Ascham School
Trial Higher School Certificate
Mathematics 4 unit

July 1999 **Time allowed: 3 hours**

Instructions to Students

1. Attempt all questions
2. All questions are of equal value
3. Answer each question in a separate booklet
4. Marks may not be awarded for careless of badly arranged work
5. Approved calculators may be used
6. Table of Standard Integrals are provided

Question 1 (15 marks)

a) Find $\int 7x\sqrt{4x^2-3}dx$ **2**

Evaluate the following definite integrals

(b) (i) $\int_0^{\sqrt{2}} \sqrt{4-x^2}dx$ **3**

(ii) $\int_0^{\pi} x\sin x dx$ **3**

(iii) $\int_2^4 \frac{dx}{x^2-4x+8}$ **3**

(iv) $\int_{-1}^{1} \frac{4+x^2}{4-x^2}dx$ **4**

Question 2 (15 marks) START A NEW BOOKLET

a) (i) Solve $x^2-3ix+4=0$ **2**

(ii) Express $\sqrt{12-5i}$ in the form $a+ib$, where a,b are real **4**

(iii) Find the locus of z, where $z=\frac{u-i}{u-2}$ **5**

α) If u is purely real

β) If u moves around a unit circle

(iv) Indicate on an Argand diagram the region in which both the following inequalities are satisfied. **4**

$$|z-(3+i)|\le 3 \text{ and } \frac{\pi}{4}\le \arg[z-(1+i)]\le\frac{\pi}{2}$$

Question 3 (15 marks) START A NEW BOOKLET

a) Let $I_n=\int_0^{\frac{\pi}{4}} \tan^n x dx$ where n is an integer and $n\ge 3$ **4**

Show that $I_n+I_{n-2}=\frac{1}{n-1}$ and hence evaluate I_5

b) (i) If $u=\frac{1+i}{\sqrt{2}}$, show that $u^4=-1$

(ii) On an Argand diagram illustrate the roots of the equation $z^4=1$ **11**

(iii) On the same diagram illustrate the roots of the equation $z^4=-1$

(iv) Hence or otherwise write down the solutions of the equation $z^8-1=0$

Question 4 (15 marks) START A NEW BOOKLET

a) The roots of the polynomial $P(x) = 4x^3 - 12x^2 + 11x - 3$ are in arithmetic sequence. Solve P(x) = 0 over the real number system. 4

b) (i) Prove that if Q(x) is a polynomial with a real root at x = a of multiplicity $r+1$ then Q'(x) has r – fold roots at x = a. 7

(ii) Solve the equation $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$ given that it has a root of multiplicity 2 over C.

c) If $z = \cos\theta + i\sin\theta$

(i) Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ 4

(ii) Hence by dividing throughout by z^2 or otherwise, solve the equation $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$, given that $|z| = 1$. 2

Question 5 (15marks) START A NEW BOOKLET

a) (i) Show that the equation of the tangent and normal at $P(a\cos\theta, b\sin\theta)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ and $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$ respectively. 3

(ii) The tangent and normal at P cut the y – axis at A and B respectively. Find the coordinates of A and B. 2

(iii) Show that the focus S lies on the circumference of the semi circle which has diameter AB. 3

b) (i) Determine the real values of k for which $\dfrac{x^2}{4+k} + \dfrac{y^2}{9+k} = 1$ defines 3

α) an ellipse

β) an hyperbola

(ii) If k = -5 in the above equation, find the eccentricity, the coordinates of the foci and the equations of the directrices of the conic. 2

(iii) Draw a neat sketch of the conic indicating all key features. 2

Question 6 (15 marks) **START A NEW BOOKLET**

a) Let $f(x)=\frac{4}{x}-x$. Provide separate half page sketches of the graphs of the following:

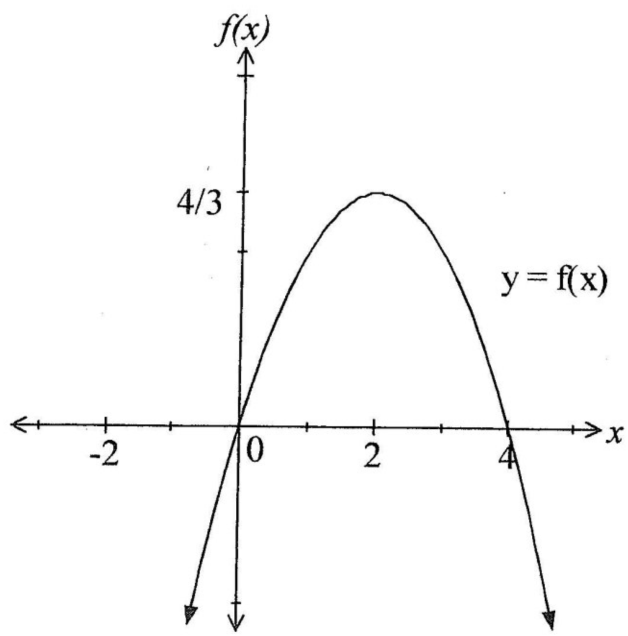
(i) $y=f(x)$ **2**

(ii) $y=\sqrt{f(x)}$ **2**

(iii) $y=e^{f(x)}$ **2**

Label each graph carefully

b)



(i) Use the diagram to find the values of a,b,c *given* $f(x)=ax^2+bx+c$ **2**

(ii) Solve $-1\le f(x)\le 1$ **3**

(iii) Hence or otherwise sketch **4**

α) $y=\ln[f(x)]$

β) $y=\cos^{-1}[f(x)]$

Question 7 (15 marks) START A NEW BOOKLET

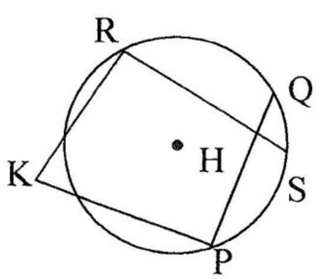
a) The base of a solid is a circular region of radius a units. Find the volume if every cross section of a plane perpendicular to a certain diameter is a square with one side lying in the base. **4**

b) Find, by the method of cylindrical shells, the volume of the solid generated when the region bounded by the curve $y = x^2 + 1$, the line $x = 2$ and the coordinate axes is rotated about the line $x = 3$. **5**

c) Find the value of x such that $\sin x = \cos 5x$ and $0 < x < \pi$ **6**

Question 8 (15 marks) START A NEW BOOKLET

a) PQ and RS are 2 chords of a circle. PQ and RS intersect at H. K is a point such that angle KPQ and angle KRS are right angles. Show that KH produced is perpendicular to QS.



6

(b) A parachutist of mass m falls to ground from a plane. Given that air resistance is proportional to the square of his speed v: **9**

(i) Draw a diagram showing clearly the forces acting on the parachutist during his free fall.

(ii) Deduce that $\frac{d}{dx}(v^2) = 2g - 2kv^2$

(iii) Show that $v^2 = \frac{g}{k} - Ae^{-2kx}$ satisfies the differential equation in part (ii) and show that $A = \frac{g}{k}$

(iv) Sketch the graph of v^2 against x and find an expression for the terminal speed of the parachutist during his free-fall.

End of Exam

Ascham
1999 4 unit Mathematics Trial – Solutions

1a) $\int 7x\sqrt{4x^2-3}\,dx = \frac{7}{8}(4x^2-3)^{3/2}\cdot\frac{2}{3} + c$

$= \frac{7}{12}\sqrt{(4x^2-3)^3} + c$

b) (i) $\int_0^{\sqrt{2}} \sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{4\cos^2\theta}\cdot 2\cos\theta\,d\theta$

let $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$
when $x = \sqrt{2}$, $\theta = \frac{\pi}{4}$
$x = 0$, $\theta = 0$

