Student Number____________________

ASCHAM SCHOOL

2011
YEAR 12

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks)
Use a SEPARATE writing booklet.

(a) Find $\int \sin^3\theta \, d\theta$. **2**

(b) (i) Express $\dfrac{3x+1}{(x+1)(x^2+1)}$ in the form $\dfrac{a}{x+1}+\dfrac{bx+c}{x^2+1}$. **2**

(ii) Hence find $\int \dfrac{3x+1}{(x+1)(x^2+1)}\,dx$. **2**

(c) Use the substitution $x=2\sin\theta$, or otherwise, to evaluate $\int_1^{\sqrt{3}} \dfrac{x^2}{\sqrt{4-x^2}}\,dx$. **3**

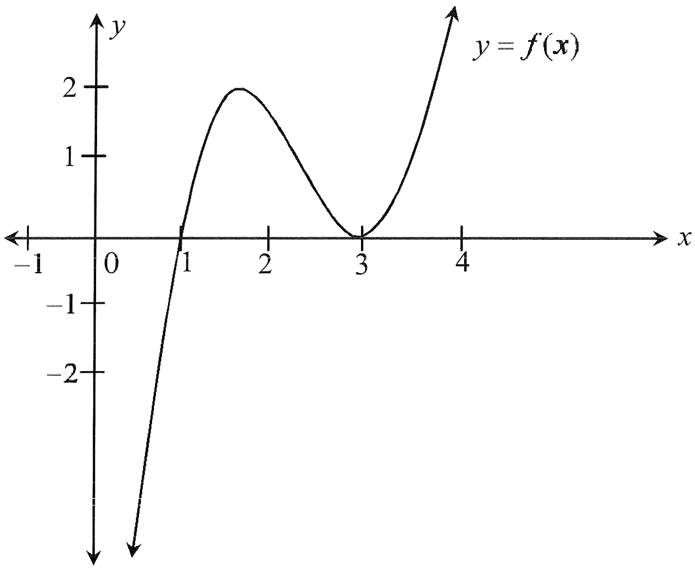
(d) Find $\int x^2\sqrt{3-x}\,dx$. **3**

(e) Evaluate $\int_0^1 \tan^{-1}\theta \, d\theta$. **3**

Marks

QUESTION 2 (15 marks)
Start a new writing booklet.

(a)



The diagram above is a sketch of the function $y = f(x)$.

On separate diagrams sketch:

(i) $y = (f(x))^2$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \ln f(x)$ **2**

(iv) $y^2 = f(x)$ **2**

(b) (i) If $f'(x) = \dfrac{2-x}{x^2}$ and $f(1) = 0$, find $f''(x)$ and $f(x)$. **3**

(ii) Explain why the graph of $f(x)$ has only one turning point and find the value of the function at that point, stating whether it is a maximum or a minimum value. **2**

(iii) Show that $f(4)$ and $f(5)$ have opposite signs and draw a sketch of $f(x)$. **2**

Marks

QUESTION 3 (15 marks)
Start a new writing booklet.

(a) Express $(\sqrt{3}+i)^8$ in the form $x+iy$. **3**

(b) On an Argand diagram, sketch the region where the inequalities **3**

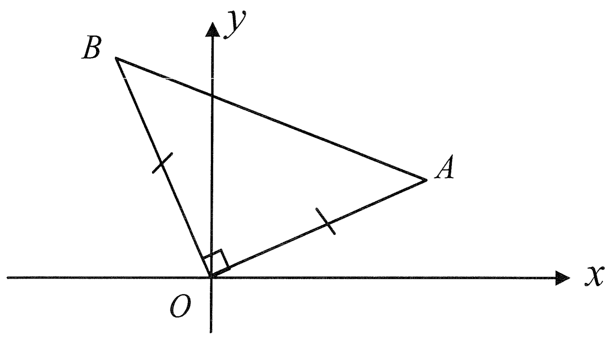
$$|z| \leq 3 \text{ and } -\frac{2\pi}{3} \leq \arg(z+2) \leq \frac{\pi}{6} \text{ both hold.}$$

(c) Show that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} = \sin\theta+i\cos\theta$. **3**

(d) (i) Express $z=\dfrac{-1+i}{\sqrt{3}+i}$ in modulus-argument form. **2**

(ii) Hence evaluate $\cos\dfrac{7\pi}{12}$ in surd form. **2**

(e) The Argand diagram below shows the points A and B which represent the complex numbers z_1 and z_2 respectively.



Given that ΔBOA is a right-angled isosceles triangle, show that $(z_1+z_2)^2 = 2z_1z_2$. **2**

Marks

QUESTION 4 (15 marks)
Start a new writing booklet.

