 Name: _______________

ASCHAM SCHOOL
2010

MATHEMATICS EXTENSION 2: YEAR 12 TRIAL EXAMINATION

TIME ALLOWED: 3 HOURS PLUS 5 MINUTES' READING TIME

EXAMINATION DATE: 30 JULY 2010

INSTRUCTIONS

ALL QUESTIONS MAY BE ATTEMPTED.
ALL QUESTIONS ARE OF EQUAL VALUE (15 MARKS).
ELEGANCE AND RIGOUR IN SOLUTIONS WILL ATTRACT HIGHER MARKS.
ALL NECESSARY WORKING MUST BE SHOWN.
MARKS MAY NOT BE AWARDED FOR CARELESS OR POORLY ARRANGED WORK.
APPROVED CALCULATORS AND TEMPLATES MAY BE USED.

COLLECTION

START EACH QUESTION IN A NEW BOOKLET.
IF YOU USE A SECOND BOOKLET FOR A QUESTION, PLACE IT INSIDE THE FIRST.
ATTACH THE GRAPH ANSWER SHEET TO Q3 ANSWERS.
WRITE YOUR NAME, TEACHER'S NAME AND QUESTION NUMBER ON EACH BOOKLET.

JMH

 Name: _______________

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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Question 1

(a) Use the Table of Standard Integrals to find $\int \frac{2x\ dx}{\sqrt{x^4+16}}$. **2**

(b) Evaluate $\int_0^1 \sin(a\pi x)\sin(b\pi x)\ dx$ where $a, b \in \mathbb{Z}$. **3**

(c) By using geometric evidence, explain why **3**

$\int_0^1 x^n\ dx + \int_0^1 x^{\frac{1}{n}}\ dx = 1$ for $n \in \mathbb{Z}^+$.

(d) Let f be a continuous function for $-5 \le x \le 10$ and let $g(x) = f(x) + 2$. **3**

If $\int_{-5}^{10} f(t)\,dt = 4$, what is $\int_{-5}^{10} g(u)\,du$? Give reasons.

(e) Consider the *lemniscate* curve given by $\left(x^2+y^2\right)^2 = 12\left(x^2-y^2\right)$. **4**

Find $\frac{dy}{dx}$.

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Question 2

(a) Find the Cartesian equation of the locus described by the equation $\text{Re}(z) = |z-2|$. **3**

Draw a sketch.

(b) (i) Sketch $|z-i| = \frac{1}{2}$. **2**

(ii) Find the greatest value of *arg z* when $|z-i| = \frac{1}{2}$. **2**

(c) Express $\sqrt{8+6i}$ in the form $a+ib$ ($a>0$) and hence solve the equation $z^2 + 2(1+2i)z - (11+2i) = 0$. Express answers in the form $x+iy$, $x, y \in \mathbb{R}$. **4**

(d) Prove that any equation of the form $x^3 - mx^2 + n = 0$, $m, n \neq 0$, cannot have a triple root. Assuming the equation has a double root, find the relation between m and n. **4**

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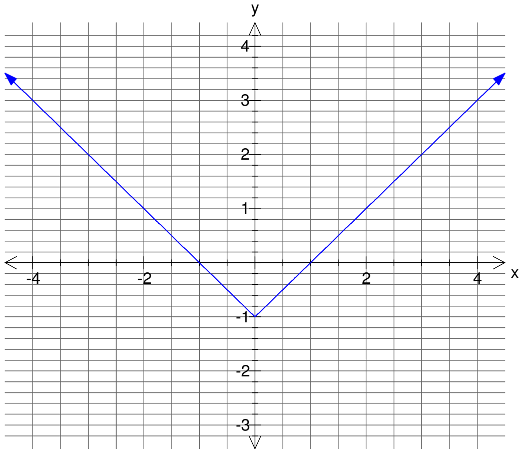
Question 3

(a) Consider the graph of the function $f(x) = |x| - 1$ shown below.



Graph the transformations of $y = f(x)$ whose equations are shown below on the answer sheet provided at the back of this examination paper. Detach and place it with your answers to Question 3. Show distinguishing features.
Sketch:

(i) $y = x\,f(x)$ **2**

(ii) $|y| = f(x)$ **2**

(iii) $y = e^{f(x)}$ **2**

(iv) $y = \sqrt{f(x)}$ **2**

(b) (i) Prove that if the line $ax + by + c = 0$ is a tangent to the circle $x^2 + y^2 = R^2$ then $R^2(a^2 + b^2) = c^2$. **3**

(ii) A straight line with equation $ax + by + c = 0$ moves so that the sum of the perpendicular distances, d_1 and d_2 from each of the points (2, 0) and (– 2, 0) to the line is always equal to 6 units. Prove that the line always touches a circle and find the equation of this circle. **4**
[You may assume for convenience that $c > 2a$ if required.]

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Question 4

(a) A solid yet buoyant life ring can be generated by rotating the ellipse $\frac{x^2}{64}+\frac{y^2}{25}=1$ around the line $x = 28$.

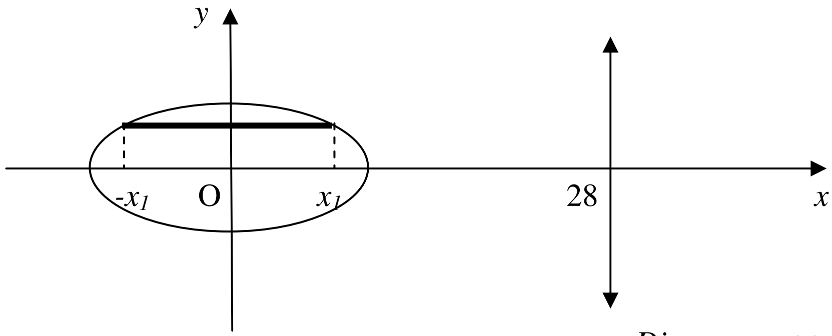

Diagram not to scale.

