ABBOTSLEIGH EXTENSION 2 TRIAL 2004

Total marks – 120

Attempt Questions 1-8

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the function $f(x) = \dfrac{x-1}{x+3}$.

(i) Sketch the graph of $y = f(x)$ showing clearly the coordinates of any points of intersection with the x axis or the y axis, and the equations of any asymptotes. **3**

(ii) Show that the line $y = x$ is a tangent to the curve $y = f(x)$ and find the coordinates of its point on contact. Draw the tangent line on the graph and show the coordinates of its point of contact. **3**

(iii) On separate axes, sketch the graphs of $y = \dfrac{1}{f(x)}$ and $y = f^{-1}(x)$. In each case, show clearly the coordinates of any points of intersection with the x axis or the y axis, the equations of any asymptotes and the line $y = x$. **5**

(b) Find

(i) $\displaystyle\int \frac{3x\,dx}{(2x^2-1)^4}$ **1**

(ii) $\displaystyle\int 2xe^x dx$ **2**

(c) Determine the minimum value of $f(x) = 4\cos^2 x - 4\sin^2 x$ **1**

Marks

QUESTION 2 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_{-2}^{2} \sqrt{4-x^2}\,dx$ 2

(b) If $w = 3+4i$ and $z = 5-2i$, find $z(|w| - \overline{w})$. 2

(c) For a particle moving on the x - axis, the acceleration at time t is given by $\frac{d^2x}{dt^2} = -\tan x$. Initially the particle is at the origin with velocity $u > 0$.

(i) Show $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ where v is the velocity of the particle. Hence find v in terms of x. 2

(ii) Explain why motion could only exist for $0 \le x < \frac{\pi}{2}$. 2

(iii) Discuss briefly the value of u for the particle to move from its initial position to near $\frac{\pi}{2}$. 1

(d) Given $z = 1 - i\sqrt{3}$

(i) Write z in modulus-argument form. 1

(ii) Hence find z^8, giving your answer in the form $x + iy$, where x and y are real. 2

(e) Sketch the following loci on separate Argand diagrams:

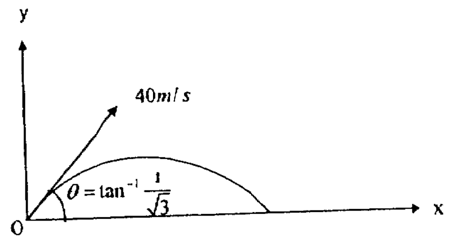
(i) $|z-1| = |z+i|$ 1

(ii) $z\overline{z} - i\overline{z} + iz = 0$ 2

Marks

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows the path of an object launched at an angle of $\theta = \tan^{-1}\frac{1}{\sqrt{3}}$ to the horizontal with an initial speed of 40ms^{-1} from O. The acceleration due to gravity is taken as 9.8ms^{-2}, and air resistance is ignored.

(i) Derive expressions for $x(t)$ and $y(t)$ where t is time in seconds. 4

(ii) Show that the path of the object is parabolic. 2

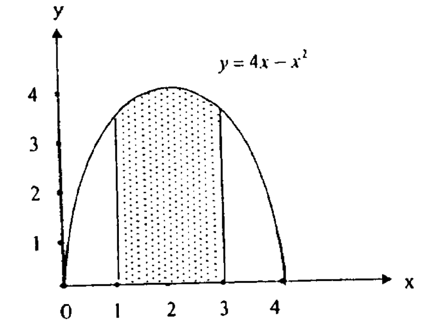
(iii) Find the angle and speed of the object at 1.5 seconds. 3

(iv) Calculate the time and range of the flight. 3

(b) If ω and ω^2 are the complex roots of unity, show that $\omega^4 + \omega^3 + 2\omega^2 = \omega^5$. 3

QUESTION 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) 5



The shaded area shown on the diagram between the curve $y = 4x - x^2$, the x axis, $x = 1$ and $x = 3$, is rotated about the y axis to form a solid. Use the method of cylindrical shells to find the volume of the solid.

(b) Show that $2 + i$ is a zero of $x^3 - 11x + 20 = 0$. Hence or otherwise solve $x^3 - 11x + 20 = 0$. 4

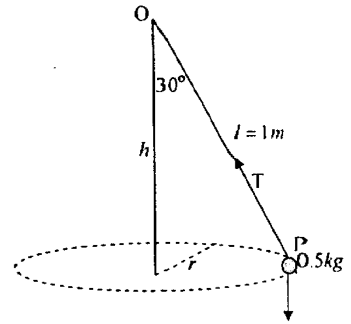
(c) The equation $x^3 + 2x - 1 = 0$ has roots α, β, γ.

(i) Find the value of $\sum \alpha$ and the value of $\sum \alpha\beta$. 2

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. 2

(iii) Find the equation with roots $\frac{1}{\alpha - 1}, \frac{1}{\beta - 1}, \frac{1}{\gamma - 1}$. 2

QUESTION 5 (15 marks) Use a SEPARATE writing booklet. **Marks**



(a) A body P of mass 0.5 kg is suspended from a fixed point O by means of a light rod of length 1 m. The mass is rotated in a horizontal circle at a constant speed and the rod makes an angle of 30° with the downward direction of the vertical using $g = 9.8\ \text{ms}^{-2}$.