(a) If $z = 1 + i$ is a root of the equation $z^3 + pz^2 + qz + 6 = 0$ where p and q are real, find p and q. **3**

(b) Show that if the polynomial $f(x) = x^3 + px + q$ has a multiple root, then $4p^3 + 27q^2 = 0$. **3**

(c) The base of a solid is the region in the first quadrant bounded by the curve $y = \sin x$, the x-axis and the line $x = \dfrac{\pi}{2}$. **3**

Find the volume of the solid if every cross-section perpendicular to the base and the $x-$ axis is a square.

(d) (i) Find the five roots of the equation $z^5 = 1$. Give the roots in modulus-argument form. **2**

(ii) Show that $z^5 - 1$ can be factorised in the form :

$$z^5 - 1 = (z-1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$$ **2**

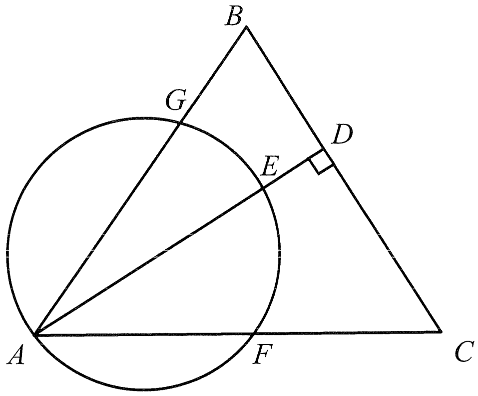
(iii) Hence show that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$. **2**

Marks

QUESTION 5 (15 marks)
Start a new writing booklet.

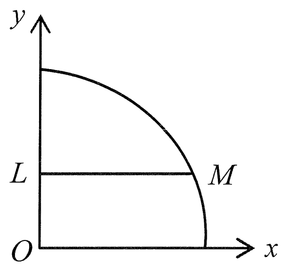
(a) The ellipse $(x-1)^2+\dfrac{y^2}{4}=1$ is rotated about the $y-$axis.
Use the method of slicing to find the volume of the solid formed by the rotation. **4**

(b) In the triangle ABC, AD is the perpendicular from A to BC. E is any point on AD and the circle drawn with AE as diameter cuts AC at F and AB at G. **4**



Prove B, G, F and C are concyclic.

(c) The diagram below shows the part of the circle $x^2+y^2=a^2$ in the first quadrant.



(i) If the horizontal line LM through $L(0,b)$, where $0<b<a$, divides the area between the curve and the coordinates axes into two equal parts, show that

$$\sin^{-1}\frac{b}{a}+\frac{b\sqrt{a^2-b^2}}{a^2}=\frac{\pi}{4}.$$ **3**

(ii) If the radius of the circle is 1 unit, show that b can be found by solving the equation
$\sin 2\theta=\dfrac{\pi}{2}-2\theta$, where $\theta=\sin^{-1}b$. **3**

(iii) Without attempting to solve the equation, how could θ (and hence b) be approximated? **1**

Marks

QUESTION 6 (15 marks)
Start a new writing booklet.

(a) An ellipse has equation $\frac{x^2}{4}+\frac{y^2}{3}=1$ with vertices $A(2,0)$ and $A'(-2,0)$. P is a point (x_1, y_1) on the ellipse.

(i) Find its eccentricity, coordinates of its foci, S and S', and the equations of its directrices. **3**

(ii) Prove that the sum of the distances SP and $S'P$ is independent of the position of P. **3**

(iii) Show that the equation of the tangent to the ellipse at P is $\frac{xx_1}{4}+\frac{yy_1}{3}=1$. **3**

(iv) The tangent at $P(x_1, y_1)$ meets the directrix at T. Prove that angle PST is a right angle. **3**

