Aschom School.

Question 1

(i) Find $\int \operatorname{cosec} x \, dx$ by using the substitution $t = \tan(\frac{x}{2})$

(ii) Find $\int \frac{dx}{x(1+x^2)}$

(iii) Find (a) $\int x\sqrt{x^2-1}\, dx$

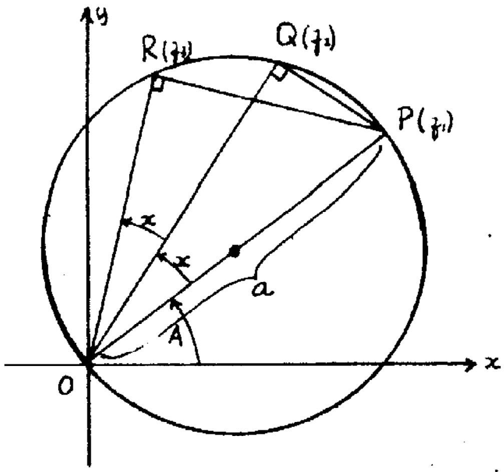
(b) $\int_1^2 x\sqrt{3x-2}\, dx$

(iv) If $I_n = \int_0^1 x^n e^x dx$ where n is a positive integer, show that

$$I_{n+1} = e - (n+1)I_n .$$

Hence evaluate $\int_0^{\frac{1}{4}} t^3 . e^{4t} \, dt$ leaving your answer in terms of e.

Question 2



(i) P is the point on the Argand diagram for which $OP = a$ and $x\hat{O}P = A$. A circle is drawn on OP as a diameter, and on it Q and R are the points such that $P\hat{O}Q = QOR = x$. The points P, Q, R represent the complex numbers z_1, z_2, z_3.

(a) Show that $z_3 = a\cos 2x[\cos(2x+A)+i\sin(2x+A)]$ and express z_2 and z_1 also in the form $z=r(\cos\theta + i\sin\theta)$.

(b) Prove that $z_1 z_3 \cos^2 x = (z_2)^2 \cos 2x$.

(ii) A point z on the Argand diagram is given by

$$z = t^2 + 2it$$

where $t = u + iv$, and $z = x + iy$.

(a) Find expressions for x and y in terms of u and v.

(b) Show that the equation of the locus of z when $v = 0$ and u varies is $y^2=4x$. Also find the locus of z when

(α) $v = 1$ and u varies

(β) $u = 0$ and v varies

Question 3

(a) Prove that the curves

$$6y = x^3 + 3x^2 - 9x - 27$$

and $$3y = x^3 - 3x^2 + 9x - 27$$

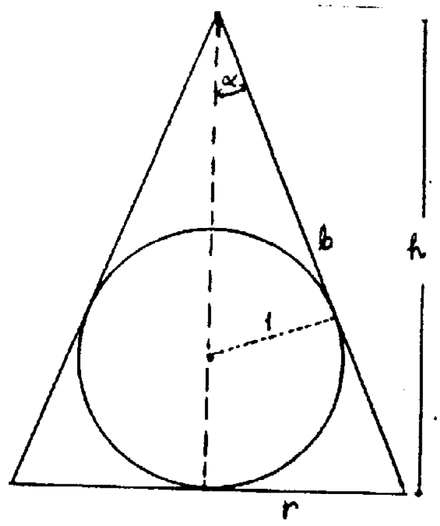
have one common point only and they cross one another there.

(b) Sketch the parts of the curves for $-3.5 \leq x \leq 3.5$ and $-10 \leq y \leq 1$. Show maximum, minimum turning points and points of inflexions clearly.

(c) Find the equation of the common tangent and the coordinates of all the points in which it meets each curve.

Question 4

(i) A right circular cone is circumscribed about a given sphere; radium - 1. The circular base also touching the sphere.



(i) Show that $r^2 = \frac{h^2}{h(h-2)}$

(ii) If the total surface area of the cone is ($S = \pi r^2 + \pi rb$) is to be a minimum, prove that the semi vertical angle α must be $\sin^{-1}(\frac{1}{3})$.

(ii) Given $u_1 = 1$, $u_2 = 5$, $u_n = 5u_{n-1} - 6u_{n-2}$ for $n = 2, 3, 4, \ldots\ldots.$

Prove that $u_n = 3^n - 2^n$ by using induction.

Question 5

(a) Deduce that the equation of the tangent at $P(x_1 y_1)$ on the hyperbola $x^2-y^2 = 1$ is $xx_1 - yy_1 = 1$.

(b) T (X,Y) is the point where the perpendicular from O the origin meets the tangent drawn at $P(x_1,y_1)$ on the hyperbola $x^2-y^2 = 1$.

Deduce that the equation of OT is $y = \dfrac{-y_1 x}{x_1}$

and find expressions for X and Y in terms of x_1 and y_1.

(c) Eliminate x_1 and y_1 from the equations in (b) and show that the equation of the locus of T as P moves on the hyperbola is $X^2 - Y^2 = (X^2 + Y^2)^2$

Question 6

(a) A is the area of the region R bounded by the upper branch of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ the x axis and the lines $x = \pm a$

(a) Show that the area of region R is

$$A = ab[\sqrt{2} + \ln(1+\sqrt{2})] \text{ sq. units.}$$

(b) S_1 is the solid whose base is the ellipse

$$E = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Cross-section perpendicular to the base E and to the minor axis of E, are plane figures similar to region R. Find the volume of S_1.

(c) The solid S_2 is obtained by rotating region R about the y axis.

Use the cylindrical shell method to computate the volume of S_2

Question 7

(i) The roots of a cubic equation are α, β and γ.

$\Sigma\alpha^t = \alpha^t+\beta^t+\gamma^t$. It is given that $\Sigma\alpha=-1$, $\Sigma\alpha^2=7$, $\Sigma\alpha^3=8$.

(a) Deduce that the equation is $x^3+x^2-3x-6 = 0$.

(b) Using the above information, evaluate $3\Sigma\alpha^4$.

(ii) At least one zero of $P(x) = 3x^3+px^2+15x+10$ is purely imaginary, p is real.

Find p and hence write P(x) as a product of lineal factors.

(iii) x_1, x_2 and x_3 are the roots of $x^3+bx+c = 0$.

Find the cubic equation whose roots are x_1x_2, x_1x_3 and x_2x_3.

Question 8

(i) D is any point on BC of triangle ABC. P is a point on the extension of side AB - towards B - such that BP = BD.

The line through PD meets AC at E.

(a) Draw a clear diagram showing the given information

(b) Prove that PE is perpendicular to AC only if

$C\hat{A}B = A\hat{C}B$

(ii) AB and CD are the parellel sides of trapezium ABCD.

AB = 10cm, CD = 6cm. E and F are points on sides AD and CB respectively such that EF is parellel to AB.

EF divides the area of trapezium ABCD such that the area of trapezium EFCD is $\frac{2}{3}$ of the area of trapezium ABCD.

Find the length of interval EF.