Student Number____________________

ABBOTSLEIGH

AUGUST 2008
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

Outcomes assessed

HSC course

- **E1** appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- **E2** chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- **E3** uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- **E4** uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
- **E5** uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
- **E6** combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
- **E7** uses the techniques of slicing and cylindrical shells to determine volumes
- **E8** applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
- **E9** communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

From the Extension 1 Mathematics Course

Preliminary course

- **PE1** appreciates the role of mathematics in the solution of practical problems
- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives that require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE1** appreciates interrelationships between ideas drawn from different areas of mathematics
- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks)
Use a SEPARATE writing booklet.

(a) Using the table of standard integrals find $\int \frac{dx}{\sqrt{x^2+7}}$ **1**

(b) By completing the square find $\int \frac{dx}{\sqrt{4x-x^2}}$ **2**

(c) Find $\int \frac{1-2x}{\sqrt{1-x^2}}dx\ , \quad |x|<1.$ **2**

(d) Find $\int \cos^3 x\, dx.$ **2**

(e) (i) Use the substitution $x=\frac{2}{3}\sin\theta$ to prove that $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\ dx = \frac{\pi}{3}.$ **3**

(ii) Hence or otherwise, find the area enclosed by the ellipse $\frac{9x^2}{4}+\frac{y^2}{4}=1.$ **2**

(f) Evaluate $\int_0^1 \tan^{-1} x\ dx$. **3**

Marks

QUESTION 2 (15 marks)
Start a new writing booklet.

(a) Given $z_1 = 3 - i$ and $z_2 = 2 + 5i$, express the following in the form $a + ib$ where a and b are real:

(i) $(\overline{z}_1)^2$ **2**

(ii) $\dfrac{z_1}{z_2}$ **2**

(iii) $|z_1 z_2|$ **2**

(b) (i) Sketch the region $|z + 1 + i| \leq 1$. **2**

(ii) Find the maximum and minimum values of $|z|$. **2**

(c) (i) The complex number $z = x + iy$ is represented by the point P. If $\dfrac{z-1}{z-2i}$ is purely imaginary, show that the locus of P is the circle $x^2 - x + y^2 - 2y = 0$. **3**

(ii) Sketch this locus showing all important features. **2**

Marks

QUESTION 3 (15 marks)
Start a new writing booklet.

(a) Sketch on separate diagrams, the graphs of:

(i) $y=(x-1)^2(x+2)$ **1**

(ii) $y^2=(x-1)^2(x+2)$ **2**

(iii) $y=\dfrac{1}{(x-1)^2(x+2)}$ **2**

(b) Sketch $y=\log_e(x+1)^2$ **2**

(c) Sketch the graph of the function $y=\dfrac{x^2-x+1}{(x-1)^2}$, clearly showing the coordinates of any points of intersection with the x and y axes, the coordinates of any turning points and the equations of any asymptotes. There is no need to investigate points of inflexion. **4**

(d) If α, β and γ are the roots of $x^3+2x^2-3x-4=0$,

(i) Evaluate $\alpha^2+\beta^2+\gamma^2$. **2**

(ii) Form the equation whose roots are $\beta\gamma$, $\alpha\gamma$ and $\alpha\beta$. **2**

Marks

QUESTION 4 (15 marks)
Start a new writing booklet.

(a) The foci of a hyperbola of eccentricity $\frac{13}{12}$ are the points $(\pm 13, 0)$.

(i) Show that the equation of the hyperbola is $\frac{x^2}{144} - \frac{y^2}{25} = 1$. **2**

(ii) Find the equation of the tangent to the hyperbola at the point $(12\sec\theta, 5\tan\theta)$. **3**

(b) (i) Show that the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$. **2**

(ii) Show that the pair of tangents drawn from the point (3, 4) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to each other. **3**

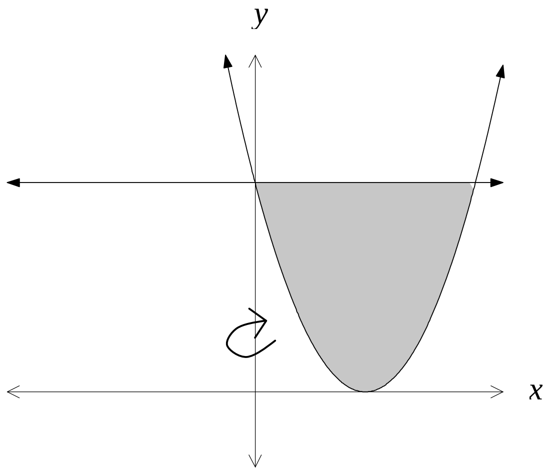
(c) (i) Verify that $\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is a root of $z^5 + z - 1 = 0$. **2**

(ii) Find the monic cubic equation with real coefficients whose roots are also the roots of $z^5 + z - 1 = 0$ but do not include α. **3**

QUESTION 5 (15 marks)
Start a new writing booklet.

(a) The base of a certain solid is a circle with radius 2. Each parallel cross-section of the solid is a square. Find the volume of the solid. **3**

(b) The area enclosed by the curve $y=(x-2)^2$ and the line $y=4$ is rotated around the y – axis. Use the method of cylindrical shells to find the volume formed. **3**



(c) (i) Show that the tangent to the rectangular hyperbola $xy=4$ at the point $T\left(2t,\dfrac{2}{t}\right)$ has equation $x+t^2y=4t$. **2**

(ii) This tangent cuts the x-axis at the point Q. Find the coordinates of Q. **1**

(iii) Show that the line through Q which is perpendicular to the tangent at T has equation $t^2x-y=4t^3$. **1**