(b) If $a+b+c=1$,

(i) Prove $a^2+b^2 \geq 2ab$. **1**

(ii) Prove $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$. **2**

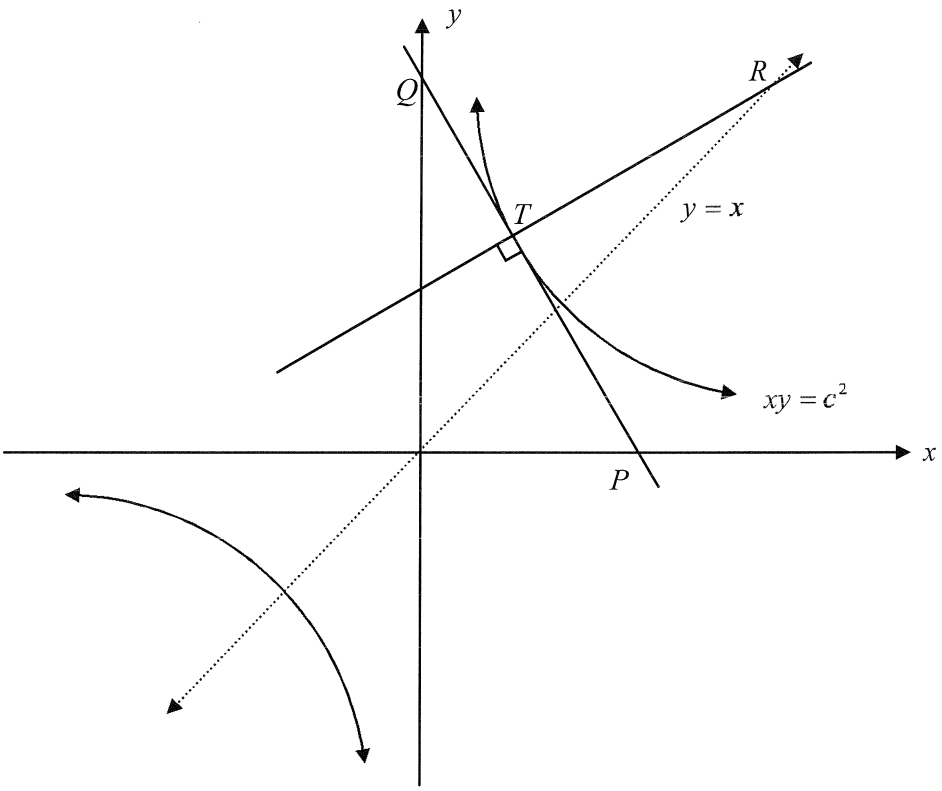
Marks

QUESTION 7 (15 marks)
Start a new writing booklet.

(a) The point $T(ct, \frac{c}{t})$ lies on the hyperbola $xy = c^2$.

The tangent at T meets the $x-$axis at P and the $y-$axis at Q.

The normal at T meets the line $y = x$ at R.



not to scale

You may assume that the tangent at T has equation $x + t^2 y = 2ct$.

(i) Find the coordinates of P and Q. **2**

(ii) Find the equation of the normal at T. **2**

(iii) Show that the $x-$coordinate of R is $x = \frac{c}{t}(t^2 + 1)$. **2**

(iv) Prove that ΔPQR is isosceles. **3**

(b) (i) If $I_n = \int \frac{dx}{\left(x^2+1\right)^n}$ prove that $I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right]$. **4**

(ii) Hence evaluate $\int_0^1 \frac{dx}{\left(x^2+1\right)^2}$. **2**

QUESTION 8 (15 marks) **Marks**
Start a new writing booklet.

(a) A plane of mass M kg on landing, experiences a variable resistive force due to air resistance of magnitude Bv^2 newtons, where v is the speed of the plane. That is, $M\ddot{x} = -Bv^2$.

(i) Show that the distance (D_1) travelled in slowing the plane from speed V to speed U under the effect of air resistance only, is given by: **4**

$$D_1 = \frac{M}{B}\ln\left(\frac{V}{U}\right)$$

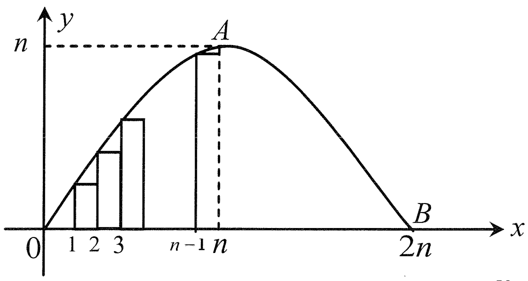
After the brakes are applied, the plane experiences a constant resistive force of A Newtons (due to brakes) as well as a variable resistive force, Bv^2. That is, $M\ddot{x} = -(A + Bv^2)$.

(ii) After the brakes are applied when the plane is travelling at speed U, show that the distance D_2 required to come to rest is given by: **4**

$$D_2 = \frac{M}{2B}\ln\left[1 + \frac{B}{A}U^2\right].$$

(iii) Use the above information to estimate the total stopping distance after landing, for a 100 tonne plane if it slows from 90 m/s to 60 m/s under a resistive force of $125v^2$ Newtons and is finally brought to rest with the assistance of a constant braking force of magnitude 75 000 Newtons. (Note: 1 Newton (N) = 1 kg. m/s^2) **2**

(b)



The diagram above represents the curve $y = n\sin\frac{\pi x}{2n}$, $0 \le x \le 2n$, where n is any integer $n \ge 2$.

The points $O(0,0)$, $A(n,n)$ and $B(2n,0)$ lie on this curve.