Find the volume of rubber required to make it by taking slices perpendicular to the y-axis, using the steps:

(i) Show that the volume of 1 slice δy thick can be approximated by $V = \pi\left(\frac{896}{5}\sqrt{25-y^2}\right)\delta y$. **3**

(ii) Hence find the volume of rubber required to make the life ring. **3**

(b) Show that $\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx, \quad n = 1, 2, 3, \ldots$ **5**

Hence evaluate $\int_0^1 x^3 e^x dx$.

(c) A particle of mass 10 kg is found to experience a resistive force, in Newtons, of one-ninth of the square of its velocity v, in metres per second, when it moves through the air.
The particle is projected vertically upwards from a point O with a velocity of $30\sqrt{3}$ m/s and the point A, vertically above O, is the highest point reached by the particle before it starts to fall to the ground again.
Assuming the value of $g = 10\,ms^{-2}$,

(i) Explain why $\ddot{x} = -10 - \frac{1}{90}v^2$. **1**

(ii) Find the time the particle takes to reach A from O. **3**

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Question 5

(a) Solve the equation $\cos 3x = \sin 7x$. **4**

(b) Consider the ellipse E with semi-major and semi-minor axes a and b.
Point $P(a\cos\theta, b\sin\theta)$ lies on E. Rectangle $PQRS$ is inscribed in E.

(i) Show that an expression for the perimeter, l, of $PQRS$ is given by $l = 4a\cos\theta + 4b\sin\theta$. **2**

(ii) Find the largest perimeter possible and the value of θ for which it occurs. **4**

(c)

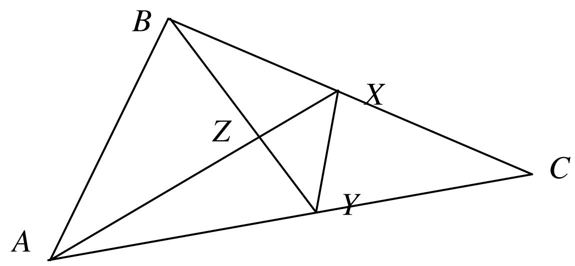


Diagram not to scale.

X and Y are points on the sides BC and AC of a triangle ABC respectively such that $\angle AXC = \angle BYC$ and $BX = XY$.
Copy the diagram into your examination booklet then,

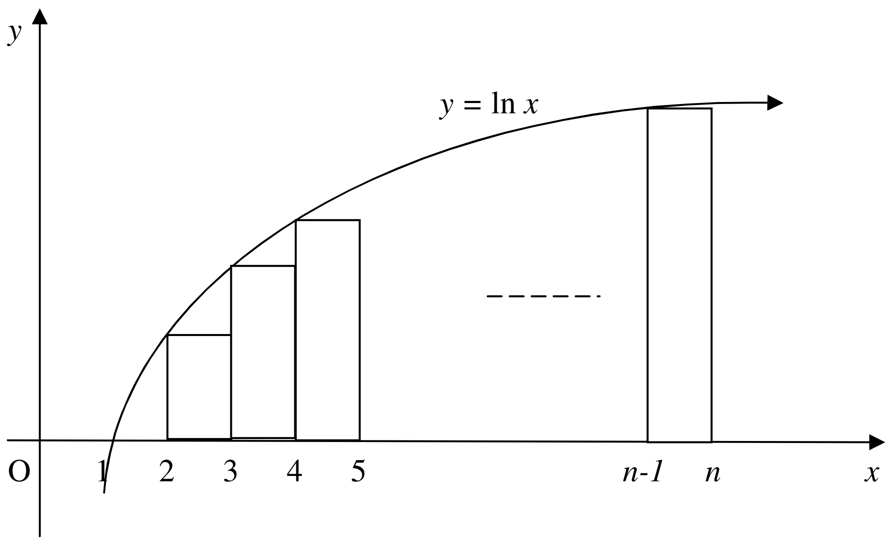
(i) prove $ABXY$ is a cyclic quadrilateral. **2**

(ii) Hence or otherwise, prove AX bisects $\angle BAC$. **3**

Question 6

(a) (i) Evaluate exactly the integral $\int_1^n \ln x \, dx$. **2**

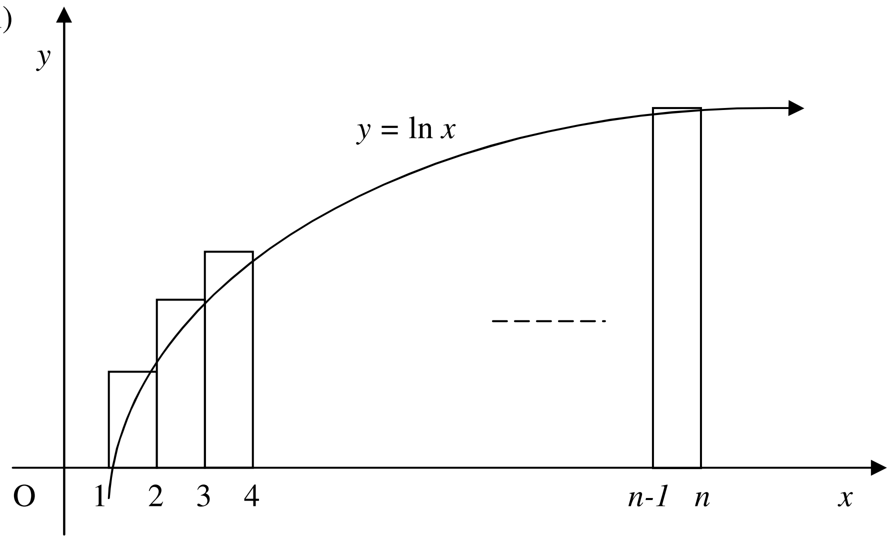
(ii) Consider the curve $y = \ln x$. The area under the curve for $1 \le x \le n$ is approximated by dividing it into rectangles under the curve each of width 1 unit.
See diagram below (*not to scale*).



