ASCHAM SCHOOL

MATHEMATICS EXTENSION 2

TRIAL EXAMINATION

2003

Time : 3 hours + 5 minutes reading time

Instructions:

Attempt all questions

All questions are of equal value

All necessary working should be shown for every question.

Full marks may not be awarded for careless or badly arranged work

A Table of Standard Integrals is provided

Approved calculators may be used

Each question should be answered in a separate booklet

Question 1

a) $(2-3i)(4+i)=p+iq$ where $p,q \in R$. Find p and q. **[1]**

b) (i) Express $z=-\sqrt{3}+i$ in modulus-argument form. **[2]**

(ii) Hence show that $z^7+64z=0$ **[2]**

c) Sketch the following subsets of the Argand diagram, showing important features and intercepts with the axes.

(i) $\{z: 1<|z|\le 3 \text{ and } 0<\arg z \le \frac{\pi}{2}\}$ **[2]**

(ii) $\{z: |z+1|+|z-1|=3\}$ **[3]**

(iii) $\{z: \arg(z-2)-\arg(z+2)=\frac{\pi}{3}\}$ **[2]**

d) Find the Cartesian form of the equation of the locus of the point z if $\operatorname{Re}\left[\frac{z-4}{z}\right]=0$ **[3]**

Question 2 *Please take a new booklet*

a) Find $\int \frac{e^{2x}}{e^x+1}dx$ **[2]**

b) Evaluate $\int \tan^3 x\, dx$ **[2]**

c) Evaluate $\int_0^{\pi} e^x \sin x\, dx$ **[3]**

d) (i) Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{x(\pi-2x)}=\frac{2}{\pi}\ln 2$ **[3]**

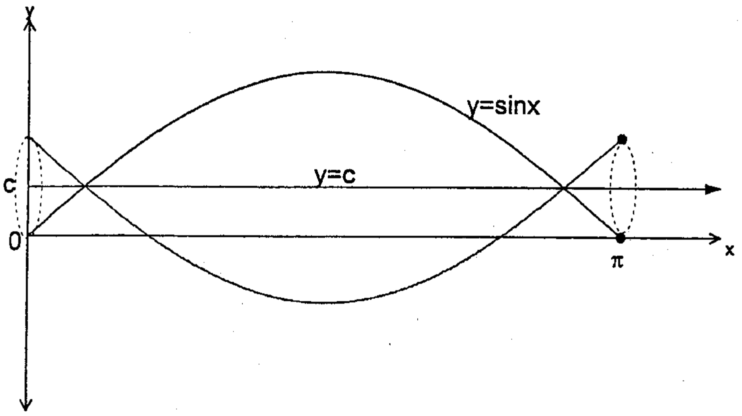
(ii) Using the substitution $u=a+b-x$, show that $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$ **[2]**

(iii) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi-2x)}dx$ **[3]**

Question 3 *Please take a new booklet*

a) A chocolate has a circular base of radius 1cm. If every section perpendicular to this base is an equilateral triangle, find the volume of chocolate needed to make a box of 40 such chocolates. **[6]**

b)



The arch $y = \sin x$, $0 \le x \le \pi$, is revolved around the line $y = c$ to generate the solid shown.

(i) Show that the volume generated is given by $\pi(\pi c^2 - 4c + \frac{\pi}{2})$ **[6]**

(ii) Find the value of c which minimises the volume. **[3]**

Question 4 *Please take a new booklet*

a) A ball of mass m is thrown vertically upwards under gravity, the air resistance to the motion being $\frac{mgv^2}{a^2}$ where the speed is v, a is a constant and g is the acceleration due to gravity.

(i) Show that during the upward motion of the ball

$$v\frac{dv}{dx} = \frac{-g}{a^2}\left(a^2 + v^2\right)$$

where x is the upward displacement. **[2]**

(ii) Show that the greatest height reached is $\frac{a^2}{2g}\ln\left(1+\frac{u^2}{a^2}\right)$ where u is the speed of projection. **[5]**

b) A curve is defined by the parametric equations $x = \cos^3\theta$, $y = \sin^3\theta$ for $0 < \theta < \frac{\pi}{4}$.