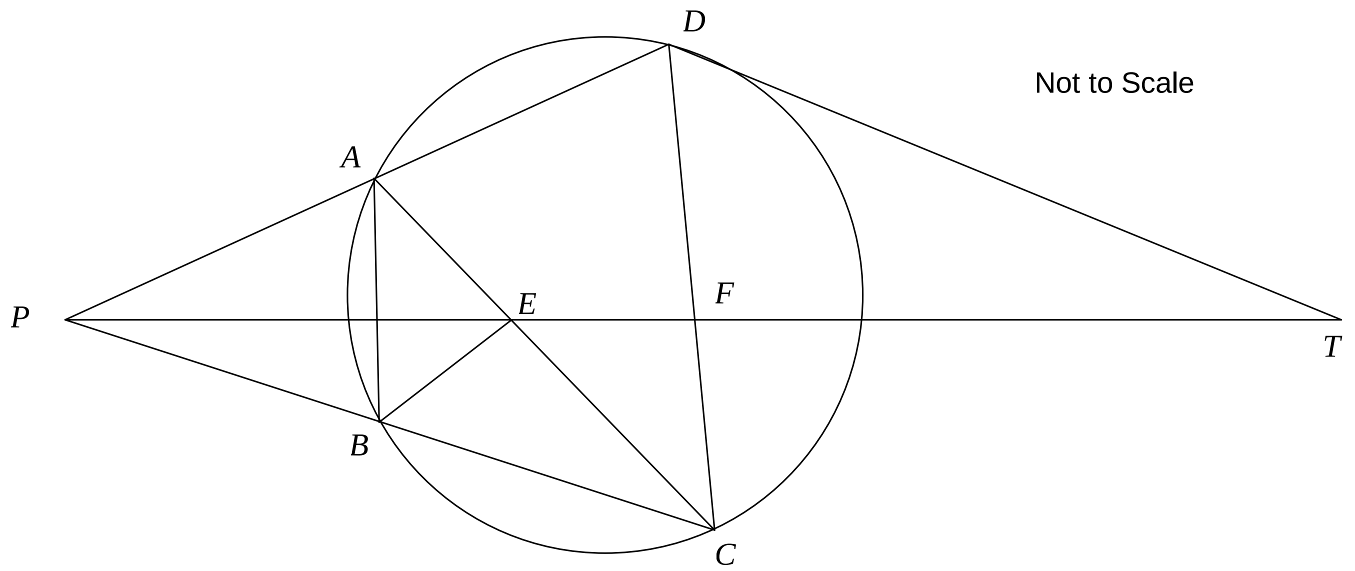
(iv) This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint of RS has coordinates $M\left(2t,-2t^3\right)$. **3**

(v) Find the equation of the locus of M as T moves on the rectangular hyperbola, stating any restrictions that may apply. **2**

Marks

QUESTION 6 (15 marks)
Start a new writing booklet.

(a) $ABCD$ is a cyclic quadrilateral. DA produced and CB produced meet at P. T is a point on the tangent at D. PT cuts CA and CD at E and F respectively. $TF = TD$.



(i) Copy the diagram and show that $AEFD$ is a cyclic quadrilateral. **3**

(ii) Show that $AEBP$ is a cyclic quadrilateral. **2**

(b) (i) If $I_n = \int_0^1 \frac{dx}{\left(1+x^2\right)^n}$ prove that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$. **3**

(ii) Hence evaluate $\int_0^1 \frac{dx}{\left(1+x^2\right)^3}$. **2**

(c) (i) Use the substitution $x = a - y$ where a is a constant to prove that

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$ **1**

(ii) Hence show that $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$ **4**

Marks

QUESTION 7 (15 marks)
Start a new writing booklet.

(a) The functions $S(x)$ and $C(x)$ are defined by the formulae:

$$S(x)=\frac{1}{2}(e^x-e^{-x}) \text{ and } C(x)=\frac{1}{2}(e^x+e^{-x}).$$

(i) Verify that $S'(x)=C(x)$. **1**

(ii) Show that $S(x)$ is an increasing function for all real values of x. **1**

(iii) Prove that $\left[C(x)\right]^2=1+\left[S(x)\right]^2$. **1**

(iv) $S(x)$ has an inverse function $S^{-1}(x)$ for all values of x. Briefly justify this statement. **1**

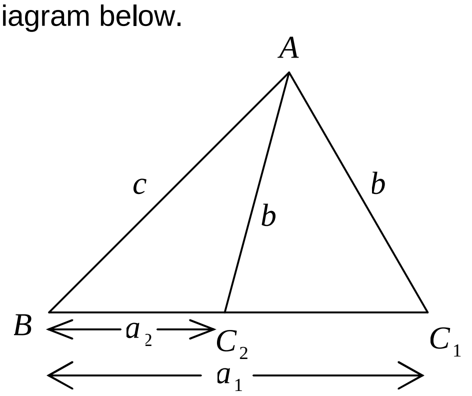
(v) Let $y=S^{-1}(x)$. Prove $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1+x^2}}$. **2**

(vi) Hence, or otherwise, show that $S^{-1}(x)=\log_e\left(x+\sqrt{1+x^2}\right)$. **2**

(b) (i) Using the remainder theorem, or otherwise, show that $x-a-b-c$ is a factor of $P\left(x\right)=(x-a)(x-b)(x-c)-(b+c)(c+a)(a+b)$. **2**

(ii) Hence, or otherwise, solve the equation $(x-2)(x+3)(x+1)-4=0$. **2**

(c) In $\triangle ABC$ the lengths b and c and $\angle B$ are given and have such values that two distinct triangles are possible as shown in the diagram below.



Show that $a_1-a_2=2\sqrt{b^2-c^2\sin^2 B}$ **3**

QUESTION 8 (15 marks) **Marks**
Start a new writing booklet.

(a) A particle of mass $1\,kg$ moves in a straight line before coming to rest. The resultant force acting on the particle directly opposes its motion and has magnitude $m(1+v)$ where v is its velocity. Initially the particle is at the origin and travelling with velocity Q where $Q>0$

(i) Show that v is related to the displacement x by the formula $x = Q - v + \log_e\left(\frac{1+v}{1+Q}\right)$. **3**

(ii) Find an expression for v in terms of t. **2**

(iii) Find an expression for x in terms of t. **1**

(iv) Show that $Q = x + v + t$ **1**

(v) Find the distance travelled and the time taken by the particle in coming to rest. **2**

(b) (i) State why, for $x<1$, the sum of n terms of the series $1+x+x^2+x^3+.......+x^{n-1}$ is $\frac{1-x^n}{1-x}$. **1**

(ii) Show that $1+2x+3x^2+.......+(n-1)x^{n-2} = \frac{(n-1)x^n - nx^{n-1}+1}{(1-x)^2}$ **2**