$= \int_0^{\pi/4} 4\cos^2\theta\,d\theta$

$= 2\int(1+\cos 2\theta)\,d\theta$

$= 2\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi/4}$

$= 1 + \pi/2$

(ii) $\int_0^{\pi} x\sin x = \int_0^{\pi} x\frac{d}{dx}(-\cos x)\,dx$

$= \left[-x\cos x\right]_0^{\pi} + \int_0^{\pi}\cos x\cdot 1\,dx$

$= \left[-x\cos x\right]_0^{\pi} + \left[\sin x\right]_0^{\pi}$

$= \pi$

(iii) $\int_2^4 \frac{dx}{x^2-4x+8} = \int_2^4 \frac{dx}{(x-2)^2+4}$

$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x-2}{2}\right)\right]_2^4$

$= \pi/8$

(iv) $\int_{-1}^{1} \frac{4+x^2}{4-x^2}\,dx$

Ne $\frac{4+x^2}{4-x^2} = -1 + \frac{8}{4-x^2}$

and $\frac{8}{4-x^2} = \frac{a}{2-x} + \frac{b}{2+x} \Rightarrow a=2,\ b=2$

$\therefore \int_{-1}^{1}\frac{4+x^2}{4-x^2}\,dx = -\int_{-1}^{1} 1\,dx + \int_{-1}^{1}\frac{2\,dx}{2-x} + \int_{-1}^{1}\frac{2\,dx}{2+x}$

$= \left[-x - 2\ln(2-x) + 2\ln(2+x)\right]_{-1}^{1}$

$= 4\ln 3 - 2$

2a) $x^2 - 3ix + 4 = 0$

$x = \frac{3i \pm\sqrt{-25}}{2}$

2(ii) let $12-5i = (x+iy)^2$ x, y [real]

$= x^2 - y^2 + 2ixy$

Thus $2xy = -5$ $x^2 - y^2 = 12$

Solving for x, y

$x^2 - \frac{25}{4x^2} = 12$

$4x^4 - 48x^2 - 25 = 0$

$x = \pm 5/\sqrt{2}$ or $\pm i/\sqrt{2}$ but x, y rea[l]

$\therefore x = \pm\frac{5}{\sqrt{2}}$, $y = \mp\frac{1}{\sqrt{2}}$

$x + iy = \frac{1}{\sqrt{2}}(5-i)$ or $\frac{1}{\sqrt{2}}(-5+i)$

(iii)(α) $z = \frac{u-i}{u-2}$ & u is purely rea[l]

method 1. changing the subject to u

$u = \frac{2z-i}{z-1}$

Since u is purely real, $\arg u = 0, \pi$

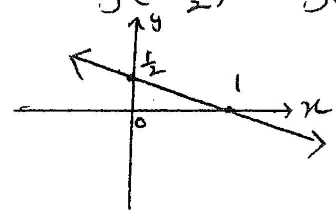
$\arg(2z-i) - \arg(z-1) = 0, \pi$

$\arg 2 + \arg(z - \frac{i}{2}) - \arg(z-1) = 0, \pi$

$\therefore \arg(z - \frac{i}{2}) = \arg(z-1)$ (Since a[...]

OR $\arg(z - \frac{i}{2}) = \arg(z-1) + \pi$

$z = x + [...]$



method 2 $u = \frac{2z-i}{z-1}$ & let $z = x+iy$

$\therefore u = \frac{2(x+iy) - i}{x+iy-1} \Rightarrow$ realise

$u = \frac{2x^2+2y^2-2x-2y - i(2y+x-[...])}{(x-1)^2+y^2}$

Since u is purely real, $2y + x - 1 = 0$

Ascham 99 4U Trial, Solutions.

5 b) (i) $\dfrac{x^2}{4+k} + \dfrac{y^2}{9+k} = 1$

α) For the conic to be an ellipse

$4+k>0$ and $9+k>0$

i.e. $k>-4$ and $k>-9 \Rightarrow k>-4$

β) For the conic to be a hyperbola

$4+k>0$ & $9+k<0$ OR $4+k<0$ and $9+k>0$

i.e. $k>-4$ & $k<-9$ OR $k<-4$ & $k>-9$

no solution $-9<k<-4$.

Thus for a hyperbola $-9<k<-4$.