(i) Show that the equation to the normal to the curve at the point P $(\cos^3\phi, \sin^3\phi)$ is $x\cos\phi - y\sin\phi = \cos 2\phi$ **[4]**

(ii) The normal at P cuts the x-axis at A and the y-axis at B. Show $AB = 2cot2\phi$ **[4]**

Question 5 *Please take a new booklet*

a) If $ax^3 + bx^2 + d = 0$ has a double root, show that $27a^2d + 4b^3 = 0$ **[3]**

b) (i) Prove that $P(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x + c$ has no real zeros if $c > 9\frac{1}{3}$ **[4]**

(ii) Explain why the largest zero of $P(x)$ is greater than 2 if $c = -2$. Find an approximation for the largest zero of $P(x)$ using one application of Newton's method. **[3]**

c) (i) P is any point inside a circle center O. M is the midpoint of chords AB through P. Find the locus of M. Explain your answer. **[3]**

(ii) Q is any point outside a circle center C. N is the midpoint of chords DE through Q. State the locus of N. **[2]**

Continued on next page

Question 6 *Please take a new booklet*

a) (i) Find the five fifth roots of unity. **[2]**

(ii) If $\omega = cis\frac{2\pi}{5}$, show that $1+\omega+\omega^2+\omega^3+\omega^4=0$ **[3]**

(iii) Show that $z_1=\omega+\omega^4$ and $z_2=\omega^2+\omega^3$ are roots of the equation $z^2+z-1=0$ **[3]**

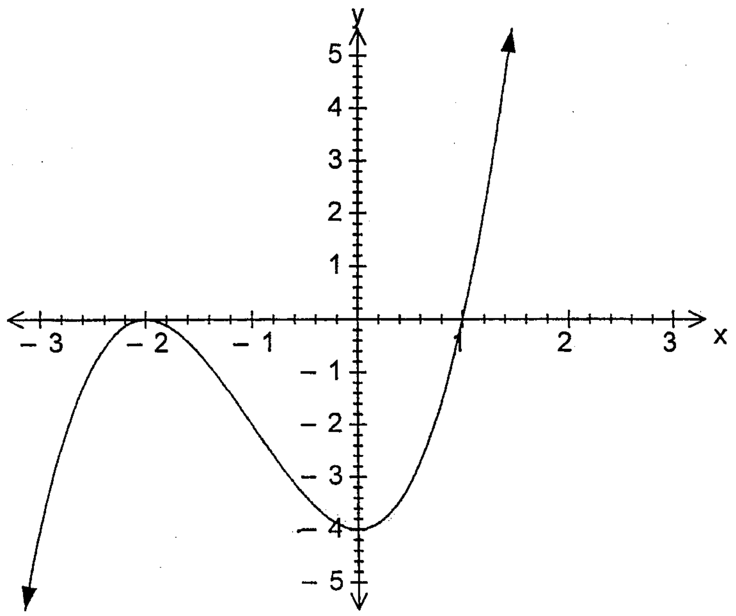
b) (i) By using the expansions of $\cos(x-y)$ and $\cos(x+y)$ show that $\sin x\sin y=\frac{1}{2}(\cos P-\cos Q)$ where $P=(x-y)$ and $Q=(x+y)$ **[3]**

(ii) Hence prove that $\sin x+\sin 3x+\sin 5x+........+\sin(2n-1)x=\dfrac{\sin^2 nx}{\sin x}$

[4]

Question 7 *Please take a new booklet*

a) The graph of $y=x^3+3x^2-4$ is sketched below



(i) Sketch the curves $y=\left|x^3+3x^2-4\right|$ and $y=\ln\left|x^3+3x^2-4\right|$ on separate axes. **[3]**

(ii) Hence or otherwise determine the value of m, where m is a constant, such that the equation $2\ln|x+2|+\ln|x-1|=m$ **[4]**

b) AB is a diameter of a circle whose centre is O and C is a point on the circumference such that ∠AOC is acute. OC is produced to meet the tangent at A in D. Let ∠CBD = α and ∠ABC = β. Prove

(i) $\tan(\alpha+\beta)=\frac{1}{2}\tan 2\beta$ **[3]**

(ii) $\tan\alpha=\tan^3\beta$ **[3]**

(iii) Calculate the value of α when AD = AB **[2]**

Question 8 *Please take a new booklet*

a) (i) Show that the condition for the line $y=mx+c$ to be tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$ **[3]**

(iii) Hence or otherwise prove that the pair of tangents from the point (3, 4) to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ are at right angles to each other. **[4]**

b) Let $I_{2n}=\int_{-1}^{1}(1-x^2)^n\,dx$ where $n\geq 0$

(i) Use the substitution $x=\sin\theta$ to show that $I_{2n}=\frac{2n}{2n+1}I_{2n-2}$ **[3]**