(iii) Hence find an expression for $1+1+\frac{3}{4}+\frac{4}{8}+.......+\frac{n-1}{2^{n-2}}$ and show that this sum is always less than 4. **3**

END OF PAPER

TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \quad = \frac{1}{a}e^{ax}, \qquad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \qquad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \qquad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

$$NOTE: \quad \ln x = \log_e x, \qquad x > 0$$

ABBOTSLEIGH

AUGUST 2008
YEAR 12
ASSESSMENT 4
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2
Solutions

Question 1

(a) $= \ln(x + \sqrt{x^2 + 7}) + C$

(b) $= \int \frac{dx}{\sqrt{4 - (x-2)^2}}$

$= \sin^{-1}\frac{x-2}{2} + C$

(c) $= \int \frac{1}{\sqrt{1 - x^2}} - 2x(1 - x^2)^{-\frac{1}{2}}\ dx$

$= \sin^{-1} x - 2\sqrt{1 - x^2} + C$

(d) $= \int \cos x\,(1 - \sin^2 x)\,dx$

$= \sin x - \frac{1}{3}\sin^3 x + C$

(e) i.

$dx = \frac{2}{3}\cos\theta\ d\theta, \qquad \begin{cases} x = 0, \sin\theta = 0 \\ x = \frac{2}{3}, \sin\theta = \frac{\pi}{2} \end{cases}$

$\therefore I = \int_0^{\frac{\pi}{2}} \frac{4}{3}\cos^2\theta\ d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{2}{3}(\cos 2\theta + 1)\,d\theta$

$= \left[\frac{1}{3}\sin 2\theta + \frac{2}{3}\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{3}$

ii.

Note that $\int_0^{\frac{2}{3}} \sqrt{4 - 9x^2}\,dx = \int_{\frac{2}{3}}^{0} \sqrt{4 - 9x^2}\,dx$

Now, the top part of the ellipse has the equation $y = \sqrt{4 - 9x^2}$

$A_{top} = 2I = \frac{2\pi}{3}, \qquad (from\ part\ i)$

Similarily, due to symmetry, the bottom part of the ellipse has the same area

$A_{total} = \frac{4\pi}{3}$

(f) Observing the graph of $y = \tan^{-1} x$ between $(0, 0)$ and $\left(1, \frac{\pi}{4}\right)$

$I = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan y\ dy$

$= \frac{\pi}{4} + [\ln(\cos x)]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

Question 2

(a) i.

$(\overline{z_1})^2 = 4 - 6i$

ii.

$\frac{z_1}{z_2} = \frac{(3-i)(2-5i)}{29}$

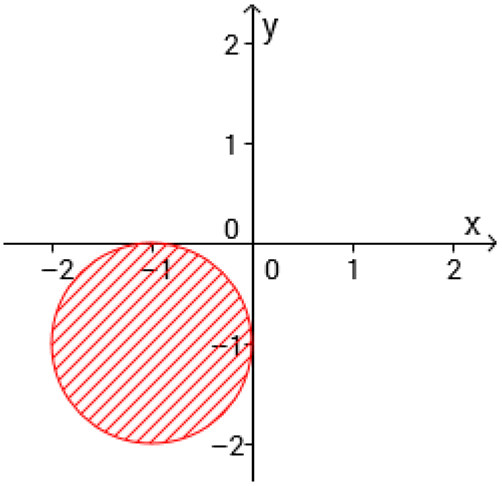
$= \frac{1-17i}{29}$

iii.

$z_1 z_2 = 1 + 17$

$|z_1 z_2| = \sqrt{290}$

(b) i.



ii.

Distance of centre of locus from $(0,0) = \sqrt{2}$

Minimum value of $|z| = \sqrt{2} - 1$, maximum value of $|z| = \sqrt{2} + 1$

(c) i.

$\frac{z-1}{z-2i} = \frac{x+iy-1}{x+iy-2i}$

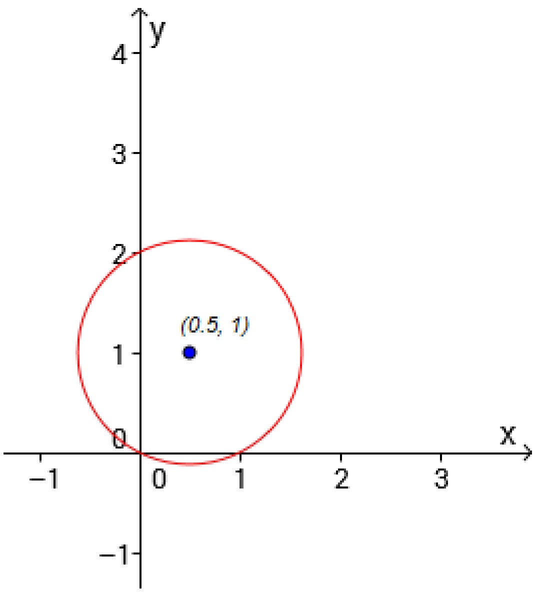
$= \frac{(x+iy-1)(x-iy+2i)}{x^2+(y-2)^2}$

$= \frac{x^2+2ix+y^2-2y-x-iy-2i}{x^2+(y-2)^2}$

Since the real part of $\frac{z-1}{z-2i}$ is 0 (i.e. the expression is purely imaginary)

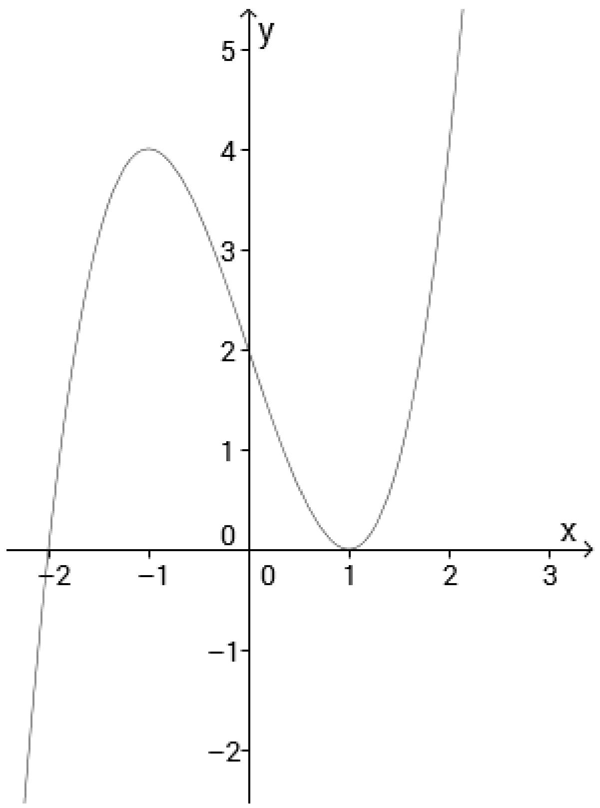
It follows that $x^2 - x + y^2 - 2y = 0$

ii.