Show that the sum of the rectangles, S_u, is given by **2**

$$S_u = \ln\left((n-1)!\right).$$

(iii)



Another approximation, S_a, is made by dividing the area into rectangles that lie above the curve. See diagram above (*not to scale*). Find a similar expression for this area, S_a. **1**

Question 6 (a) continued

(iv) Hence explain why

$$\ln\left((n-1)!\right) < n\ln n - n + 1 < \ln\left(n!\right)$$ **1**

(v) Further, show that **2**

$$\left(n-1\right)! < n^n e^{1-n} < n!$$

(b)

(i) Use the substitution $t = \tan\frac{x}{2}$ to prove $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} = \frac{\pi}{3\sqrt{3}}$. **3**

(ii) Show that $\int_0^{2a} f\left(x\right)dx = \int_0^{a} \{f\left(x\right) + f\left(2a-x\right)\}\,dx$. **2**

[Hint: let $w = 2a - x$.]

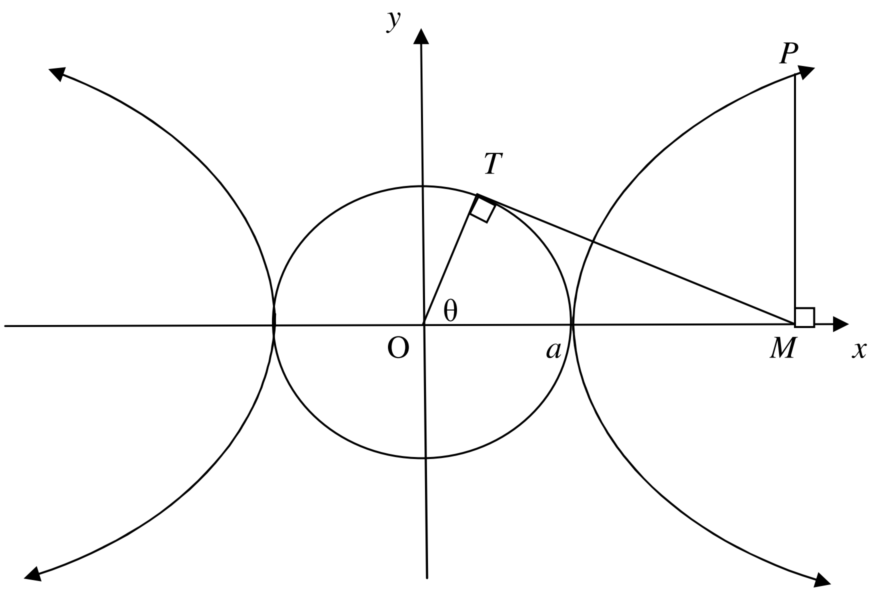
(iii) Hence or otherwise evaluate $\int_0^{\pi} \frac{x\,dx}{2+\sin x}$. **2**

Question 7

(a) Show that:
$\cot 2x - \tan 2x = 2\cot 4x$ **5**

Hence prove by Mathematical Induction that for $n = 1, 2, 3, \ldots$
$\tan x + 2\tan 2x + 4\tan 4x + \ldots + 2^{n-1}\tan\left(2^{n-1}x\right) = \cot x - 2^n \cot\left(2^n x\right)$.

(b)



Consider the diagram (*not to scale*) of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and circle $x^2 + y^2 = a^2$. *T* lies on the circle and *P* lies on the hyperbola.

(i) Show that P has coordinates $P(a\sec\theta, b\tan\theta)$. **2**

Q lies on the hyperbola and has coordinates $Q(a\sec\phi, b\tan\phi)$.

(ii) If $\theta + \phi = \frac{\pi}{2}$, $\theta \neq \frac{\pi}{4}$, show that the chord PQ has equation $ay = b(\cos\theta + \sin\theta)x - ab$. **4**

(iii) Show that every such chord passes through a fixed point and find its coordinates. **2**

(iv) Show that as θ approaches $\frac{\pi}{2}$, the chord PQ approaches a line parallel to an asymptote. **2**

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Question 8

(a) Sketch the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, where $a>0$ for $x\ge 0$, $y\ge 0$. **7**

Show that the curve touches the x, y axes.

The tangent at $P(x_0, y_0)$ on it has equation $y_0^{\frac{1}{3}}x+x_0^{\frac{1}{3}}y=x_0^{\frac{1}{3}}y_0^{\frac{1}{3}}a^{\frac{2}{3}}$.

This tangent cuts the *x* and *y* axes at *A* and *B* respectively. Show that the line segment *AB* is independent of the position of *P*.

(b) Let $\rho=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$.

The complex number $\alpha=\rho+\rho^2+\rho^4$ is a root of the quadratic equation $x^2+ax+b=0$ where *a* and *b* are real.

(i) Prove that $1+\rho+\rho^2+...+\rho^6=0$. **2**

(ii) The second root of the quadratic equation is β. Express β in terms of positive powers of ρ. Justify your answer. **2**

(iii) Find the values of the coefficients *a* and *b*. **2**

(iv) Deduce that $-\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}=\frac{\sqrt{7}}{2}$. **2**

End of fun!

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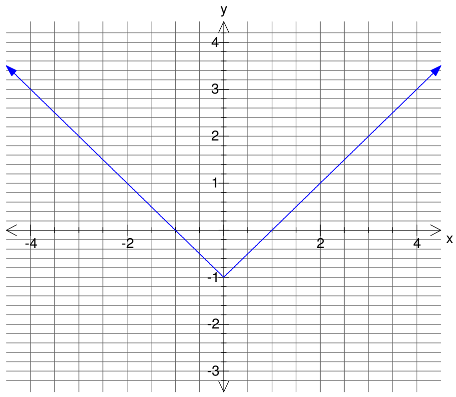
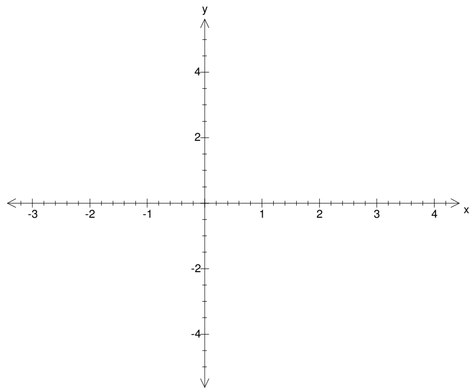
Year 12 Ext 2 Graphing Transformations Answer sheet – DETACH and PLACE in QUESTION 3 booklet

