 < \frac{x(1-x)^3}{1+x} < x(1-x)^3$$ **3**

(iii) Deduce that $\dfrac{1}{2}\displaystyle\int_0^1 x(1-x)^3\,dx < \int_0^1 \frac{x(1-x)^3}{1+x}\,dx < \int_0^1 x(1-x)^3\,dx$ **1**

(iv) Given that $\displaystyle\int_0^1 \frac{x(1-x)^3}{1+x}\,dx = \frac{67}{12} - 8\ln 2$, deduce that $\dfrac{83}{120} < \ln 2 < \dfrac{667}{960}$ **4**

QUESTION 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A projectile is fired from the origin O with velocity V and angle of elevation α, where α is acute.

(i) By letting $g =$ acceleration due to gravity and $k = \dfrac{V^2}{2g}$, derive the Cartesian equation of the parabolic path of the projectile. Show that as a quadratic equation in $\tan\alpha$, its Cartesian equation is

$$x^2\tan^2\alpha - 4kx\tan\alpha + \left(4ky + x^2\right) = 0$$ **4**

(ii) Show that the projectile can pass through the point (X, Y) in the first quadrant by firing at two different initial angles α_1 and α_2 if

$$X^2 < 4k^2 - 4kY$$ **2**

(iii) Let $\tan\alpha_1$ and $\tan\alpha_2$ be the two real roots of the quadratic equation in part (i). Show that $\tan\alpha_1 \tan\alpha_2 > 1$, and hence explain why it is impossible for <u>both</u> α_1 and α_2 to be less than 45°. **3**

(b) It is given that $A > 0, B > 0$ and n is a positive integer.

(i) Divide $A^{n+1} - A^nB + B^{n+1} - B^nA$ by $A - B$ **2**

(ii) Deduce that $A^{n+1} + B^{n+1} \ge A^nB + B^nA$ **1**

(iii) Show by induction, that $\left(\dfrac{A+B}{2}\right)^n \le \dfrac{A^n + B^n}{2}$ **3**

End of paper