(i) By considering the areas of the lower rectangles of width 1 from $x = 0$ to $x = n$, prove that **3**

$$\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \dots + \sin\frac{\pi(n-1)}{2n} < \frac{2n}{\pi}.$$

(ii) Hence or otherwise, explain why $2n\sum_{r=1}^{n-1}\sin\frac{\pi r}{2n} < \frac{\pi n^2}{2}$. **2**

END OF PAPER

SOLUTIONS & MARKING SCHEME
Ext2 TRIAL HSC ASCHAM 2011

Question 1

a) $\int \sin^3\theta \, d\theta = \int \sin\theta (\sin^2\theta) \, d\theta$

$= \int \sin\theta (1 - \cos^2\theta) \, d\theta$

$= \int \sin\theta \, d\theta + \int \cos^2\theta (-\sin\theta) \, d\theta$

$= -\cos\theta + \dfrac{\cos^3\theta}{3} + C$

$\overset{1}{\checkmark}$ $\overset{1}{\checkmark}$

by substitution: where $u = \cos\theta$

$\dfrac{du}{d\theta} = -\sin\theta$

$\int \cos^2\theta (-\sin\theta) \, d\theta = \int u^2 \, du$

$= \dfrac{u^3}{3} = \dfrac{\cos^3\theta}{3}$

b) (i) $3x + 1 = a(x^2+1) + (bx + c)(x+1)$

$= ax^2 + a + bx^2 + (b+c)x + c$

$= (a+b)x^2 + (b+c)x + a + c$

$\therefore a + b = 0$ ①

$b + c = 3$ ②

$a + c = 1$ ③

from ①: $a = -b$ ④

Sub ④ into ③:

$$\begin{cases} -b + c = 1 & ⑤ \\ b + c = 3 & ② \end{cases}$$

② + ⑤: $2c = 4$

$\therefore c = 2$ ⑥

Sub ⑥ into ②: $a + 2 = 1 \Rightarrow a = -1$

$b = 1$

$\therefore \dfrac{3x+1}{(x+1)(x^2+1)} = \dfrac{-1}{x+1} + \dfrac{x+2}{x^2+1}$

$\overset{1}{\checkmark}$ $\overset{1}{\checkmark}$

(ii) $\int \frac{3x+1}{(x+1)(x^2+1)}\,dx = \int \frac{-1}{x+1}\,dx + \int \frac{x+2}{x^2+1}\,dx$

$= -\ln(x+1) + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx$

$= -\ln(x+1) + \frac{1}{2}\ln(x^2+1) + 2\tan^{-1}x + C$

1 ✓ 1 ✓

(c) $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}} \times 2\cos\theta\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\sin^2\theta\,d\theta$ 1 ✓

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2(2\sin^2\theta)\,d\theta$

since $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

when $x = 1$,

$1 = 2\sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

when $x = \sqrt{3}$

$\sqrt{3} = 2\sin\theta$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \theta = \frac{\pi}{3}$

$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1-\cos 2\theta)\,d\theta$ since $\cos 2\theta = 1 - 2\sin^2\theta$

1 ✓

$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2\left[\left(\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}\right) - \left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right)\right]$

$= 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$ 1 ✓

(d) $\int x^2 \sqrt{3-x}\, dx$ | let $u = 3-x$, $\therefore x = 3-u$ and $dx = -du$

$= \int (3-u)^2 \sqrt{u}\,(-du)$

$= -\int (9-6u+u^2)\sqrt{u}\, du$

$= -\int (9u^{1/2} - 6u^{3/2} + u^{5/2})\, du$

$= -\dfrac{9u^{3/2}}{(3/2)} + \dfrac{6u^{5/2}}{(5/2)} - \dfrac{u^{7/2}}{(7/2)} + c$

$= -6\sqrt{(3-x)^3} + \frac{12}{5}\sqrt{(3-x)^5} - \frac{2}{7}\sqrt{(3-x)^7} + c$

1 ✓ 1 ✓ 1 ✓

(e) let $\int_0^1 1.\tan^{-1}\theta\, d\theta = \int_0^1 u\, dv$

where $u = \tan^{-1}\theta$, $v = \theta$
$du = \dfrac{1}{1+\theta^2} d\theta$, $dv = 1 d\theta$

$\int_0^1 u\,dv = [uv]_0^1 - \int_0^1 v\,du$

$= [\theta . \tan^{-1}\theta]_0^1 - \int_0^1 \dfrac{\theta}{1+\theta^2} d\theta$ 1 ✓