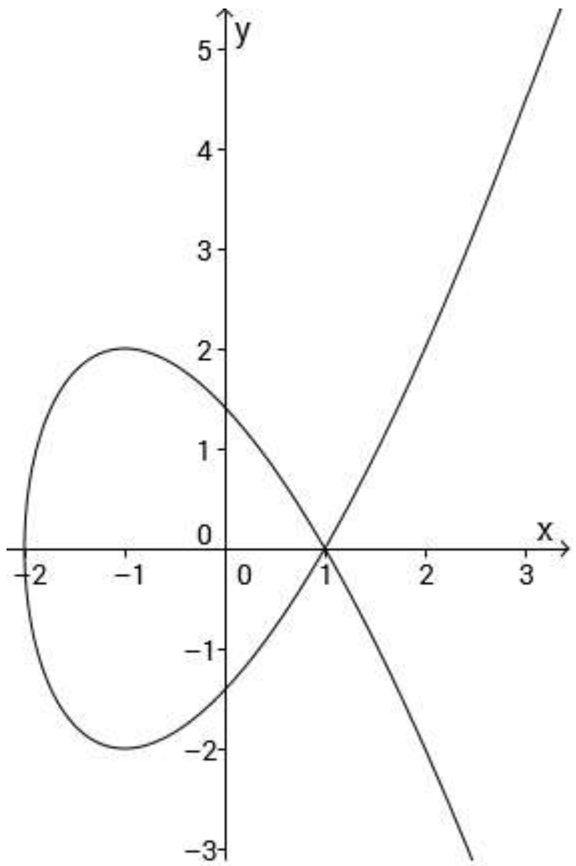


Question 3

(a) i.

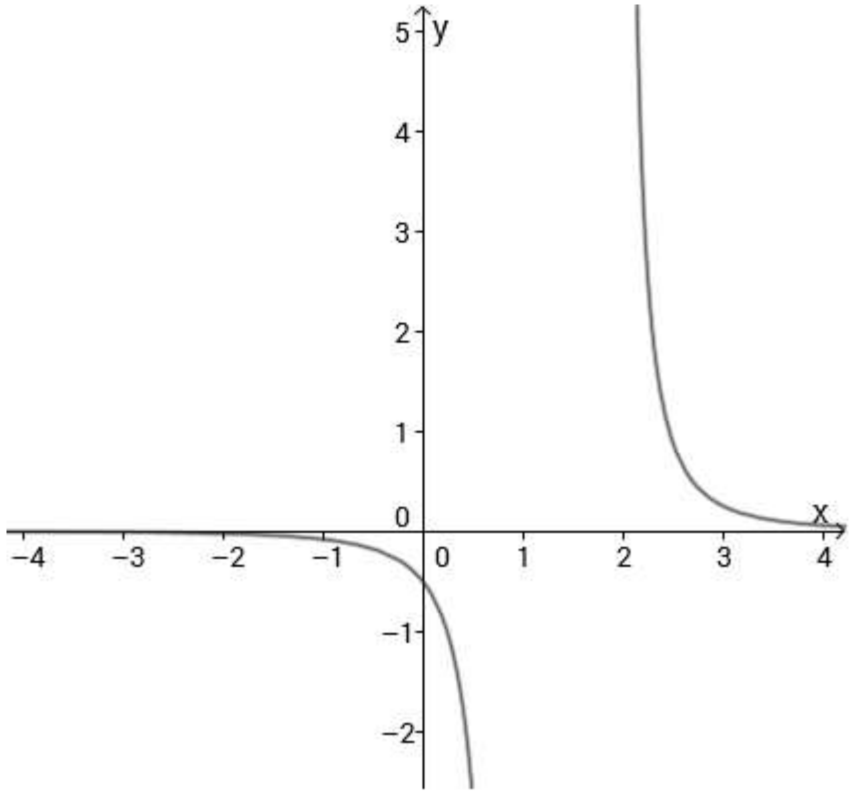


ii.