(ii) If $k=-5$ the conic is a hyperbola

$\dfrac{y^2}{4} - \dfrac{x^2}{1} = 1$ (conjugate of $\dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$)

To find e: let $a=2$ $b=1$

$b^2 = a^2(e^2-1)$

$e^2 = 5/4$

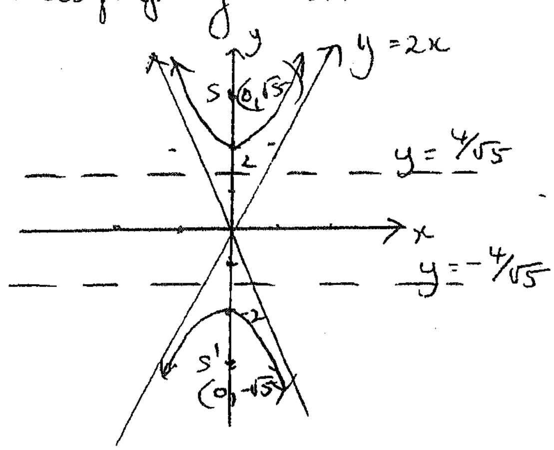
$e = \dfrac{\sqrt5}{2}$

foci $(0, \pm ae) = (0, \pm\sqrt5)$

directrices: $y = \pm\dfrac{a}{e} = \dfrac{4}{\sqrt5}$

asymptotes same as $\dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$

∴ asymp. $y = \pm 2x$



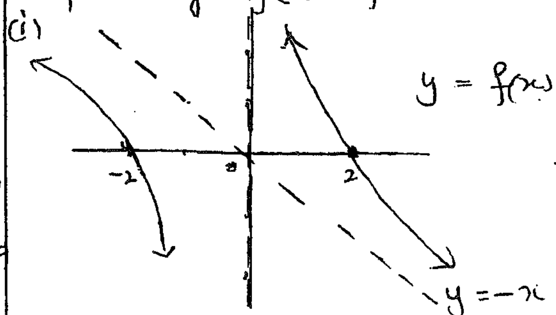
6. a) $f(x) = \dfrac{4}{x} - x$

• $\lim_{x\to\infty} f(x) = -x$, $\lim_{x\to 0} f(x) = \dfrac{4}{x}$

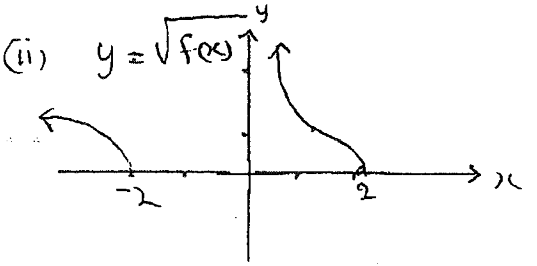
• intercepts: $x \neq 0$, $y=0 \Rightarrow x = \pm 2$

• symmetry: $f(-x) = -f(x)$ ∴ odd fn.

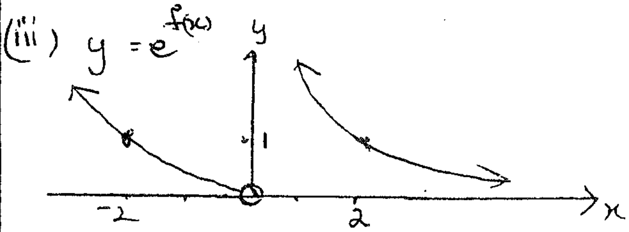
(i)



