Question 1

a) If $z = \frac{1}{2}(1 + i)$ write the modulus and argument for each of the numbers z and z^2. Hence or otherwise show on an Argand Diagram the point represented by $1 + z + z^2$.

b) Given z is a complex number and $\bar{z}$ is its conjugate solve

$$z\bar{z} - 2z + 2\bar{z} = 5 - 4i$$

c) If $z = \frac{z_1}{z_2}$ where $z_1 = i$ and $z_2 = -2i - 2$

Find (i) $|z|$

(ii) $\arg z$

(iii) z^6.

d) Sketch the following loci

(i) $|z - 4| > |z|$

(ii) $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$

(iii) $\text{Arg}\left(\frac{1}{z}\right) < \arg\left(\frac{z}{i}\right)$ for $\arg(z) > 0$

Question 2 (Start a new book)

a) Find (i) $\int \frac{7x - 10}{\sqrt{x^2 - 6x + 5}}\,dx$

(ii) $\int \sin 5x \cos x \, dx$

b) Evaluate (i) without integrating $\int_{-1}^{1} x^2 \sin x \, dx$

(ii) $\int_{0}^{\frac{\pi}{4}} \frac{2 \, dx}{13 - 5\cos 2x}$

c) Show $\int e^{ax} \sin 3x \, dx = \frac{e^{ax}}{a^2 + 9}[a \sin 3x - 3\cos 3x]$ and hence

evaluate $\int_{0}^{2\pi} e^{2x} \sin 3x \, dx$.

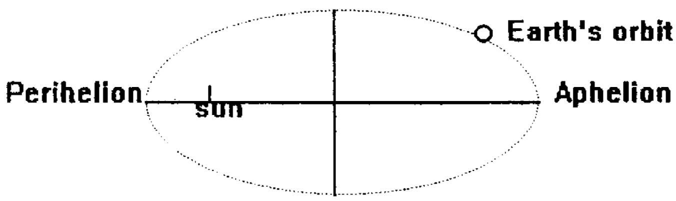
Question 3 (Start a new book)

a) For the curve $36x^2 - 9y^2 = 36$

(i) Write down the eccentricity, the co-ordinates of the foci, the equations of the directrices and the equations of the asymptotes.
(ii) Sketch the curve indicating the foci, directrices and asymptotes
(iii) Write the equation of the conjugate hyperbola.

b) The foci of an ellipse are S (4, 0) and S′ (-4,0) and any point P on the ellipse is such that SP + S′P = 10. Find the equation of the ellipse.

c)



The orbit of the earth is an ellipse of eccentricity 0.0167 with the sun at one focus and the major axis 297 million kilometres in length.
What is the Earth's (i) aphelion (farthest distance from the sun)
(ii) perihelion (closest distance to the sun)?

d) (i) If $P(cp, \frac{c}{p})$ is any point on $xy = c^2$ determine in terms of parameter p the co-ordinates of Q if PQ is a diameter of $xy = c^2$.

(ii) Prove that in a rectangular hyperbola $xy = c^2$ the lines joining any point $T(ct, \frac{c}{t})$ on the curve to the ends of a diameter PQ are equally inclined to one of the asymptotes.

Question 4 (Start a new book)

a) If p,q are roots of $\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x} = 0$ and if $a^2 + b^2 = 4ab$ prove that $p^2 + q^2 = 6pq$.

b) Given $P(z) = z^4 - 2z^3 + az - 9$ where a is real and $P(z) = 0$ has $z = 1 + i\sqrt{2}$ as one of its roots, find a and solve $P(z) = 0$ for z over the complex numbers.

c) (i) If a is a multiple root of the polynomial equation $P(x) = 0$ prove that a is a root of $P'(x) = 0$.

(ii) Find all roots of the equation $50x^3 + 35x^2 - 12x - 9 = 0$ given that it has a multiple root.

Question 5 (Start a new book)

a) Using the result $(a - b)^2 > 0$ (that is a,b are both positive and $a \neq b$) deduce that $a^2 - ab + b^2 > ab$.
Hence show that if a,band c are positive

(i) $a^3 + b^3 > ab(a + b)$

(ii) $2(a^3 + b^3 + c^3) > a^2(b + c) + b^2(a + c) + c^2(a + b)$

b) The area enclosed by the curve $y = (x + 2)^2$ and the line $y = 4$ is rotated about the y-axis. Using cylindrical shells find the volume formed.

c) The base of a solid S is the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. All cross-sections perpendicular to the base are isosceles triangles whose bases are parallel to the major axis (x-axis) of the ellipse. and whose perpendicular heights are h. Find the volume of S.

Question 6 (Start a new book)

a) LMN is an isosceles triangle with LM = LN and P is a point on the bisector of angle L. MP produced meets LN at Q and NP produced meets LM at R. Prove that M, N, Q and R are concyclic.

b) A body of mass 5kg is dropped from a height at which the gravitational acceleration is g. Assuming that air resistance is proportional to the speed v, the constant of proportion being $\frac{1}{8}$ show that $\ddot{x} = g - \frac{1}{40}v$ and find:

(i) the velocity after time t.

(ii) the terminal velocity.

(iii) the distance the body has fallen after time t.

c) (i) Use de Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$ and $\sin 3\theta$ in terms of $\sin\theta$.
Use the former result to solve the equation $8x^3 - 6x + 1 = 0$.

(ii) Deduce that α) $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$

β) $\sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8$

Question 7 (Start a new book)

a) Let $f(x) = \frac{\pi}{2} + \sin^{-1}(x-1)$ and $g(x) = \frac{\pi}{2}x$.

(i) Sketch $y = f(x)$ and $y = g(x)$ on the same co-ordinate axes.

(ii) Hence evaluate $\int_0^2 [f(x) - g(x)]\,dx$ giving reasons for your answer.

b) Let $t(x) = (1 + \cot^4 x)^{-1}$ where $0 \le x \le \frac{\pi}{2}$ and where $t(0) = 0$.

(i) Prove that $t(x) + t(\frac{\pi}{2} - x) = 1$ for $0 \le x \le \frac{\pi}{2}$.

(ii) Sketch $y = t(x)$ for $0 \le x \le \frac{\pi}{2}$ given that a horizontal tangent exists at $x = 0$. Show clearly the property established in (i) above.

(iii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} t(x)\,dx$.

Question 8 (Start a new book)

a) Sketch the locus of z if $|w| = 1$ and $z = \dfrac{w+1}{1-w}$.

b) If $x + \dfrac{1}{x} = 1$ show that $x^7 + \dfrac{1}{x^7} = 1$.

c) A body, projected vertically upwards with a speed U returns with speed V. Assuming constant acceleration due to gravity, g and that air resistance is proportional to the square of the speed show that the total time taken is $\dfrac{1}{kW}\left[\tan^{-1}\dfrac{U}{W} + \dfrac{1}{2}\log_e\left|\dfrac{W+V}{W-V}\right|\right]$ where $W = \sqrt{\dfrac{g}{k}}$.