(ii) Show that $I_6=\frac{32}{35}$ **[2]**

(iii) Deduce that $I_{2n}=\frac{2^{2n+1}(n!)^2}{(2n+1)!}$ **[3]**

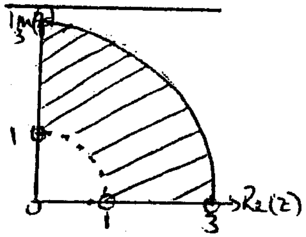
End of Examination

1a) $(2-3i)(4+i)$
$= 8-10i-3i^2$
$= 11-10i$
$p=11,\ q=-10$

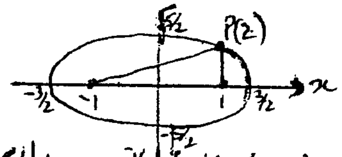
b)i) $z=-\sqrt{3}+i$
$|z|=\sqrt{3+1}$
$=2$
$\arg z = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$
$=\frac{5\pi}{6}$.
$z = 2\operatorname{cis}\frac{5\pi}{6}$

ii) $z^7+64z = 2^7\left(\operatorname{cis}\frac{5\pi}{6}\right)^7 + 64\times 2\operatorname{cis}\frac{5\pi}{6}$
$= 128\operatorname{cis}\frac{35\pi}{6} + 128\operatorname{cis}\frac{5\pi}{6}$
$= 128\left(\operatorname{cis}\left(-\frac{\pi}{6}\right)+\operatorname{cis}\frac{5\pi}{6}\right)$
$= 128\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} - \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
$=0$

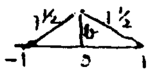
c). (i)



ii) $|z+1|+|z-1|=3$

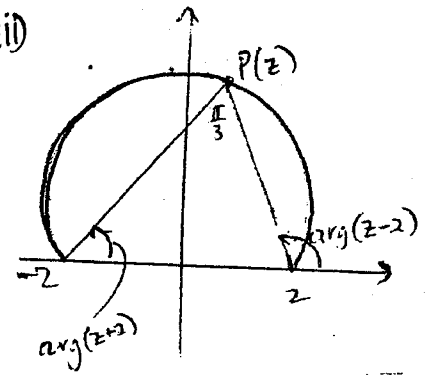


Ellipse with focii $(\pm1,0)$
$a+1+a-1=3 \Rightarrow a=\frac{3}{2}$



$b=\sqrt{\left(\frac{3}{2}\right)^2-1^2}$
$=\frac{\sqrt{5}}{2}$

iii)



d) $\operatorname{Re}\left(\frac{z-4}{z}\right)=0$.
Let $z=x+iy$
$\frac{z-4}{z} = \frac{x+iy-4}{x+iy}\times\frac{x-iy}{x-iy}$
$= \frac{x^2+y^2-4x+4iy}{x^2+y^2}$
$\operatorname{Re}\left(\frac{z-4}{z}\right) = 1-\frac{4x}{x^2+y^2}=0$
$x^2+y^2=4x$.

2a) $\int\frac{e^{2x}}{e^x+1}\,dx = \int e^x - \frac{e^x}{e^x+1}\,dx$
$= e^x - \ln(e^x+1)+c$.

b) $\int\tan^3x\,dx = \int\tan x(\sec^2x-1)\,dx$
$=\int\tan x\sec^2x - \frac{\sin x}{\cos x}\,dx$
$=\frac{1}{2}\tan^2x + \ln|\cos x| + c$

c) $\int_0^{\pi} e^x\sin x\,dx = I$ $\quad\begin{cases} u=\sin x \\ du=\cos x\,dx \\ dv=e^x dx \\ v=e^x\end{cases}$
$I = \left[e^x\sin x\right]_0^{\pi} - \int_0^{\pi}e^x\cos x\,dx$
$= 0 - \int_0^{\pi}e^x\cos x\,dx$ $\quad\begin{matrix} u=\cos x \\ du=-\sin x \\ dv=e^x dx \\ v=e^x\end{matrix}$
$= -\left[e^x\cos x\right]_0^{\pi} - \int_0^{\pi}e^x\sin x\,dx$.
$= -\left[e^{\pi}(-1)-1\right] - I$