(ii) $y = \sqrt{f(x)}$



(iii) $y = e^{f(x)}$



b) (i) $y = ax^2 + bx + c$

$x=0$ $y=0 \Rightarrow c=0$ ①

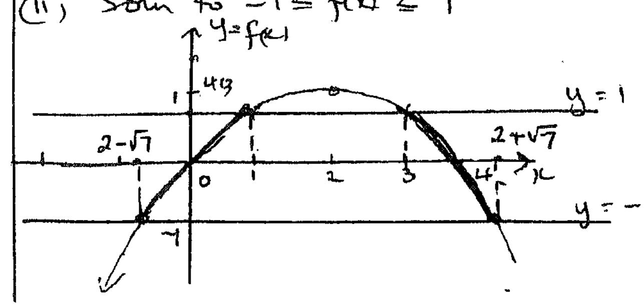
$x=4$ $y=0 \Rightarrow 16a+4b=0$ ②

$x=2$, $y=4/3 \Rightarrow \frac{2}{3} = 2a+b$ ③

∴ on soln: $a = -1/3$ $b = 4/3$

Thus $y = -\dfrac{x}{3}(x-4)$

(ii) Soln to $-1 \le f(x) \le 1$



6 (ii) To solve $-1 \le f(x) \le 1$

when $f(x) = 1$ $\frac{x}{3}(4-x) = 1$ $\Rightarrow x = 1, 3$

$f(x) = -1$

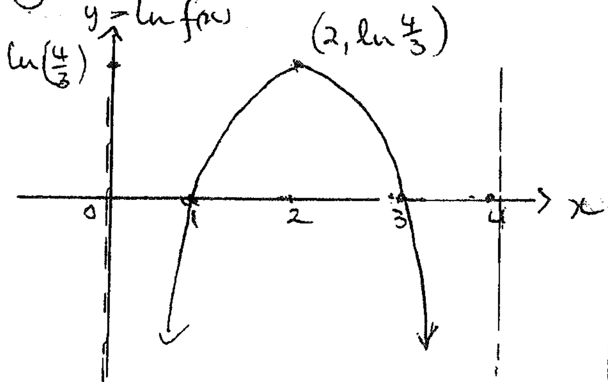
$x^2 - 4x - 3 = 0$

$x = 2 \pm \sqrt{7}$

Thus we see from the graph $y = f(x)$ that $-1 \le f(x) \le 1$

for $2-\sqrt{7} \le x \le 1$ OR $3 \le x \le 2+\sqrt{7}$

(iii) α) $y = \ln[f(x)]$



when $f(x) = 0$ $\ln f(x)$ is undefined.

$f(x) = 4/3$ $\ln f(x) = \ln 4/3$

$f(x) = 1$ $\ln f(x) = 0$

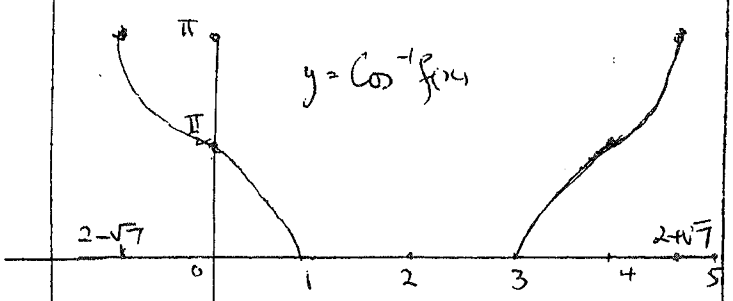
β) $y = \cos^{-1}[f(x)]$

for $-1 \le f(x) \le 1 \Rightarrow 0 \le \cos^{-1} f(x) \le \pi$

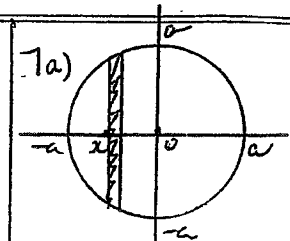
Thus for $2-\sqrt{7} \le x \le 1$ $\cos^{-1} f(x) \le \pi$

$3 \le x \le 2+\sqrt{7} \Rightarrow 0 \le \cos^{-1} f(x) \le \pi$

check: $\cos^{-1} 0 = \frac{\pi}{2}$ $\therefore$ $y = \frac{\pi}{2}$ at $x = 0, 4$



7a)



typical $\delta V = [2f(x)]^2 \delta x$

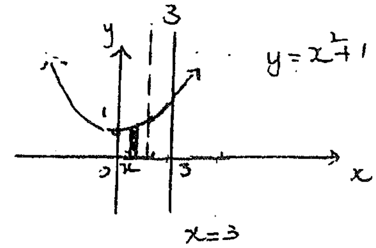
$= 4y^2 \delta x$

Total $V = \int_{-a}^{a} 4(a^2 - x^2)\,dx$

$= 8\left[a^2 x - \frac{x^3}{3}\right]_0^a$

$= \frac{16a^3}{3}\ u^3$

b)