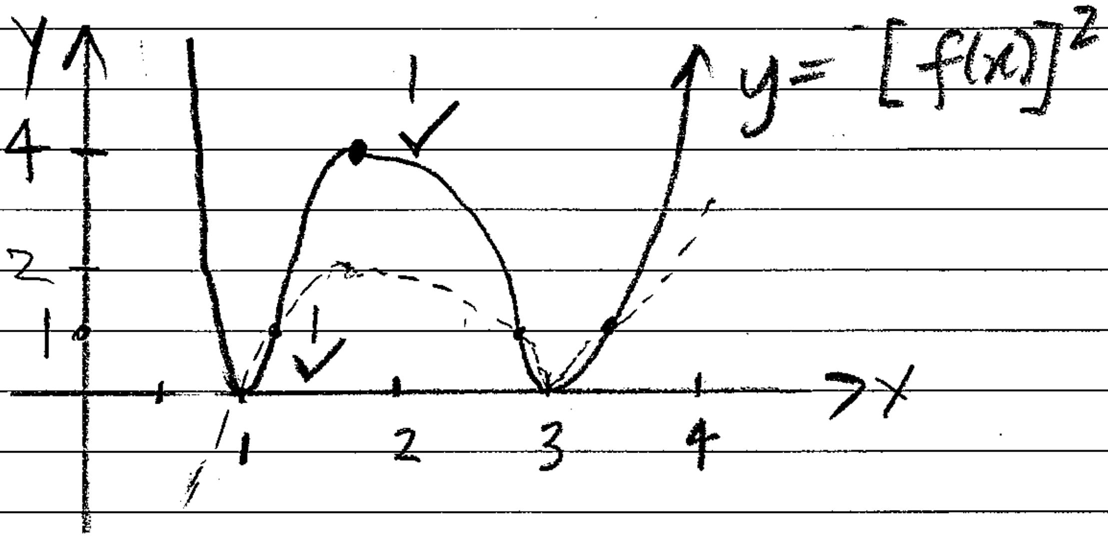
$= \tan^{-1} 1 - \left[\frac{1}{2}\ln(1+\theta^2)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

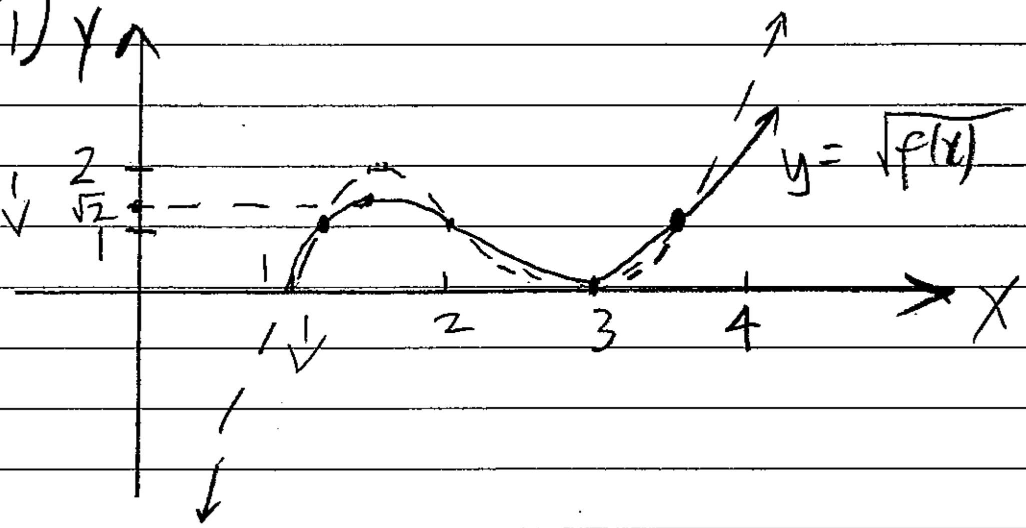
1 ✓ 1 ✓

Question 2

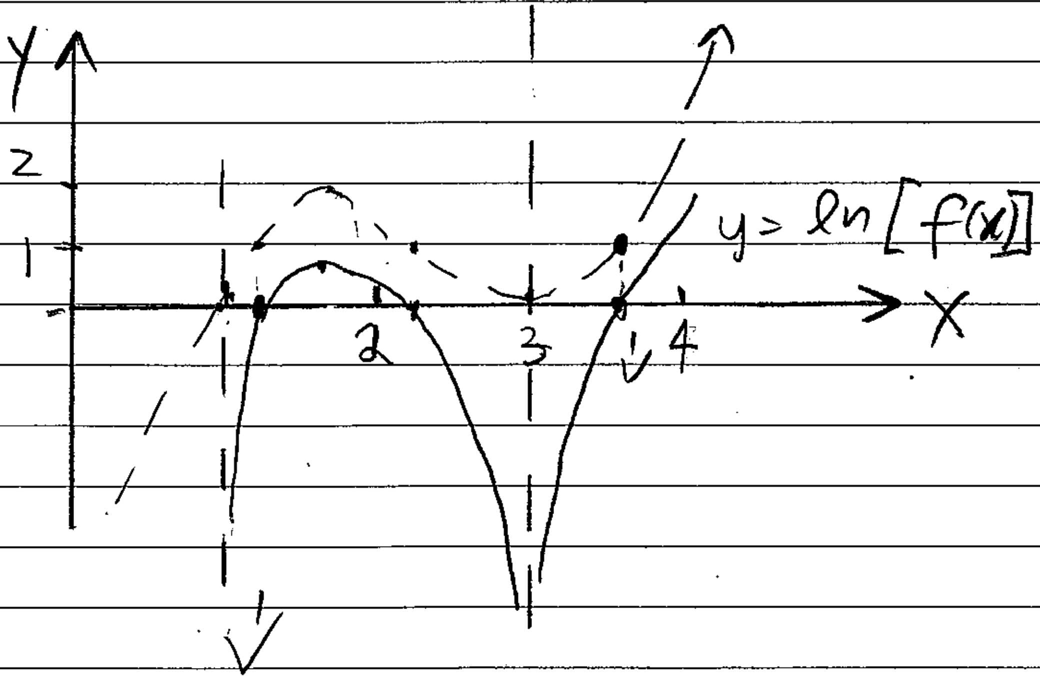
i)