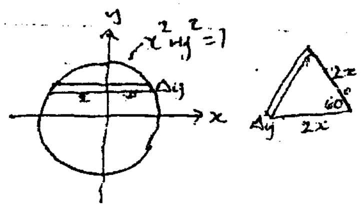
d)(i) $\frac{1}{x(\pi-2x)} = \frac{A}{x}+\frac{B}{\pi-2x}$.
$1 = A(\pi-2x)+Bx$
$A=\frac{1}{\pi}\quad B=\frac{2}{\pi}$
$\int_{\pi/6}^{\pi/3}\frac{dx}{x(\pi-2x)} = \int_{\pi/6}^{\pi/3}\frac{1/\pi}{x}+\frac{2/\pi}{\pi-2x}\,dx$
$=\frac{1}{\pi}\left(\ln x - \ln(\pi-2x)\right)\Big]_{\pi/6}^{\pi/3}$
$=\frac{1}{\pi}\left[\ln\frac{x}{\pi-2x}\right]_{\pi/6}^{\pi/3}$
$=\frac{1}{\pi}\left[0-\ln\frac{1}{4}\right]$
$=\frac{2}{\pi}\ln 2$.

ii) Let $u=a+b-x$ $\quad\begin{matrix} x=a & x=b \\ u=b & u=a\end{matrix}$
$du=-dx$
$\int_a^b f(a+b-x)\,dx = \int_b^a f(u).-du$
$=\int_a^b f(u)\,du$
$=\int_a^b f(x)\,dx$.

(iii) $\int_{\pi/6}^{\pi/3}\frac{\cos^2x}{x(\pi-2x)}\,dx = I$
$I = \int_{\pi/6}^{\pi/3}\frac{\cos^2\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}{\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\left(\pi-2\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right)}\,dx$
$=\int_{\pi/6}^{\pi/3}\frac{\sin^2x}{\left(\frac{\pi}{2}-x\right)(2x)}$
$=\int_{\pi/6}^{\pi/3}\frac{1-\cos^2x}{(\pi-2x)x}\,dx$
$=\int^{\pi/3}\frac{1}{\quad} - \frac{\cos^2x}{\quad}\,dx$

$I = \frac{1}{\pi}\ln 2$

Q3 a)



$\Delta V = \frac{1}{2}\cdot 2x \cdot 2x \sin 60° \, \Delta y$

$= 2x^2 \cdot \frac{\sqrt{3}}{2}\Delta y$

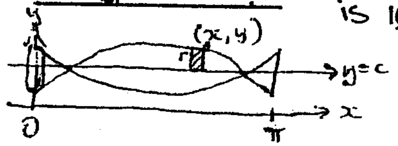
$V = \int_{-1}^{1} \sqrt{3}(1-y^2)\,dy$

$= 2\sqrt{3}\int_0^1 (1-y^2)\,dy$

$= 2\sqrt{3}\left[y - \frac{y^3}{3}\right]_0^1$

$= \frac{4}{3}\sqrt{3}$ cm^3 ∴ Total vol of choc is $\frac{160\sqrt{3}}{3}$ cm^3

b).



(i) $r = y - c$

$\Delta V = \pi r^2 \Delta x$

$= \pi (y-c)^2 \Delta x$

$V = \pi\int_0^{\pi} (\sin x - c)^2\,dx$

$= \pi\int_0^{\pi} \sin^2 x - 2c\sin x + c^2\,dx$

$= \pi\int_0^{\pi} \frac{1}{2}(1-\cos 2x) - 2c\sin x + c^2\,dx$

$= \pi\left[\frac{x}{2} - \frac{1}{4}\sin 2x + 2c\cos x + c^2 x\right]_0^{\pi}$

$= \pi\left[\frac{\pi}{2} - 4c + c^2\pi\right]$ units^3

(ii) Min occurs when $V' = 0, V'' > 0$

$V' = \pi[0 - 4 + 2c\pi] = 0$

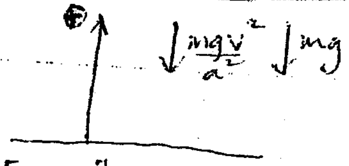
$c = \frac{4}{2\pi}$

$= \frac{2}{\pi}$

$V'' = 2\pi^2 > 0$

∴ Minimum occurs when $c = \frac{2}{\pi}$

Q4 a)