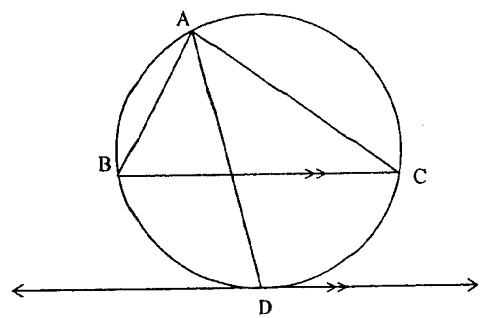
(i) Resolve the vertical and horizontal forces at P and show that $\tan\theta = \frac{v^2}{rg}$ where v is the linear velocity of P and r is the radius of the circle. 3

(ii) Find the tension (T) in the rod 1

(iii) Find the linear velocity of P. 1

(iv) Find the period of the motion. 1

(b) 3



A tangent intersects the circle above at D. BC is a chord to the circle parallel to the tangent. A is a point on the circle. Prove that $\angle DAB = \angle DAC$.

Marks

QUESTION 5 (continued)

(c) The velocity of a particle moving in a straight line is given by $v^2 = 4(8 + 2x - x^2)$ where x is the displacement in metres from the origin.

(i) Show the motion is Simple Harmonic Motion **1**

(ii) Determine the centre of the motion **1**

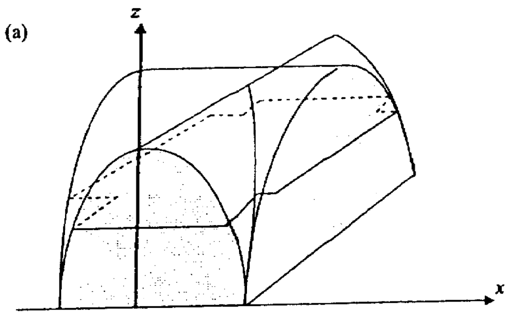
(iii) Determine the rest positions of the particle. **2**

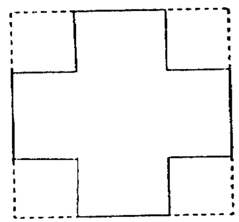
(d) A satellite travels in a circular orbit of radius 20 000 km around the earth, taking 15 hours to complete a revolution. Find the angular velocity of the satellite. **2**

Marks

QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

(a)





A sandstone cap on the corner of a fence is shown above, formed in the shape of two intersecting parabolic cylinders.

On the front face, the equation of the parabola is $z = 4 - x^2$, where x is the horizontal distance measured from the mid-point of the base of the front face, and z is the height.

The shape of a horizontal slice of thickness dz taken at height z is also shown. It is a square with four smaller squares removed, one from each corner.

(i) Find x in terms of z. **1**

(ii) Show that the volume is $V = \int_0^4 \left(4^2 - 4(2 - \sqrt{4-z})^2\right) dz$. **2**

(iii) Hence find the volume of stone in the cap. **4**

(b) A rock of mass 5 kg is propelled vertically upward into the air from the ground with initial speed 12ms^{-1}. The rock is subject to air resistance of $\frac{v^2}{2}$ Newtons in the opposite direction to its velocity, v ms^{-1}. The rock is also subject to downward gravitational force of 50 Newtons. Thus the equation of motion of the rock until it reaches it highest point is $\ddot{x} = -\frac{v^2}{10} - 10$, where x metres is the height of the rock above the ground when its velocity is $v\,\text{ms}^{-1}$.

(i) Using $\ddot{x} = v\frac{dv}{dx}$, show that $v^2 = 244e^{-\frac{x}{5}} - 100$ while the rock is rising. **3**

(ii) Find the maximum height reached by the rock. **2**

(iii) Using $\ddot{x} = \frac{dv}{dt}$, find the time taken by the rock to reach maximum height. **3**

Marks

QUESTION 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that the tangent to the rectangular hyperbola $xy = 4$ at the point $T\left(2t, \frac{2}{t}\right)$ has equation $x + t^2 y = 4t$. **2**

(ii) This tangent cuts the x – axis at point Q. Show that the line through Q which is perpendicular to the tangent at T has equation $t^2 x - y = 4t^3$. **2**

(iii) This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint M of RS has coordinates $M(2t, -2t^3)$. **2**

(iv) Find the equation of the locus of M as T moves on the rectangular hyperbola, stating any restrictions that may apply. **2**

(b) Find the roots of the equation $(2+i)z^2 - 4z + (2-i) = 0$ expressing any complex roots in the form $a + bi$ where a and b are real. **3**

(c) Given the function $f(x) = 2\cos^{-1}\left(x^2 - 1\right)$

(i) State the domain and range of $f(x)$. **2**

(ii) Hence make a neat sketch of $f(x)$. **2**

Marks

QUESTION 8 (15 marks) Use a SEPARATE writing booklet.

(a) The line $y = x$ meets a directrix of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ at the point V in the first quadrant. Tangents from V meet the ellipse at P (x_1, y_1) and Q (x_2, y_2). The eccentricity of the ellipse is e.

(i) Show this information on a sketch. **1**

(ii) Given that the chord of contact of tangents from the point (x_0, y_0) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has equation $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$, deduce that the equation of PQ is $\frac{x}{ae} + \frac{y}{ae(1-e^2)} = 1$ and verify that PQ is a focal chord of the ellipse. **3**

(iii) Show that x_1 and x_2 are roots of the equation $(2-e^2)x^2 - 2ae(1-e^2)x + a^2(e^2 - e^4 - 1) = 0$ **2**

(b) (i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{dx}{\sin x} = \ln 3$ **3**

(ii) Use the substitution $u = \pi - x$ to show that $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - x}{\sin x}dx$ **2**

(iii) Hence find the exact value of $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}dx$ **1**

(c) Show that the greatest coefficient of $(2+3x)^{12}$ is $\left(\binom{6}{6}\binom{6}{1} + \binom{6}{5}\binom{6}{2} + \binom{6}{4}\binom{6}{3}\right)(2)^6(3)^7$. You may assume the result $t_{k+1} \geq t_k$ in $(a+bx)^n$ is $(n-r+1)b \geq ar$. **3**

End of Paper