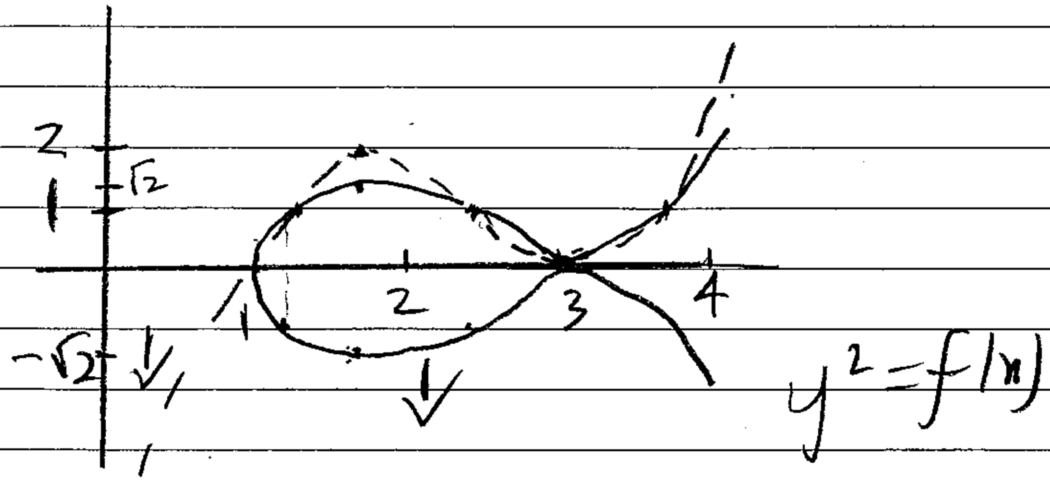
(ii)



(iii)



(iv)



(b) (i) $f'(x) = \dfrac{2-x}{x^2} = 2x^{-2} - x^{-1}$

$f''(x) = -4x^{-3} + x^{-2}$ ✓

$= -\dfrac{4}{x^3} + \dfrac{x}{x^3} = \dfrac{x-4}{x^3}$

$f(x) = \int \left(2x^{-2} - \dfrac{1}{x}\right) dx$

$= \dfrac{2x^{-1}}{-1} - \ln x + C$

$= -\dfrac{2}{x} - \ln x + c$ ✓

$f(1) = -\dfrac{2}{1} - \ln(1) + C = 0$

$= -2 + C = 0 \Rightarrow C = 2$ ✓

$f(x) = -\dfrac{2}{x} - \ln x + 2$

(ii) $f(x)$ has a turning point when $f'(x) = 0$

$f'(x) = 0$ when $\frac{2-x}{x^2} = 0$

i.e. when $x = 2$

this is the only point when $f(x)$ is stationary since it is the only point where the gradient is zero.

This means that there is only one turning point.