$F = m\ddot{x}$

(i) $m\ddot{x} = -\frac{mgv^2}{a^2} - mg$

$\ddot{x} = -\frac{g}{a^2}v^2 - g$

$\frac{dv}{dt} = -\frac{g}{a^2}(v^2+a^2)$

$\frac{dv}{dx}\cdot\frac{dx}{dt} = -\frac{g}{a^2}(a^2+v^2)$

$v\cdot\frac{dv}{dx} = \frac{-g}{a^2}(a^2+v^2)$

(ii) $\therefore \frac{dv}{dx} = \frac{-g}{a^2}\left(\frac{a^2}{v}+v\right)$

$\frac{dx}{dv} = \frac{-a^2}{g}\left(\frac{v}{a^2+v^2}\right)$

$x = -\frac{a^2}{g}\int \frac{v}{a^2+v^2}\,dv$

$= -\frac{a^2}{2g}\ln(v^2+a^2) + c$

when $x = 0$, $v = u$

$\Rightarrow c = \frac{a^2}{2g}\ln(u^2+a^2)$

$x = \frac{-a^2}{2g}\ln(v^2+a^2) + \frac{a^2}{2g}\ln(u^2+a^2)$

$x = -\frac{a^2}{2g}\left[\ln a^2 - \ln(u^2+a^2)\right]$

$= \frac{a^2}{2g}\ln\frac{u^2+a^2}{a^2}$

$= \frac{a^2}{2g}\ln\left(1+\frac{u^2}{a^2}\right)$

b) $x = \cos^3\theta \qquad \frac{dx}{d\theta} = -3\cos^2\theta.\sin\theta$

$y = \sin^3\theta \qquad \frac{dy}{d\theta} = 3\sin^2\theta.\cos\theta$

$\frac{dy}{dx} = \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta}$

$= -\tan\theta$

$m_{normal} = \cot\phi$ at P.

Eqn of normal:

$y - \sin^3\phi = \frac{\cos\phi}{\sin\phi}(x - \cos^3\phi)$

$\sin\phi\, y - \sin^4\phi = \cos\phi\, x - \cos^4\phi$

$x\cos\phi - y\sin\phi = \cos^4\phi - \sin^4\phi$

$= (\cos^2\phi - \sin^2\phi)$

$= \cos 2\phi$

(ii) $y_A = 0$, $x_A = \frac{\cos 2\phi}{\cos\phi}$

$x_B = 0$, $y_B = \frac{-\cos 2\phi}{\sin\phi}$

$AB = \sqrt{\frac{\cos^2 2\phi}{\cos^2\phi} + \frac{\cos^2 2\phi}{\sin^2\phi}}$

$= \cos 2\phi\sqrt{\frac{\sin^2\phi+\cos^2\phi}{\sin^2\phi\cos^2\phi}}$

$= \frac{\cos 2\phi}{\sin\phi\cos\phi}$

$= \frac{\cos 2\phi}{\frac{1}{2}\sin 2\phi}$

$= 2\cot 2\phi$

Q5a) $P(x) = ax^3 + bx^2 + d = 0$

$P'(x) = 3ax^2 + 2bx$

Double root so root of $P(x)$ also root of $P'(x)$

$3ax^2 + 2bx = 0$

$x(3ax + 2b) = 0$

$x = 0$ or $x = -\frac{2b}{3a}$

$x = 0$ is not a root of $P(x)$

$\therefore x = -\frac{2b}{3a}$ is

$\therefore P\left(-\frac{2b}{3a}\right) = a\cdot\left(\frac{-2b}{3a}\right)^3 + b\left(-\frac{2b}{3a}\right)^2 + d = 0$

$-\frac{8ab^3}{27a^3} + \frac{4b^3}{9a^2} + d = 0$

$\times 27a^2$

$-8b^3 + 12b^3 + 27a^2d = 0$

$27a^2d + 4b^3 = 0$

b). $P(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x + c = 0$

$P'(x) = x^3 - x^2 - 4x + 4$

$= x^2(x-1) - 4(x-1)$

$= (x-1)(x-2)(x+2).$

tps when $P'(x) = 0$

$x = \pm 2$ or 1

$P''(x) = 3x^2 - 2x - 4$

$\left.\begin{array}{l} P''(2) > 0 \\ P''(-2) > 0 \\ P''(1) < 0 \end{array}\right\}$ min tps at $x = \pm 2$.

If $P(x)$ has no real roots then $P(\pm 2) > 0$.