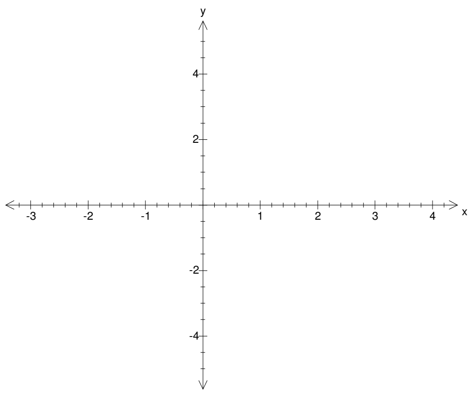
$y = f(x)$



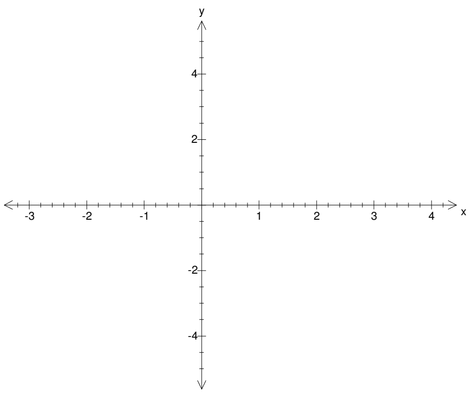
$y = x\,f(x)$



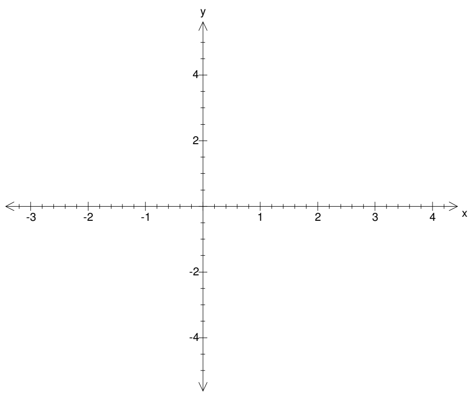
$|y| = f(x)$



$y = e^{f(x)}$



$y = \sqrt{f(x)}$



Ascham Extension 2 Mathematics Trial 2010 Solutions

Solutions to
Title: Ascham 2010 Ext 2 Maths Trial Yr12 /120

Q1 a) $\int \frac{2x\,dx}{\sqrt{x^4+16}} = \ln\left(x^2+\sqrt{x^4+16}\right)+C$ (2)

b) $\int_0^1 \sin(a\pi x)\sin(b\pi x)\,dx$

$= \int_0^1 \frac{1}{2}\left[\cos(a-b)\pi x - \cos(a+b)\pi x\right]dx$

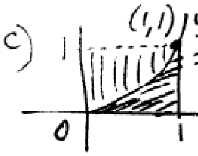
$= \frac{1}{2}\left[\frac{\sin(a-b)\pi x}{(a-b)\pi} - \frac{\sin(a+b)\pi x}{(a+b)\pi}\right]_0^1$

$= \frac{1}{2}\left[\frac{\sin(a-b)\pi}{(a-b)\pi} - \frac{\sin(a+b)\pi}{(a+b)\pi} - \left(\frac{\sin 0}{(a-b)\pi} - \frac{\sin 0}{(a+b)\pi}\right)\right]$ (3)

since $a, b \in \mathbb{Z}$ then $\sin(a\pm b)\pi = 0$

$= 0$ unless $a = \pm b$ then undefined

c)



Since $\int x^{\frac{1}{n}}dx = \int y^{\frac{1}{n}}dy$ then

$\int_0^1 x^n dx + \int_0^1 x^{\frac{1}{n}}dx$ = Sum of areas under curve and to right of y-axis

They are inverse functions so sum $= 1.$ (3)

d) $\int_{-5}^{10} f(t)dt = 4 \quad \therefore \int_{-5}^{10} g(u)du$ (3)

$= \int_{-5}^{10} g(x)\,dx = \int_{-5}^{10} f(x) + 2\,dx$

$= \int_{-5}^{10} f(x)dx + \int_{-5}^{10} 2\,dx = 4 + \left[2x\right]_{-5}^{10}$

$= 4 + 30 = 34$

e) $(x^2+y^2)^2 = 12(x^2-y^2)$

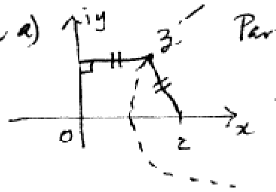
$\therefore 2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 12\left(2x-2y\frac{dy}{dx}\right)$

$4(x^2+y^2)x + 4(x^2+y^2)y\frac{dy}{dx} = 24x - 24y\frac{dy}{dx}$

$4(x^2+y^2)y\frac{dy}{dx} + 24y\frac{dy}{dx} = 24x - 4(x^2+y^2)x$

$\therefore \frac{dy}{dx} = \frac{24x - 4(x^2+y^2)x}{4(x^2+y^2)y + 24y}$ (4)

Q2 a)



Parabola focus $x = 2$, $y = 0$

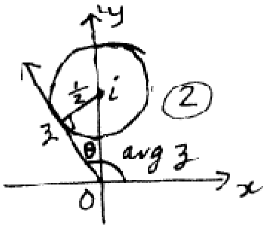
directrix $x = 0$

$a = 1$

Vertex $(1, 0)$ (3)

$y^2 = 4(x-1)$.

b)



(2)

Find θ:

$\sin\theta = \frac{\frac{1}{2}}{1} = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ (2)

$\therefore$ Max $\arg z = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$

c) Let $\sqrt{8+6i} = a+ib$, $a > 0$.

$\therefore (a+ib)^2 = 8+6i$

$a^2 + 2aib + i^2b^2 = 8+6i$

$\therefore a^2 - b^2 = 8 \qquad 2ab = 6 \Rightarrow ab = 3.$

By inspection, $a = 3$, $b = 1$.