when $x = 2$, $f(x) = -\frac{2}{2} - \ln 2 + 2$

$f(2) = 1 - \ln 2 = 0.30685\ldots$

$f''(2) = \frac{2-4}{2^3} = -\frac{1}{4} < 0$

$\therefore f(x)$ is concave down at $x = 2$

hence $(2, 1-\ln 2)$ is a maximum turning point. ✓

(iii) $f(4) = -\frac{2}{4} - \ln 4 + 2$

$= 1\frac{1}{2} - \ln 4$

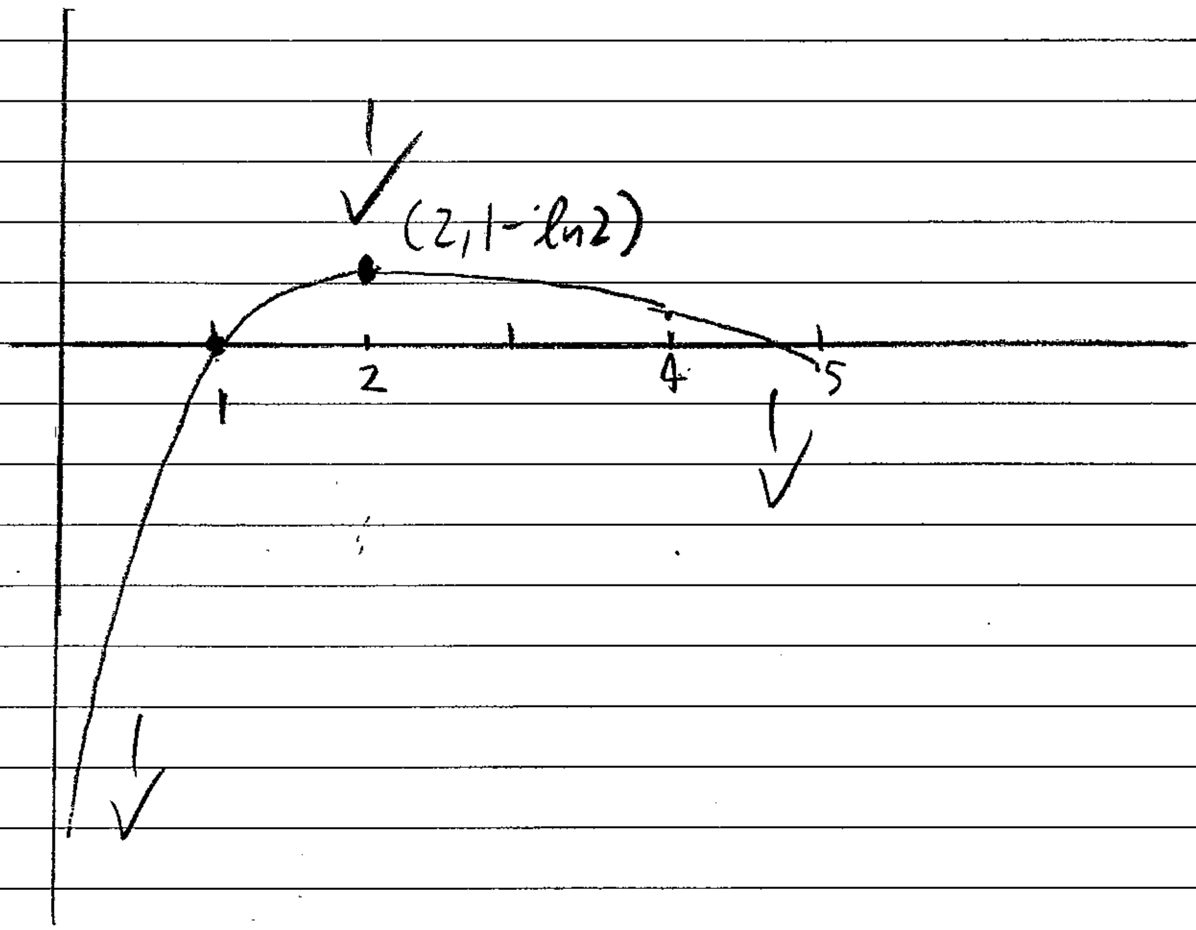
$= 0.11371\ldots > 0$

$f(5) = -\frac{2}{5} - \ln 5 + 2$

$= 1\frac{3}{5} - \ln 5$

$= -0.009438\ldots < 0$ ✓

f(x) is only defined for $x > 0$
since $\ln(x)$ is only defined for $x > 0$.



Question 3

(a) $(\sqrt{3}+i)^8 =$

let $z = \sqrt{3} + i$

$|z| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= \sqrt{4} = 2$

$z = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\cos\theta + i\,2\sin\theta$

$\left.\begin{array}{l}\cos\theta = \frac{\sqrt{3}}{2} \\ \sin\theta = \frac{1}{2}\end{array}\right\} \theta = \frac{\pi}{6}$ ✓[1]

$\therefore z = 2\,\mathrm{cis}\,\frac{\pi}{6}$

$(\sqrt{3}+i)^8 = \left(2\,\mathrm{cis}\frac{\pi}{6}\right)^8 = 2^8\,\mathrm{cis}\left(8\times\frac{\pi}{6}\right)$

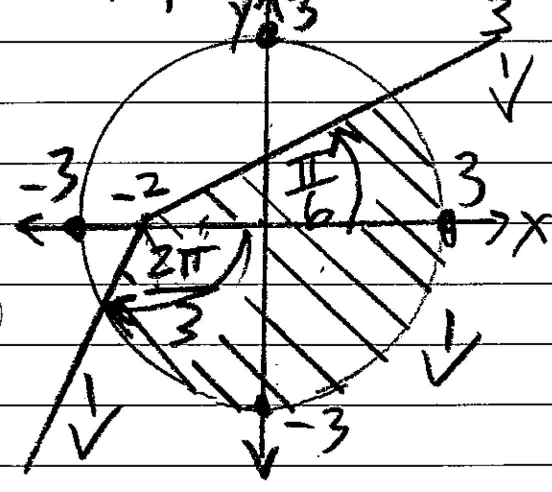
$= 256\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= 256\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= -128 - i\,128\sqrt{3}$