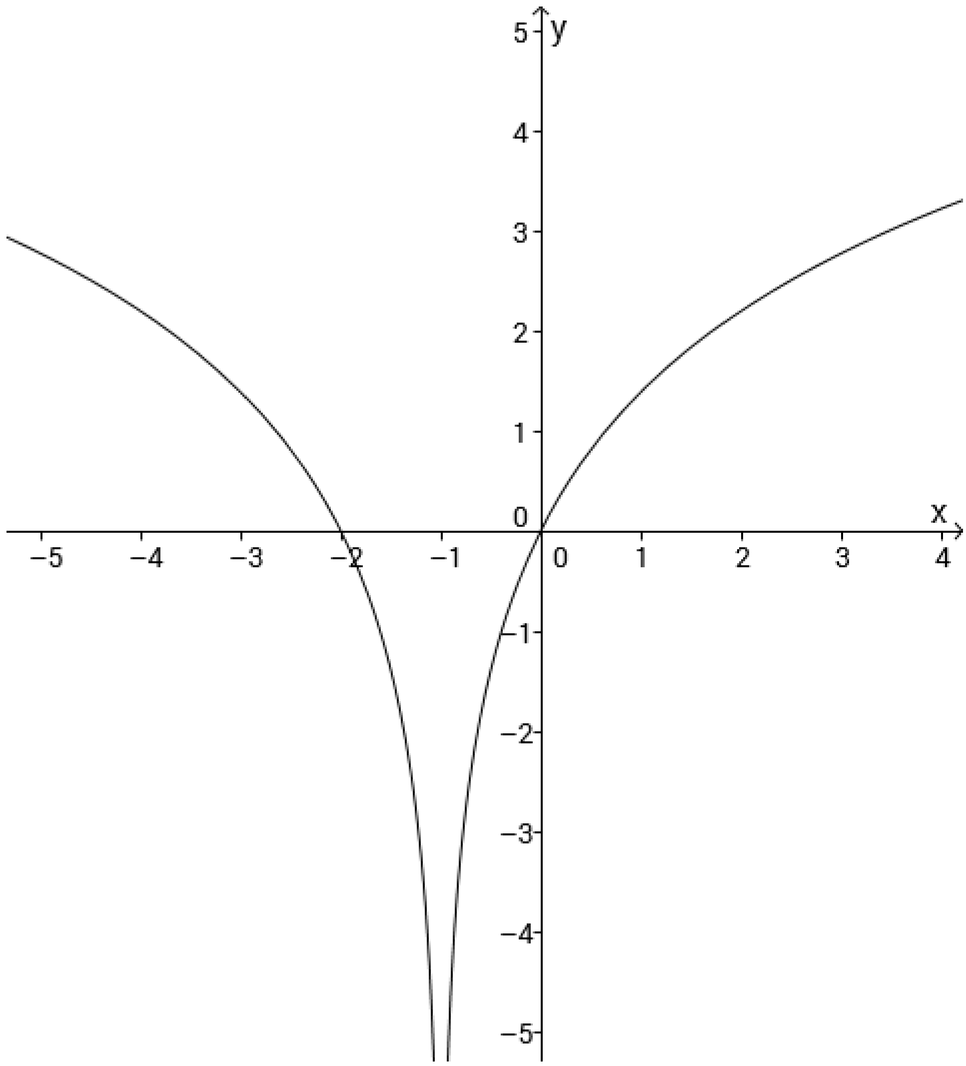


Question 3 (continued)

(a) iii.



(b)



Question 3 (continued)

(c) $y = x - 1 + \frac{x}{(x-1)^2}$

$\frac{dy}{dx} = -\frac{x+1}{(x-1)^3}, \qquad \frac{d^2y}{dx^2} = \frac{2(x+2)}{(x-1)^4}$

$At\ S.Ps, \qquad \frac{dy}{dx} = 0$

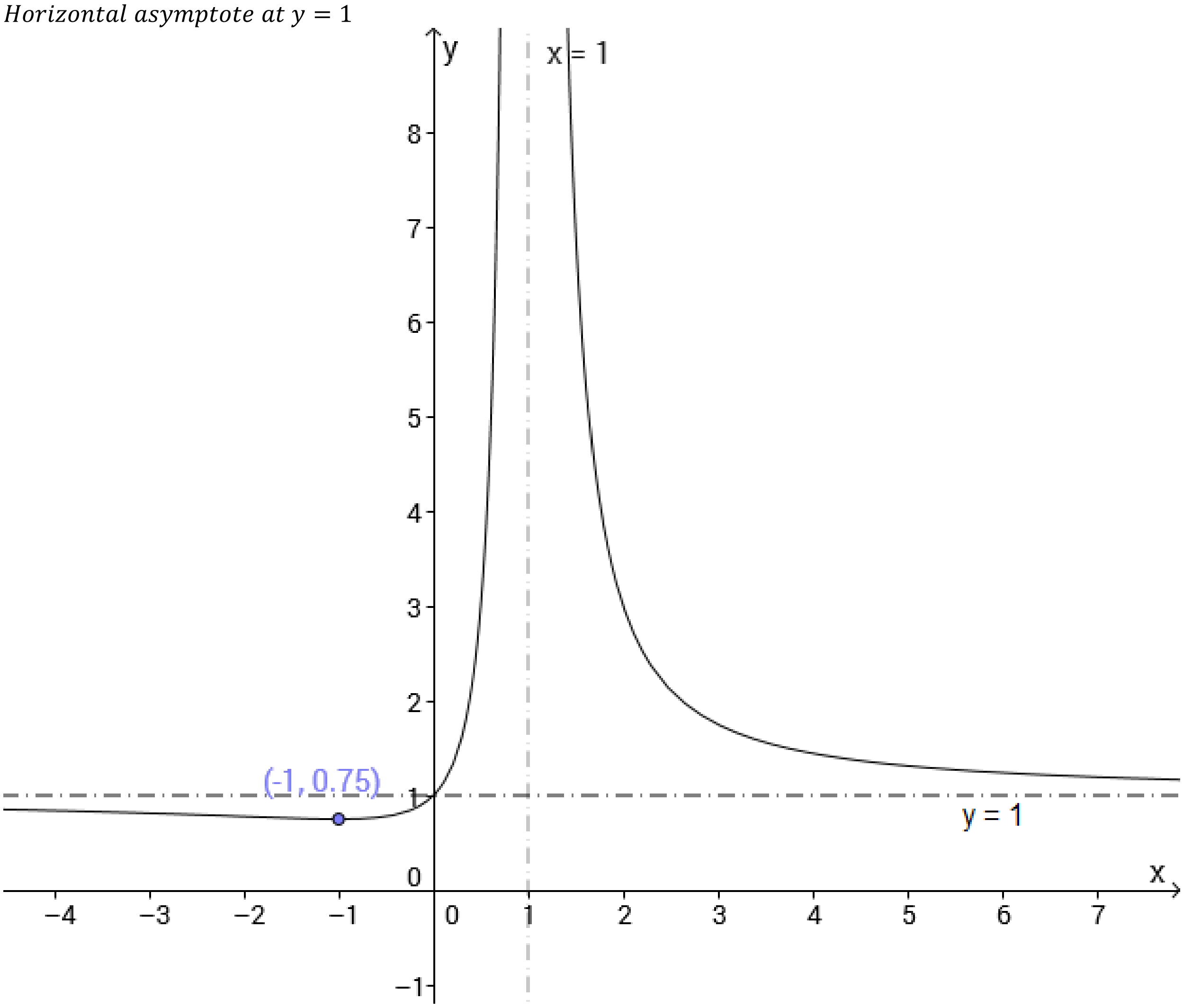
$\begin{cases} x = -1 \\ y = 0.75 \end{cases}$

$y'' > 0 \therefore \min T.P$

Vertical asymptote at $x = 1$

$As\ x \to \pm\infty, \qquad f(x) \to 1$

$Horizontal\ asymptote\ at\ y = 1$



(d) i.