$\therefore \sqrt{8+6i} = 3+i$

$z^2 + 2(1+2i)z - (11+2i) = 0$

$\therefore z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-2(1+2i) \pm \sqrt{4(1+2i)^2 - 4(1)(-(11+2i))}}{2(1)}$

$= \frac{-2-4i \pm \sqrt{4(1+4i-4) + 44 + 8i}}{2}$

$= \frac{-2-4i \pm \sqrt{4(4i-3) + 44 + 8i}}{2}$

$= \frac{-2-4i \pm \sqrt{-12+16i+44+8i}}{2}$

$= \frac{-2-4i \pm \sqrt{32+24i}}{2}$ (4)

$= \frac{-2-4i \pm 2\sqrt{8+6i}}{2}$

$= \frac{-2-4i \pm 2(3+i)}{2}$

$= \frac{-2-4i \pm (6+2i)}{2} = \frac{4-2i}{2}$ or $\frac{-8-6i}{2}$

$= 2-i$ or $-4-3i$

Q2 cont'd d) If $P(x)=0$ has triple root α, then $P(\alpha)=P'(\alpha)=P''(\alpha)=0$

$P(\alpha)=\alpha^3-m\alpha^2+n=0$

$P'(\alpha)=3\alpha^2-2\alpha m=0$, $P''(\alpha)=6\alpha-2m=0$

$\therefore \alpha(3\alpha-2m)=0 \quad \therefore \alpha=\frac{m}{3}$

$\therefore \alpha=0$ or $\alpha=\frac{2m}{3}\neq\frac{m}{3}$ $\therefore$ No.

If double root then $P(\alpha)=P'(\alpha)=0$

$\therefore$ If $\alpha=0$ $P(\alpha)\neq 0$. Try $\alpha=\frac{2m}{3}$

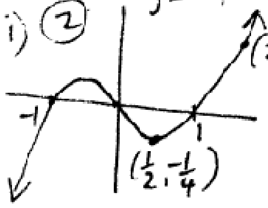
$P(\alpha)=\frac{8m^3}{27}-m.\frac{4m^2}{9}+n=0$

$\therefore \frac{8m^3}{27}-\frac{12m^3}{27}+n=0$

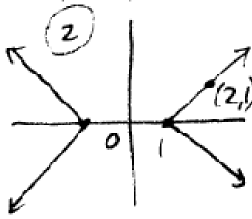
(4) $\therefore n=\frac{4m^3}{27}$ or $4m^3=27n$.

Q 3. a) i) $y=xf(x)$ ii) $|y|=f(x)$

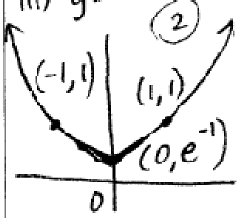
i) (2)

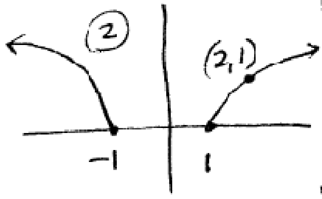


(2)



iii) $y=e^{f(x)}$ (2) IV) $y=\sqrt{f(x)}$ (2)





b)(i) Solve sim: $y=\frac{-(ax+c)}{b} \Rightarrow x^2+y^2=R^2$

$\therefore x^2+\left(-\frac{(ax+c)}{b}\right)^2=R^2$

$b^2x^2+a^2x^2+2acx+c^2-b^2R^2=0$

If tangent then $\Delta=0$

$\therefore (2ac)^2-4(a^2+b^2).(c^2-b^2R^2)=0$

$4a^2c^2-4(a^2c^2-a^2b^2R^2+b^2c^2-b^4R^2)=0$

$\therefore 4a^2b^2R^2-4b^2c^2+4b^4R^2=0$

Q3 b)(i) cont'd $\therefore R^2(a^2+b^2)=c^2$ QED

(3) OR (easier) Show $\left|\frac{a(0)+b(0)+c}{\sqrt{a^2+b^2}}\right|^2=R^2$

$\therefore c^2=R^2(a^2+b^2)$ QED

ii) $d_1+d_2=6$

$\left|\frac{a(2)+b(0)+c}{\sqrt{a^2+b^2}}\right|+\left|\frac{a(-2)+b(0)+c}{\sqrt{a^2+b^2}}\right|=6$

$\therefore \left|\frac{2a+c}{\sqrt{a^2+b^2}}\right|+\left|\frac{-2a+c}{\sqrt{a^2+b^2}}\right|=6$ (4)

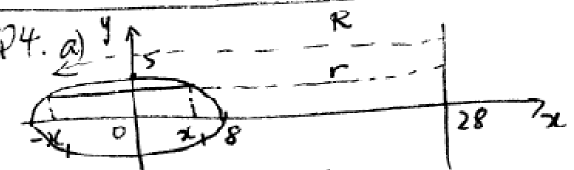
For convenience $c>2a$ then if $c>0$

$2a+c+-2a+c=6\sqrt{a^2+b^2}$

$4c^2=36(\sqrt{a^2+b^2})$ [Use similar argument if $c\leq 0$]

$\therefore$ Using (i) $R^2=\frac{36}{4}$ $\therefore$ circle equation is $x^2+y^2=\frac{36}{4}=9$ $\therefore x^2+y^2=9$.

Q4. a)



i) $V_{1slice}=\pi(R^2-r^2)\delta y$

$=\pi(R+r)(R-r)\delta y$

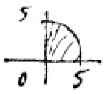
$R+r=(28+x_1)+(28-x_1)=56$

$R-r=(28+x_1)-(28-x_1)=2x_1$

Since $\frac{x_1^2}{64}+\frac{y_1^2}{25}=1$ then $x_1=\sqrt{64\left(1-\frac{y_1^2}{25}\right)}$ $(x_1>0)$

$\therefore x_1=\frac{8}{5}\sqrt{25-y_1^2}$

$\therefore V_{1slice}=2\pi\times 56\times\frac{8}{5}\sqrt{25-y^2}\,\delta y$



$=\frac{896\pi}{5}\sqrt{25-y^2}\,\delta y$ (3)

(ii) $\therefore V \doteq \sum_{y=-5}^{5}\frac{896\pi}{5}\sqrt{25-y^2}\,\delta y$

$=2\int_0^5\frac{896}{5}\pi\sqrt{25-y^2}\,dy$

$=2.\frac{896}{5}\pi.\frac{1}{4}5^2\pi$ (3)

$=2240\pi^2$ u^3.