$\frac{1}{4}\times 2^4 - \frac{1}{3}\times 2^3 - 2\times 2^2 + 4\times 2 + c > 0$

$\therefore c > -1\tfrac{1}{3}$

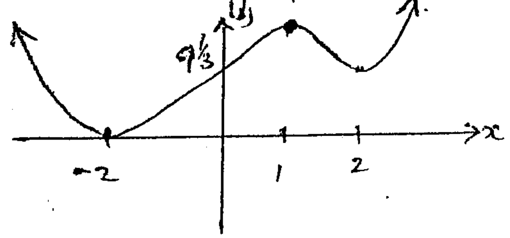
and

$\frac{1}{4}(-2)^4 - \frac{1}{3}\times(-2)^3 - 2(-2)^2 + 4\times(-2) + c > 0$

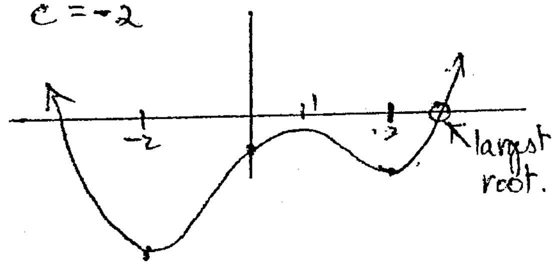
$\therefore c > 9\tfrac{1}{3}$

$\therefore \underline{c > 9\tfrac{1}{3}}$

(ii) If $c = 9\tfrac{1}{3}$ the graph of $P(x)$ is



If $c = -2$



$P(2) < 0$, $P(3) = 3.25 > 0$

Take $x_0 = 2.5$

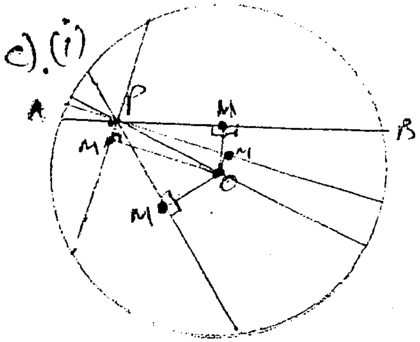
$x_1 = 2.5 - \frac{P(2.5)}{P'(2.5)}$

$= 2.5 - \frac{0.0572916}{3.375}$

$= 2.483$

~~the largest root~~ $\doteqdot 2.483$ (3dp).

c).(i)



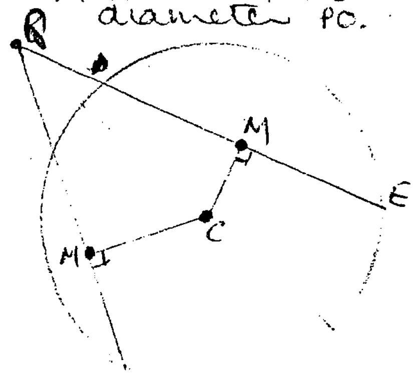
The locus of M is the circle with diameter OP.

The mid pt of any chord AB is the foot of the perpendicular from O.

$\therefore O\hat{M}A = O\hat{M}P = 90^\circ$

$\therefore$ Since angle in semicircle is 90° M lies on the circle with diameter PO.

(ii)



The locus of M is the arc of the circle with diameter QC which lies inside the circle centre C.

$2I = \frac{2}{\pi}\ln 2$

$\underline{I = \frac{1}{\pi}\ln 2}$

$= \pi\left[\frac{\pi}{2} - 0 + 2c(-1) + c^2\pi - 2c\right]$

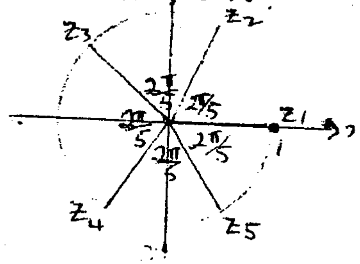
$= \pi\left[\frac{\pi}{2} - 4c + c^2\pi\right]$ units^3

iii) Min area ...

Max ht when $v = 0$.

$x = -\frac{a^2}{2g}\left[\ln a^2 - \ln(u^2 + a^2)\right]$

circle as shown



(ii) $z_1 = i$

$z_2 = \text{cis}\frac{2\pi}{5} = \omega$

$z_3 = \text{cis}\frac{4\pi}{5} = \left(\text{cis}\frac{2\pi}{5}\right)^2 = \omega^2$

$z_4 = \text{cis}\frac{6\pi}{5} = \left(\text{cis}\frac{2\pi}{5}\right)^3 = \omega^3$

$z_5 = \text{cis}\frac{8\pi}{5} = \left(\text{cis}\frac{2\pi}{4}\right)^4 = \omega^4$

using de Moivre's theorem.