$\Sigma\alpha = -2, \qquad \Sigma\alpha\beta = -3, \qquad \alpha\beta\gamma = 4$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2 \times \Sigma\alpha\beta$

$= 4 + 6$

$= 10$

ii.

Let the roots be A, B and C

$A + B + C = \alpha\beta + \beta\gamma + \gamma\alpha = -3$

$AB + BC + AC = \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha+\beta+\gamma) = 6$

$ABC = (\alpha\beta\gamma)^2 = 16$

Therefore the equation required is $x^3 + 3x^2 + 6x - 16 = 0$

Question 4

(a) i.

$e = \frac{13}{12}, \qquad ae = \pm 13$

$a = \pm 12, \qquad a^2 = 144$

Using $b^2 = a^2(e^2 - 1)$, noting that foci lie on x axis

$b^2 = 25$

Therefore equation is $\frac{x^2}{144} - \frac{y^2}{25} = 0$

ii.

$25x^2 - 144y^2 = 0$

$25x - 144y\frac{dy}{dx} = 0$ (implicit differentiation and then dividing all terms by 2)

$\frac{dy}{dx} = \frac{25x}{144y}$

At $(12\sec\theta, 5\tan\theta), m_T = \frac{5\sec\theta}{12\tan\theta}$

$12y\tan\theta - 60\tan^2\theta = 5x\sec\theta - 60\sec^2\theta$

$5x\sec\theta - 12y\tan\theta - 60 = 0$

(b) i.

Intersect $y = mx + c$ with ellipse gives

$b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$

$(a^2m^2 + b^2)x^2 + (2a^2cm)x + a^2c^2 - a^2b^2 = 0$

When $\Delta = 0$, $\quad y = mx + c$ is a tangent (one solution)

$4a^4c^2m^2 - 4(a^2m^2 + b^2)(a^2c^2 - a^2b^2) = 0$

$a^4c^2m^2 - a^4m^2c^2 + a^2b^2c^2 - a^2b^4 - a^4b^2m^2 = 0$

$a^2b^2c^2 - a^2b^4 - a^4b^2m^2 = 0$

$c^2 = a^2m^2 + b^2$ (rearranging and dividing by $a^2b^2, a \neq 0, b \neq 0$)

ii.

Note that $c = 4 - 3m$

For each of the tangents in the form $y = mx + c$

$c^2 = a^2m^2 + b^2$, from (a)

$c^2 = 16m^2 + 9$

$(4 - 3m)^2 = 16m^2 + 9$

$16 - 24m + 9m^2 = 16m^2 + 9$

$7m^2 + 24m - 7 = 0$

product of roots $= \alpha\beta = m_1m_2 = -\frac{7}{7} = -1, \qquad \therefore$ Tangents are perpendicular

(c) i.

$\alpha = \text{cis}\frac{\pi}{3}, \qquad \alpha^5 = \beta = \text{cis}\frac{5\pi}{3} = \text{cis} -\frac{\pi}{3}$

Test $P(\alpha)$

$\alpha^5 + \alpha - 1 =$

$= 0, \qquad$ since $\alpha + \beta = 1$

ii.

Original equation is $z^5 + 0z^4 + 0z^3 + 0z^2 + z - 1 = 0$

The other root is $\beta = \text{cis} -\frac{\pi}{3}, \qquad \alpha + \beta = 1, \alpha\beta = 1$

Method 1:

α and β are roots to $z^2 - z + 1 = 0, \qquad$ Long division gives $z^3 + z^2 - 1 = 0$

Method 2:

Let the other roots be A, B and C

$A + B + C = -1, \qquad AB + AC + BC = 0, \quad ABC = 1$

which gives the required equation $z^3 + z^2 - 1 = 0$

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Question 5

(a) Take a typical strip of width Δx, height of $2\sqrt{4-x^2}$

$\Delta V = 4(4-x^2)\Delta x$

$$V = \lim_{\Delta x \to 0} \sum_{-2}^{2} 4(4-x^2)\Delta x$$

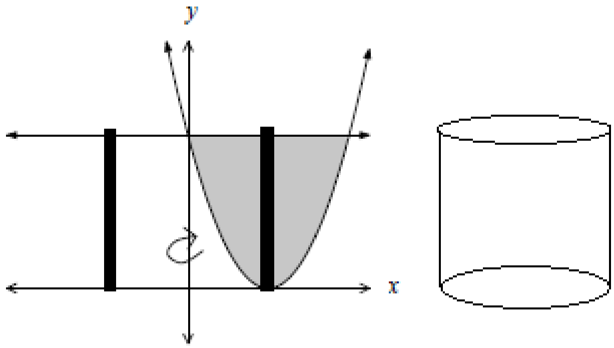
$$V = 4\int_{-2}^{2} 4 - x^2 \ dx$$

$$= 8\int_{0}^{2} 4 - x^2 \ dx$$

$$= 8\left[4x - \frac{1}{3}x^3\right]_0^2$$

$$= \frac{128}{3} \text{ cubic units.}$$

(b)



Radius of typical shell = Δx

Height of typical shell $= (4 - y) = 4 - (x-2)^2 = 4 - x^2 + 4x$

Circumference of typical shell = 2πx

$\Delta V = 2\pi x(4x - x^2)\Delta x$

$$V = 2\pi \times \lim_{\Delta x \to 0} \sum_{0}^{4} (4x^2 - x^3)\Delta x$$

$$V = 2\pi \int_0^4 4x^2 - x^3 \ dx$$

$$= 2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4$$

$$= \frac{128}{3}\pi \text{ cubic units.}$$

(c) i.

$$y = \frac{4}{x}$$

$$\frac{dy}{dx} = -\frac{4}{x^2}$$

$$m_T\ (at\ x = 2t) = -\frac{1}{t^2}$$

$$y - \frac{2}{t} = -\frac{1}{t^2}(x - 2t)$$

$t^2y - 2t = -x + 2t$

$x + t^2y = 4t$

ii.