Q4 cont'd b) $\int x^n e^x\,dx = uv - \int v\,du$

where $u = x^n$, $dv = e^x dx$

$du = nx^{n-1}dx$, $v = e^x$

$\therefore \int x^n e^x dx = x^n e^x - \int e^x . n x^{n-1} dx$

$= x^n e^x - n\int x^{n-1}e^x dx$

$\therefore \int_0^1 x^3 e^x dx = \left[x^3 e^x\right]_0^1 - 3\int_0^1 x^2 e^x dx$

$n=3 \quad = e - 3\left[\left[x^2 e^x\right]_0^1 - 2\int_0^1 x^1 e^x dx\right]$

$= e - 3\left[e - 2\left[\left[xe^x\right]_0^1 - 1\int_0^1 x^0 e^x dx\right]\right]$

$= e - 3e + 6\left[e - \left[e^x\right]_0^1\right]$

$= -2e + 6e - 6\left[e-1\right]$ (5)

$= 6 - 2e$

c) $\uparrow + x$ (i) $ma = -mg - \frac{1}{9}v^2$

$\therefore \ddot{x} = -g - \frac{1}{90}v^2$ (1)

$\therefore g = 10 \quad \ddot{x} = -10 - \frac{1}{90}v^2$

(ii) $\frac{dv}{dt} = -\frac{(900 + v^2)}{900}$ (3)

$\therefore \frac{dt}{dv} = -\frac{900}{900 + v^2}$

$t = \int_{30\sqrt{3}}^{0} -\frac{900}{900 + v^2}\,dv$

$= -900 . \frac{1}{30}\left[\tan^{-1}\frac{v}{30}\right]_{30\sqrt{3}}^{0}$

$= -30\left[\tan^{-1}0 - \tan^{-1}\sqrt{3}\right]$

$= 30 . \frac{\pi}{3}$

$t = 10\pi$ seconds.

Q5 a) $\cos 3x = \sin 7x$

$\cos 3x = \cos\left(\frac{\pi}{2} - 7x\right)$

$\therefore 3x = 2k\pi \pm \left(\frac{\pi}{2} - 7x\right)$, $k \in \mathbb{Z}$

Case 1: $3x = 2k\pi + \frac{\pi}{2} - 7x$

$\therefore 10x = \frac{4k\pi + \pi}{2}$

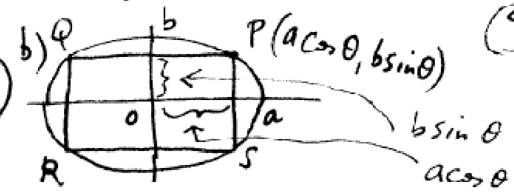
$\therefore x = \frac{\pi(4k+1)}{20}$

Case 2: $3x = 2k\pi - \left(\frac{\pi}{2} - 7x\right)$

$\therefore -4x = \frac{4k\pi - \pi}{2}$

or $x = \frac{\pi}{8}(1-4k)$ (4)

b)



(i) Perimeter $= 2PQ + 2PS$

$= 2(2a\cos\theta + 2b\sin\theta)$ (2)

$\therefore l = 4a\cos\theta + 4b\sin\theta$

(ii) $\frac{dl}{d\theta} = -4a\sin\theta + 4b\cos\theta$

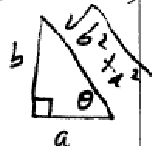
$\frac{d^2 l}{d\theta^2} = -4a\cos\theta - 4b\sin\theta$

Max l occurs when $\frac{dl}{d\theta} = 0$ and $\frac{d^2 l}{d\theta^2} < 0$. (4)

$\therefore 0 = -4a\sin\theta + 4b\cos\theta \quad 0 < \theta < \frac{\pi}{2}$ (1st Quad)

$\therefore \tan\theta = \frac{b}{a}$

$\therefore \theta = \tan^{-1}\left(\frac{b}{a}\right)$ since $a, b > 0$



$\therefore$ Test $\frac{d^2 l}{d\theta^2}$

$= -4(a\cos\theta + b\sin\theta)$

$= -4\left(a . \frac{a}{\sqrt{a^2+b^2}} + \frac{b . b}{\sqrt{a^2+b^2}}\right)$

$= -4\left(\frac{a^2+b^2}{\sqrt{a^2+b^2}}\right) = -4\sqrt{a^2+b^2} < 0$

$\therefore$ Max l when $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

Max $l = 4\sqrt{a^2+b^2}$ units.

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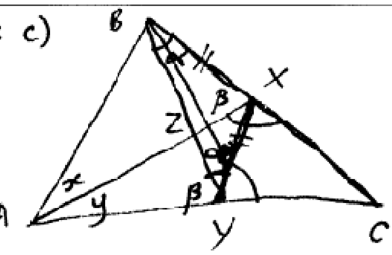
Q5 contd: c)



i) RTP: ABXY is a cyclic quad.

Let $\angle XBY = \angle XYB = \alpha$ (base $\angle$s isosceles Δ BXY equal)

Let $\angle BXA = \beta$.

$\therefore \angle BYA = \beta$. (both supplementary to equal $\angle$s given) (straight lines BC, AC)

(2)

$\therefore$ BXYA is cyclic quad (equal $\angle$s β standing on same arc BA)

ii) RTP: $\angle BAX$ $(=x)$ and $\angle XAY$ $(=y)$ are equal.