$1+\omega+\omega^2+\omega^3+\omega^4 = \text{sum of roots of } z^5-1=0$

$= -\frac{0}{1}$

$= 0$

(iii)

The eqn with roots z_1 and z_2 is

$z^2 - (z_1+z_2)z + z_1z_2 = 0$

$z^2 - (\omega+\omega^4+\omega^2+\omega^3)z + (\omega+\omega^4)(\omega^2+\omega^3) = 0$

$z^2 - (-1)z + \omega^3+\omega^4+\omega^6+\omega^7 = 0$ from (ii)

$z^2 + z + \omega+\omega^2+\omega^3+\omega^4 = 0$ since $\omega^6 = \omega^5\omega = \omega$

$z^2 + z - 1 = 0$

$\omega^7 = \omega^5\omega^2 = \omega^2$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

subtracting

$\cos(x-y) - \cos(x+y) = 2\sin x\sin y$

$\sin x\sin y = \frac{1}{2}\{\cos(x-y) - \cos(x+y)\}$

$= \frac{1}{2}(\cos P - \cos Q)$

where $P = x-y$, $Q = x+y$.

(ii) Prove $\sin x + \sin 3x + \sin 5x + \dots \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}$

$\times$ by $\sin x$

Prove $\sin x\sin x + \sin 3x\sin x + \sin 5x\sin x + \dots \sin(2n-1)\sin x = \sin^2 nx$

LHS $= \frac{1}{2}(\cos 0 - \cos 2x) + \frac{1}{2}(\cos 2x - \cos 4x) + \dots + \frac{1}{2}(\cos(2n-2)x - \cos 2nx)$

$= \frac{1}{2}(1 - \cos 2nx)$

$= \frac{1}{2}(1 - (1 - 2\sin^2 nx))$

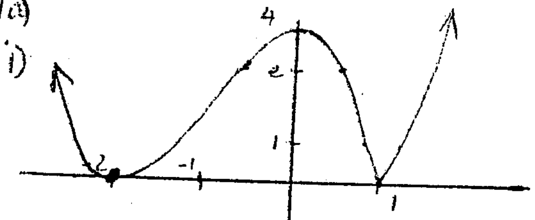
$= \frac{1}{2}\cdot 2\sin^2 nx$

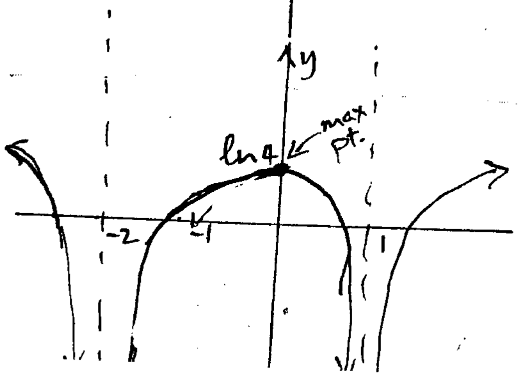
$= \sin^2 nx$

$=$ RHS

Q7a)

i)





(ii) If $y = 2\ln|x+2| + \ln|x-1| = m$

then $\frac{dy}{dx} = 0$

$y = \ln|x+2|^2|x-1|$

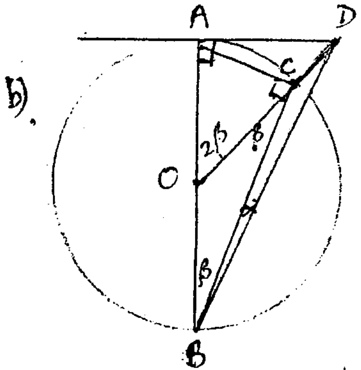
$= \ln|x^3+3x^2-4|$ as sketched.

Gradient of $y = \ln|x^3+3x^2-4|$

$= 0$ when

$y = m = \ln 4$

b).



(i) Prove $\tan(\alpha+\beta) = \frac{1}{2}\tan 2\beta$.

$D\hat{A}O = 90°$ (∠ between rad & tan).

$A\hat{C}B = 90°$ (∠ in semi circle).

$\tan(\alpha+\beta) = \frac{AD}{AB} = \frac{AD}{2AO}$

$A\hat{O}C = 2\beta$ (ext ∠ of isos Δ, radii OB = OC).

$\frac{1}{2}\tan 2\beta = \frac{AD}{2AO}$

$\therefore \tan(\alpha+\beta) = \frac{1}{2}\tan 2\beta$.

(ii) Prove $\tan\alpha = \tan^3\beta$.