At Q, $y = 0$

∴ Q is $(4t, 0)$

iii.

$m_{QS} = t^2$

$y = t^2(x - 4t)$

$t^2x = 4t^3 + y, \qquad (*)$

iv.

Intersect xy = 4 with $(*)$

$t^2x(x - 4t) = 4$

$t^2x^2 - 4t^3x - 4 = 0$

Sum of roots $= x_1 + x_2 = 4t$

$$x_M = \frac{x_1 + x_2}{2} = 2t$$

Similarily

$y(4t^3 + y) = 4t^2$

$y^2 + 4t^3y - 4t^2 = 0$

Sum of roots $= y_1 + y_2 = -4y^3$

$$y_M = \frac{y_1 + y_2}{2} = -2t^3$$

v.

$x = 2t$

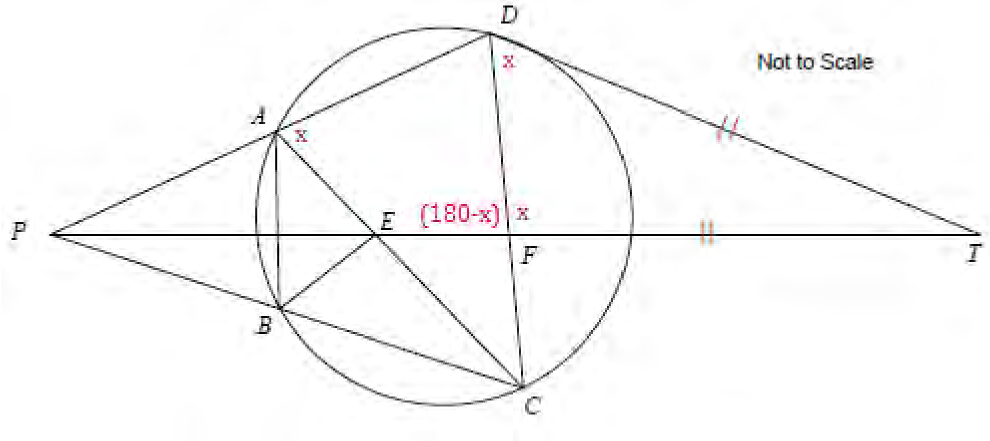
$y = -2t^3$

$x^3 = 8t^3$

∴ Locus of M is $x^3 = -4y$

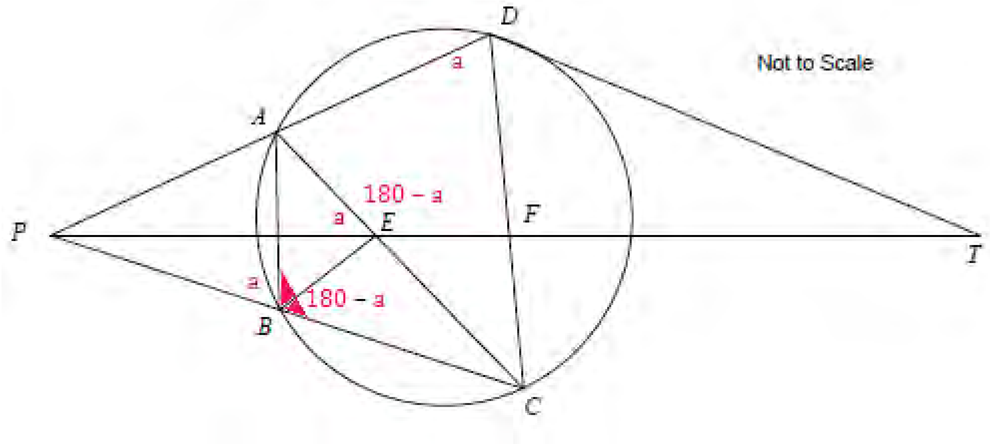
Question 6

(a) i.



Let $\angle CAD = x$
$\angle CDT = x,$ (angle in the alternate segment theorem)
$\angle DFT = x,$ (angles opposite equal sides of isosceles triangle equal)
$\angle EFD = 180 - x,$ (angles in a straight angle)
$\therefore$ $AEFD$ is a cyclic quadrilateral, (opposite angles supplementary)

ii.



$Let\ \angle PEA = a$
$\angle AEF = 180 - a,$ (angles on a straight line)
$\angle ADF = a,$ (opposite angles in cyclic quadrilateral are supplementary & part i)
$\angle ABC = 180 - a,$ (opposite angles in cyclic quadrilateral are supplementary)
$\angle ADF = a,$ (angles on a straight line)
$\therefore$ $AEBP$ is a cyclic quadrilateral, (angles standing on the same arc AP are equal)

(b) i.

$$I_n = \int_0^1 (1+x^2)^{-n} \frac{d}{dx}(x)\, dx$$

$$I_n = \left[x(1+x^2)^{-(n)}\right]_0^1 + 2n\int_0^1 x^2(1+x^2)^{-(n+1)}\, dx$$

$$I_n = 2^{-n} + 2n\int_0^1 (1+x^2)(1+x^2)^{-(n+1)}\ dx - 2n\, I_{n+1}$$

$$2nI_{n+1} = 2^{-n} + (2n-1)\, I_n$$

$$I_{n+1} = \frac{1}{n(2)^{(n+1)}} + \frac{2n-1}{2n} I_n$$

ii.

$$I_1 = \int_0^1 (1+x^2)^{-1}\, dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4}$$

$$I_2 = \frac{1}{4} + \frac{\pi}{8}$$

$$I_3 = \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right) \ = \frac{1}{4} + \frac{3\pi}{32}$$

Question 6 (continued)

(c) i.

$$I = \int_0^a f(x)dx$$

Let $x = a - y$

$dx = -dy$

$\begin{cases} x = a, y = 0 \\ x = 0, y = a \end{cases}$

$$I = \int_a^0 f(a-y) - dy$$

$$= \int_0^a f(a-y)\, dy$$

$$= \int_0^a f(a-x)\, dx\,, \qquad \text{changing variable to } x$$

ii.

$$I = \int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx$$

$$\therefore I = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$$

$$I = \int_0^\pi \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$$

$$2I = \pi \int_0^\pi \frac{\sin x}{1+\cos^2 x}\, dx$$

Let $u = \cos x$

$\begin{cases} x = \pi, u = -1 \\ x = 0, u = 1 \end{cases}$

$du = -\sin x\, dx$

$$2I = \pi \int_1^{-1} -\frac{1}{1+u^2}\, du$$

$$2I = \pi \int_{-1}^{1} \frac{1}{1+u^2}\, du$$

$$2I = \pi[\tan^{-1} u]_{-1}^{1}$$

$$2I = \pi\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$\therefore I = \frac{\pi^2}{4}$$

Question 7

(a) i.