Proof: $y = \alpha$ ($\angle$s standing on arc XY equal)

(3) $x = \alpha$ ($\angle$s standing on arc XB equal)

$\therefore x = y$ (both $= \alpha$)

$\therefore$ AX bisects $\angle BAY$ QED

Q6 a) i) $\int_1^n \ln x \, dx = uv - \int v \, du$

$= \left[x \ln x\right]_1^n - \int_1^n x \cdot \frac{1}{x} \, dx$ (2)

$= n \ln n - 0 - \left[x\right]_1^n$

$= n \ln n - n + 1$. (2)

ii) $S_u = l_2 b_2 + l_3 b_3 + \ldots + l_{n-1} b_{n-1}$

$= (\ln 2) \times 1 + (\ln 3) \times 1 + \ldots + (\ln(n-1)) \times 1$

$= \ln(2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1))$

$= \ln((n-1)!)$

iii) $S_a = l_1 b_1 + l_2 b_2 + \ldots + l_{n-1} b_{n-1}$

$= 1 \times \ln 2 + 1 \times \ln 3 + 1 \times \ln 4 + \ldots + 1 \times \ln n$

$= \ln(2 \cdot 3 \cdot 4 \cdot \ldots \cdot n)$ (1)

$= \ln((n)!)$

iv) Approx area $S_u <$ Real area $< S_a$

$\therefore \ln(n-1)! < n \ln n - n + 1 < \ln n!$ (1)

v) $\therefore \ln(n-1)! < \ln n^n - \ln e^n + \ln e < \ln n!$

$\ln(n-1)! < \ln\left(\frac{n^n \cdot e}{e^n}\right) < \ln n!$ (2)

$\therefore (n-1)! < n^n e^{1-n} < n!$ QED

b) $t = \tan \frac{x}{2}$

$\therefore x = 2\tan^{-1} t$

$\therefore dx = \frac{2dt}{1+t^2}$

$\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \sin x}$

If $x = \frac{\pi}{2}$, $t = 1$

$x = 0$, $t = 0$

$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{2 + \frac{2t}{1+t^2}}$

$= \int_0^1 \frac{2dt}{1+t^2} \times \frac{1+t^2}{2 + 2t^2 + 2t}$

$= \int_0^1 \frac{dt}{t^2 + t + 1} = \int_0^1 \frac{dt}{t^2 + t + \frac{1}{4} + \frac{3}{4}}$

$= \int_0^1 \frac{dt}{(t + \frac{1}{2})^2 + \frac{3}{4}} = \left[\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t + \frac{1}{2}}{\sqrt{3}/2}\right)\right]_0^1$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{3}{2} \times \frac{2}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{2} \times \frac{2}{\sqrt{3}}\right)\right]$

$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{3\sqrt{3}}$ QED (3)

ii) If $w = 2a - x$ then $dw = -dx$

and if $x = a$, $w = a$

$x = 0$, $w = 2a$

So $\int_0^a f(x) + f(2a - x) \, dx$

$= \int_0^a f(x) \, dx - \int_{2a}^a f(w) \, dw$ (2)

$= \int_0^a f(x) \, dx + \int_a^{2a} f(w) \, dw$

$= \int_0^a f(x) \, dx + \int_a^{2a} f(x) \, dx = \int_0^{2a} f(x) \, dx$.

(since $\int f(\square) \, d\square = \int f(\Delta) \, d\Delta$.)

Q6 cont'd (b) (iii) if $a = \frac{\pi}{2}$

$\therefore \int_0^{\pi} \frac{x\,dx}{2+\sin x} = \int_0^{\frac{\pi}{2}} \frac{x\,dx}{2+\sin x} + \int_0^{\frac{\pi}{2}} \frac{\pi - x}{2+\sin(\pi - x)}\,dx$ from (ii)

$= \int_0^{\frac{\pi}{2}} \frac{x}{2+\sin x} + \frac{\pi - x}{2+\sin x}\,dx$

$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\,dx = \pi \cdot \frac{\pi}{3\sqrt{3}}$ (from (i))

$= \frac{\pi^2}{3\sqrt{3}}$ (2)

Q7 a) $\cot 2x - \tan 2x = \frac{\cos 2x}{\sin 2x} - \frac{\sin 2x}{\cos 2x}$

$= \frac{\cos^2 2x - \sin^2 2x}{\sin 2x \cos 2x} = \frac{\cos 4x}{\frac{1}{2}\sin 4x}$

$= 2\cot 4x$ QED (2)

RTP: $P(n)$: $\tan x + 2\tan 2x + \ldots + 2^{n-1}\tan(2^{n-1}x) = \cot x - 2^n \cot 2^n x$.

Proof: Let $n = 1$: $P(1)$ LHS $= \tan x$

RHS $= \cot x - 2\cot 2^1 x$

$= \tan x$ (from above identity)

(3) $\therefore P(1)$ true.

Assume $P(k)$ true i.e.

$\tan x + 2\tan 2x + \ldots + 2^{k-1}\tan 2^{k-1}x = \cot x - 2^k \cot 2^k x$.

RTP: $P(k+1)$ true i.e.

$\tan x + 2\tan 2x + \ldots + 2^k \tan 2^k x = \cot x - 2^{k+1}\cot 2^{k+1}x$

Proof: Consider the LHS of $P(k+1)$

$= \tan x + 2\tan 2x + \ldots + 2^{k-1}\tan 2^{k-1}x + 2^k \tan 2^k x$

Q7 a) cont'd $= \cot x - 2^k \cot 2^k x + 2^k \tan 2^k x$ from $P(k)$

$= \cot x - 2^k[\cot 2^k x - \tan 2^k x]$

$= \cot x - 2^k[\cot 2(2^{k-1}x) - \tan 2(2^{k-1}x)]$

$= \cot x - 2^k[2\cot 4(2^{k-1}x)]$ from above

$= \cot x - 2^{k+1}\cot 2^{k+1}x$ = RHS of $P(k+1)$.

$\therefore P(n)$ true by Math. Induction, $n \geq 1$.

b) (i) P should satisfy $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (2)

$\therefore \frac{a^2\sec^2\theta}{a^2} - \frac{b^2\tan^2\theta}{b^2} = 1$ QED.

(ii) $\theta + \phi = \frac{\pi}{2}$, $\theta \neq \frac{\pi}{4}$. Chord PQ: (4)

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$

$\therefore \frac{b(\tan\phi - \tan\theta)}{a(\sec\phi - \sec\theta)} = \frac{y - b\tan\theta}{x - a\sec\theta}$

$\therefore \frac{\frac{\sin\phi}{\cos\phi} - \frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\phi} - \frac{1}{\cos\theta}} = \frac{ay - ab\frac{\sin\theta}{\cos\theta}}{bx - ab\cdot\frac{1}{\cos\theta}}$

if $\theta + \phi = \frac{\pi}{2}$ then $\sin\phi = \cos(\frac{\pi}{2} - \phi) = \cos\theta$ etc.