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$\frac{1}{2}\tan 2\beta = \frac{1}{2}\cdot\frac{2\tan\beta}{1-\tan^2\beta}$

$\therefore \frac{\tan\beta}{1-\tan^2\beta} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ from (i)

$\cancel{\tan\beta} - \tan\alpha\tan^2\beta = \tan\alpha + \cancel{\tan\beta} - \tan^2\beta\tan\alpha - \tan^3\beta$

$\tan^3\beta = \tan\alpha$.

(iii) If AD = AB.

$\alpha+\beta = 45°$ (∠ sum isos rt ∠'d Δ).

$\tan(\alpha+\beta) = \frac{1}{2}\tan 2\beta = 1$.

$\tan 2\beta = 2$

[redacted]

The roots lie on the unit circle as shown

z_3 z_2

Q8 a) $y = mx + c$ ①

(i). $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ②

Intersect when

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$(a^2m^2+b^2)x^2 + 2mca^2x + (a^2c^2 - a^2b^2) = 0$

If line is a tangent Δ = 0

$4m^2c^2a^4 - 4(a^2m^2+b^2)(a^2c^2-a^2b^2) = 0$

$m^2c^2a^4 - a^4m^2c^2 + a^4b^2m^2 - a^2b^2c^2 + a^2b^4 = 0$

$c^2 = \frac{a^4b^2m^2 + a^2b^4}{a^2b^2}$

$= a^2m^2 + b^2$

(ii) Prove tangents from (3, 4) to $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at rt ∠s.

$a = 4$, $b = 3$

tangents are $y = mx + c$

$= mx \pm \sqrt{a^2m^2+b^2}$ from (i)

$= mx \pm \sqrt{16m^2+9}$

These pass through (3, 4)

$\therefore 4 = 3m \pm \sqrt{16m^2+9}$

$(4-3m)^2 = 16m^2+9$

$16 - 24m + 9m^2 = 16m^2 + 9$.

$7m^2 + 24m - 7 = 0$

Two values of m satisfy this, m_1 and m_2

and $m_1 m_2$ = product of roots

$= \frac{-7}{7}$

$= -1$

b) (i) $\cos(x-y) = \cos x\cos y + \sin x\sin y$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

subtracting

Let $x = \sin\theta$

$dx = \cos\theta \, d\theta$

$x = \pm 1, \quad \theta = \pm\frac{\pi}{2}$

$$I_{2n} = \int_{-\pi/2}^{\pi/2} \cos^{2n}\theta \cos\theta \, d\theta$$

$u = \cos^{2n}\theta$

$du = 2n \cos^{2n-1}\theta \cdot -\sin\theta \, d\theta$

$dv = \cos\theta \, d\theta$

$v = \sin\theta.$

$$I_{2n} = \Big[\sin\theta \cos^{2n}\theta\Big]_{-\pi/2}^{\pi/2} + 2n\int_{-\pi/2}^{\pi/2} \sin^2\theta \cos^{2n-1}\theta \, d\theta$$

$$= 0 + 2n \int_{-\pi/2}^{\pi/2} (1-\cos^2\theta)\cos^{2n-1}\theta \, d\theta.$$

$$= 2n \int_{-\pi/2}^{\pi/2} \cos^{2n-1}\theta - \cos^{2n+1}\theta \, d\theta.$$

$$= 2n \, I_{2n-2} - 2n \, I_{2n}$$

$$(2n+1) I_{2n} = 2n \, I_{2n-2}$$

$$I_{2n} = \frac{2n}{2n+1} I_{2n-2}.$$

$$= \frac{6}{7} I_4.$$

$$I_6 = \frac{6}{7} \cdot \frac{4}{5} I_2$$

$$= \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_0$$

$$= \frac{16}{35} \int_{-1}^{1} (1-x^2)^0 dx$$

$$= \frac{16}{35} \Big[x\Big]_{-1}^{1}$$

$$= \frac{32}{35}.$$

$$I_{2n} = \frac{2n(2n-2)(2n-4) \ldots\ldots 4 \cdot 2}{(2n+1)(2n-1)(2n-3) \ldots\ldots 5 \cdot 3} I_0 \times \frac{2n(2n-2)(2n-4)\ldots 4 \cdot 2}{2n(2n-2)(2n-4)\ldots 4 \cdot 2}$$

$$= \frac{\left[2n \cdot 2(n-1) \cdot 2(n-2) \cdot 2(n-3) \ldots 2\times 2 \times 2\times 1\right]^2}{(2n+1)!} I_0$$

$$= \frac{(2^n n!)^2 \cdot 2}{(2n+1)!}$$

$$= \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$