$S'(x) = \frac{1}{2}(e^x + e^{-x})$
$= C(x)$

ii.
$C(x) > 0 \; \forall \; x$
$S'(x) \geq 0 \; \forall \; x$

iii.

$LHS = \left(\frac{1}{2}(e^x + e^{-x})\right)^2$
$\frac{1}{4}(e^{2x} + e^{-2x}) + \frac{1}{2}$
$= 1 - \frac{1}{2} + \frac{1}{4}(e^{2x} + e^{-2x})$
$= 1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2$
$= RHS$

iv.
For every x value there is a unique y value.

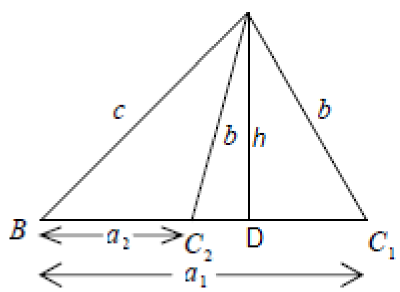
v.
$x = \frac{1}{2}(e^y - e^{-y})$
$\frac{dx}{dy} = \frac{1}{2}(e^y + e^{-y})$, using part i.
$= C(y)$
$= \sqrt{1 + \frac{1}{2}(e^{-y} - e^{-y})}$, using part iii.
$= \sqrt{1 + x^2}$, from line 1
$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$

vi.
$(0, 0)$ lies on S(x)
$\therefore$ by reflective property of inverse $(0,0)$ lies on $S^{-1}(x)$
$y = \int \frac{dx}{\sqrt{1 + x^2}}$
$y = \log_e\left(x + \sqrt{1 + x^2}\right) + C$
Substituting the point $(0,0)$ gives $C = 0$
$\therefore y = S^{-1}(x) = \log_e\left(x + \sqrt{1 + x^2}\right)$

(b) i.
If $(x - a - b - c)$ is a factor of $P(x)$ then $P(a + b + c) = 0$
$P(a + b + c) = (a + b + c - a)(a + b + c - b)(a + b + c - c) - (b + c)(c + a)(a + b)$
$= (b + c)(c + a)(a + b) - (b + c)(c + a)(a + b)$
$= 0$

ii.
$P(x) = (x - a)(x - b)(x - c) - (a + b)(c + a)(b + c)$
Let $a = 2, \quad b = -3, \quad c = -1$
$P(x) = (x - 2)(x + 3)(x + 1) - 4$
$\therefore$ from part i, $(x + 2)$ is a factor.
$P(x) = x^3 + 2x^2 - 5x - 10$
Long division gives $x^2 - 5$
Solutions to $P(x) = 0$
are $x = -2, \quad x = \sqrt{5}, \quad x = -\sqrt{5}$

(c)



Produce perpendicular from A
$h = c \sin B$
$C_1C_2 = 2\sqrt{b^2 - h^2}$, noting that $C_2D = C_1D$
$\therefore a_1 - a_2 = 2\sqrt{b^2 - c^2 \sin^2 B}$

Question 8

(a) i.

$F = -m(1+v)$
$a = -(1+v)$
$v\frac{dv}{dx} = -(1+v)$
$\frac{dv}{dx} = -\frac{1+v}{v}$
$\frac{dx}{dv} = \frac{1}{1+v} - \frac{1+v}{1+v}$
$x = \ln(1+v) - v + C$

At $x = 0, \quad v = Q$
$C - Q + \ln(1+Q) = 0$
$C = Q - \ln(1+Q)$
$\therefore x = \ln(1+v) - \ln(1+Q) + Q - v$
$x = \ln\left(\frac{1+v}{1+Q}\right) + Q - v$

ii.
$a = -(1+v)$
$\frac{dv}{dt} = -(1+v)$
$\frac{dt}{dv} = -\frac{1}{(v+1)}$
$t = -\ln(1+v) + C$

At $t = 0, \quad v = Q$
$C = \ln(1+Q)$
$t = \ln\left(\frac{1+Q}{1+v}\right)$

iii.
$e^{-t} = \frac{1+v}{1+Q}$
$v = (Q+1)e^{-t} - 1 \ (*)$
$\frac{dx}{dt} = (Q+1)e^{-t} - 1$
$x = -t - (Q+1)e^{-t} + C$
At $t = 0, \quad x = 0$
$C = (Q+1)$
$x = Q + 1 - t - (Q+1)e^{-t}$

iv.
$Q = x + t + (Q+1)e^{-1} - 1$
$Q = x + v + t, \quad$ using $(*)$

v.
Find when $v = 0$
$x = Q - \ln(1+Q)$
$t = \ln(1+Q)$

(b) i.

Geometric series for $r < 1$ is $\frac{a(1-r^n)}{1-r}$
In this series, $\quad a = 1, r = 1$
which gives the sum as
$\frac{1-x^n}{1-x}$

ii.
$1 + x + x^2 + \ ... + x^{n-1} = \frac{(1-x^n)}{1-x}$
Differentiate both sides
$1 + 2x + 3x^2 + \ ... + (n-1)x^{n-2} = \frac{n(x-1)x^{n-1} + 1 - x^n}{(1-x)^2}$
$1 + 2x + 3x^2 + \ ... + (n-1)x^{n-2} = \frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$

iii.
Let $x = \frac{1}{2}$
$LHS = 1 + 1 + \frac{3}{4} + \frac{(n-1)}{2^{n-2}}$
$= 4\left(-\left(\frac{1}{2}\right)^{n+1} - n\left(\frac{1}{2}\right)^{n-1} + 1\right), \quad n > 0$
$= 4 - (2^{1-n})(4n+1)$
$\leq 4 \ \forall\ n > 0$