Let δV be the vol of a typical cylindrical shell of thickness δx

inner radius $= 3 - (x + \delta x)$

outer radius $= 3 - x$

$\delta V = \pi\{(3-x)^2 - [3-(x+\delta x)]^2\}y$

$= \pi\{6\delta x - 2x\delta x - \delta x^2\}y$

$= \pi(6-2x)y\,\delta x$ $[\delta x^2$ …

$= \pi(6-2x)(x^2+1)\,\delta x$

Thus

$V = \int_0^2 6x^2 - 2x^3 + 6 - 2x\,dx$

$= \left[2x^3 - \frac{x^4}{2} + 6x - x^2\right]_0^2$

$= 16\pi\ u^3$

c) $\sin x = \cos 5x$ $0 < x < \pi$

$\cos(\frac{\pi}{2} - x) = \cos 5x$

$\therefore \frac{\pi}{2} - x = 2n\pi \pm 5x$ (general formula)

$\frac{\pi}{2} - x = 2n\pi + 5x$ OR $\frac{\pi}{2} - x = 2n\pi - 5x$

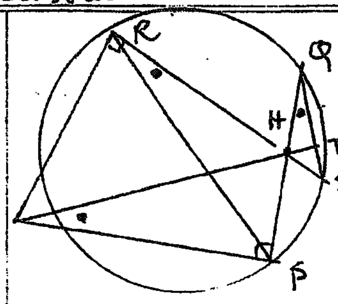
$x = \frac{1}{6}(\frac{\pi}{2} - 2n\pi)$ OR $x = \frac{1}{4}(2n\pi - \pi/2)$

$n = 0$ $x = \frac{\pi}{12}$, OR $n = 1$, $x = 3\pi/8$

$n = -1$ $x = \frac{5\pi}{12}$ $n = 2$ $x = 7\pi/8$

Ascham
6-4u Trial Soln 1999.

8a



Aim: to prove
$KH \perp QS$

Constr: join RP
Extend KH to meet QS at T

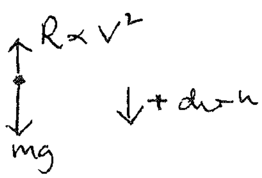
Now If KPTQ is a cyclic quad then
$K\hat{P}Q = K\hat{T}Q = 90°$

Proof: To show KPTQ is a cyclic quad
In Circle RQSP,
$S\hat{Q}P = S\hat{R}P$ (angles standing on same arc SP)

But $T\hat{R}P = S\hat{R}P$ (Since HRKP is cyclic opp ∠'s = 90°, ∠'s standing on same arc HP)

Thus $S\hat{Q}P = T\hat{K}P$ (angles equal to equal angles)
Thus KPTQ is a cyclic quad
$\therefore K\hat{P}Q = K\hat{T}Q = 90°$
$\therefore$ KH prod. is $\perp$ QS

b)



Let $R = mkV^2$ (choosing mk as constant of proportionality)

$\therefore m\ddot{x} = mg - mkv^2$
$\ddot{x} = g - kv^2$
$\frac{d}{dx}\frac{1}{2}v^2 = g - kv^2$
$\frac{d}{dx}v^2 = 2g - 2kv^2$

(iii) given $v^2 = \frac{g}{k} - Ae^{-2kx}$

$\frac{d}{dx}v^2 = \frac{d}{dx}\left(\frac{g}{k} - Ae^{-2kx}\right)$

$\frac{d}{dx}v^2 = 2Ake^{-2kx}$
$= 2k \cdot Ae^{-2kx}$
$= 2k\left(\frac{g}{k} - v^2\right)$
$= 2g - 2kv^2$

Thus $v^2 = \frac{g}{k} - Ae^{-2kx}$ satisfies
$\frac{d}{dx}v^2 = 2g - 2kv^2$

Now when $x = 0$, $v = 0$
$\therefore 0 = \frac{g}{k} - Ae^0$
$\therefore A = \frac{g}{k}$ & $v^2 = \frac{g}{k} - \frac{g}{k}e^{-2kx}$
$= \frac{g}{k}\left(1 - e^{-2kx}\right)$

(iv) $\lim_{x\to\infty} v^2 = \lim_{x\to\infty} \frac{g}{k}\left(1 - e^{-2k}\right.$
$= \frac{g}{k}$

Thus as $x \to \infty$ $v^2 \to \frac{g}{k}$,
$\to v \to \pm\sqrt{g/k}$
$\therefore$ terminal speed $= \sqrt{\frac{g}{k}}$