$\therefore \frac{\frac{\cos\theta\cos\theta - \sin\theta\sin\theta}{\sin\theta\cos\theta}}{\frac{\cos\theta - \sin\theta}{\sin\theta\cos\theta}} = \frac{\frac{ay\cos\theta - ab\sin\theta}{\cos\theta}}{\frac{bx\cos\theta - ab}{\cos\theta}}$

$\frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta} = \frac{ay\cos\theta - ab\sin\theta}{bx\cos\theta - ab}$

$\frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta - \sin\theta} = \frac{a(y\cos\theta - b\sin\theta)}{b(x\cos\theta - a)}$

$\therefore bx\cos^2\theta + bx\cos\theta\sin\theta - ab\cos\theta - ab\sin\theta = ay\cos\theta - ab\sin\theta$

$\therefore ay = b(\cos\theta + \sin\theta)x - ab$ QED.

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Q7 cont'd: iii) $\therefore$ Chord has eqn

$y = \frac{b}{a}(\cos\theta + \sin\theta)x - b = mx + K$

$\therefore$ m can vary $\therefore$ fixed y-int.

$\therefore$ fixed point is $(0, -b)$. (2)

iv) As $\theta \to \frac{\pi}{2}$ $\quad m \to \frac{b}{a}(0+1) = \frac{b}{a}$

(2)

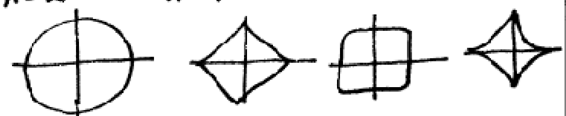
but asymptotes are $y = \pm\frac{b}{a}x$

$\therefore$ chord // asymptote if $m = \frac{b}{a}$

Q8 a) $x^{2/3} + y^{2/3} = a^{2/3}$, $x, y \geq 0$, $a > 0$

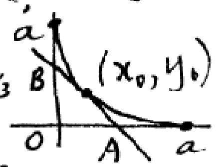
Recall $|x|^n + |y|^n = 1$ cases:

n=2 $\quad$ n=1 $\quad$ n>2 $\quad$ n<1



$\therefore$ In this case, $n = \frac{2}{3} < 1$ (7)

$\therefore$ Graph is



If $y = 0$ $x^{2/3} = a^{2/3}$

$\therefore x = a$

$x = 0$ $\quad y^{2/3} = a^{2/3} \Rightarrow y = a$

(NB: "touches" means tangential so should check $\frac{dy}{dx}$ etc.)

Tangent $y_0^{1/3}x + x_0^{1/3}y = x_0^{1/3}y_0^{1/3}a^{2/3}$

cuts x-axis at A, when $y = 0$:

$y_0^{1/3}x + 0 = x_0^{1/3}y_0^{1/3}a^{2/3}$

$\therefore A(x_0^{1/3}a^{2/3}, 0)$

Similarly B is $B(0, y_0^{1/3}a^{2/3})$

$\therefore AB = \sqrt{(x_0^{1/3}a^{2/3} - 0)^2 + (0 - y_0^{1/3}a^{2/3})^2}$

$= \sqrt{x_0^{2/3}a^{4/3} + y_0^{2/3}a^{4/3}}$

$= \sqrt{a^{4/3}(x_0^{2/3} + y_0^{2/3})} = \sqrt{a^{4/3} \times a^{2/3}}$

$= a$ indep. of P. since (x_0, y_0) lies on curve.

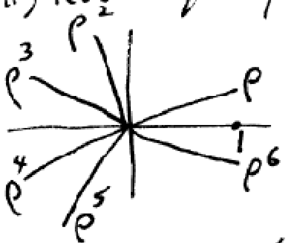
b) $\rho = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ $\quad \rho^7 = \cos 2\pi + i\sin 2\pi = 1$

$\therefore \rho$ is a root of $z^7 - 1 = 0$ (2)

$\therefore$ i) $z^7 - 1 = 0 = (z-1)(z^6 + z^5 + z^4 + \ldots + 1)$

but ρ is a root $\therefore \rho^6 + \rho^5 + \rho^4 + \ldots + 1 = 0$

ii) Roots equally spaced:



$\therefore \overline{\rho^6} = \rho$

$\overline{\rho^5} = \rho^2$

$\overline{\rho^4} = \rho^3$

(2)

Now the complex roots of equations with real coefficients come in conjugate pairs

$\therefore$ if $\alpha = \rho + \rho^2 + \rho^4$ is a root

then $\beta = \overline{\alpha} = \overline{\rho + \rho^2 + \rho^4}$

$= \overline{\rho} + \overline{\rho^2} + \overline{\rho^4}$

$\therefore \quad \beta = \rho^6 + \rho^5 + \rho^3$.

iii) Now for $x^2 + ax + b = 0$

Sum of roots $\alpha + \beta = \frac{-a}{1}$

$\therefore \rho + \rho^2 + \rho^4 + \rho^6 + \rho^5 + \rho^3 = -a$

$\therefore \quad -1 = -a$ $\quad$ (from (i))

$\therefore a = 1$ (2)

Product of roots $\alpha\beta = \frac{b}{1}$

$\therefore (\rho + \rho^2 + \rho^4)(\rho^6 + \rho^5 + \rho^3) = b$

$\rho^7 + \rho^6 + \rho^4 + \rho^8 + \rho^7 + \rho^5 + \rho^{10} + \rho^9 + \rho^7 = b$

$1 + \rho^6 + \rho^4 + \rho + 1 + \rho^5 + \rho^2 + \rho^3 + 1 = b$

$\underbrace{1 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6}_{0} + 2 = b$

$\therefore b = 2$.

iv) So roots of $x^2 + x + 2 = 0$ are

$x = \frac{-1 \pm \sqrt{1^2 - 4.1.2}}{2} = \frac{-1 \pm i\sqrt{7}}{2}$. (2)

$\therefore Im(\alpha) = Im(\rho + \rho^2 + \rho^4)$

$= \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$

$= \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$.