✓[1] ✓[1]

(b) $|z| \leq 3$ $\quad -\frac{2\pi}{3} \leq \arg(z+2) \leq \frac{\pi}{6}$



(c) $\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$

$$LHS = \frac{(1+\sin\theta) + i\cos\theta}{(1+\sin\theta) - i\cos\theta} \times \frac{(1+\sin\theta) + i\cos\theta}{(1+\sin\theta) + i\cos\theta}$$ 1 ✓

$$= \frac{(1+\sin\theta)^2 - \cos^2\theta + 2i(1+\sin\theta)\cos\theta}{(1+\sin\theta)^2 + \cos^2\theta}$$ 1 ✓

$$= \frac{(1+\sin\theta)^2 - (1-\sin^2\theta) + 2(1+\sin\theta)\,i\cos\theta}{(1+\sin\theta)^2 + (1-\sin^2\theta)}$$

$$= \frac{(1+\sin\theta)^2 - (1+\sin\theta)(1-\sin\theta) + (1+\sin\theta)\,2i\cos\theta}{(1+\sin\theta)^2 + (1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{(1+\sin\theta)\big((1+\sin\theta) - (1-\sin\theta) + 2i\cos\theta\big)}{(1+\sin\theta)\big((1+\sin\theta) + (1-\sin\theta)\big)}$$

$$= \frac{2\sin\theta + 2i\cos\theta}{2} = \sin\theta + i\cos\theta = RHS$$

1 ✓

(d) (i) $z = \dfrac{-1+i}{\sqrt{3}+i}$

let $z_1 = -1+i$, $r_1 = \sqrt{(-1)^2+1^2} = \sqrt{2}$, $\left.\begin{array}{l}\cos\theta_1 = -\frac{1}{\sqrt{2}}\\ \sin\theta_1 = \frac{1}{\sqrt{2}}\end{array}\right\}\theta_1 = \frac{3\pi}{4}$

$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= \sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$

let $z_2 = \sqrt{3}+i$, $r_2 = \sqrt{(\sqrt{3})^2+1^2} = 2$, $\left.\begin{array}{l}\cos\theta_2 = \frac{\sqrt{3}}{2}\\ \sin\theta_2 = \frac{1}{2}\end{array}\right\}\theta_2 = \frac{\pi}{6}$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= 2\,\mathrm{cis}\,\frac{\pi}{6}$

$$z = \frac{z_1}{z_2} = \frac{\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)}{2\operatorname{cis}\left(\frac{\pi}{6}\right)}$$

$$= \frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{3\pi}{4} - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{9\pi}{12} - \frac{2\pi}{12}\right)$$

$$= \frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{7\pi}{12}\right)$$

1 ✓ 1 ✓

(ii) $$z = \frac{-1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{-\sqrt{3}+1}{(3+1)} + \frac{(\sqrt{3}-1)i}{(3+1)}$$

$$= \left(\frac{1-\sqrt{3}}{4}\right) + \left(\frac{\sqrt{3}-1}{4}\right)i$$ 1 ✓

equating real parts from (i)

$$\frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{12}\right) = \frac{1-\sqrt{3}}{4}$$

$$\therefore \cos\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$$ 1 ✓

(e) From the diagram, $z_2 = iz_1$ (1) 1 ✓

since multiplying z_1 by i rotates it 90° in an anticlockwise direction.

$$\therefore (z_1 + z_2)^2 = (z_1 + iz_1)^2 \quad \text{(from (1))}$$

$$= z_1^2(1+i)^2 = z_1^2(1+2i-1)$$

$$= 2iz_1^2 = 2z_1(iz_1)$$ 1 ✓

$$= 2z_1z_2 \quad \text{(from (1))}$$

as required.

Question 4

(a) If $z_1 = 1+i$ is a root, then
$z_2 = \bar{z}_1 = 1-i$ is also a root.

$z_1 z_2 z_3 = -6$ (product of roots)

$(1+i)(1-i)z_3 = -6$
$2z_3 = -6$ $\quad z_3 = -3$ 1 ✓

$z_1 z_2 + z_1 z_3 + z_2 z_3 = q$ (sum product of roots)

$(1+i)(1-i) + (1+i)(-3) + (1-i)(-3) = q$

$\therefore q = 2 - 3 - 3 - 3i + 3i$
$\underline{q = -4}$ 1 ✓

$z_1 + z_2 + z_3 = -p$ (sum of roots)

$(1+i) + (1-i) + (-3) = -p$
$-1 = -p$

$\underline{\therefore p = 1}$ 1 ✓

(b) If $f(x)$ has a multiple root α
then $f'(\alpha)$ has the same root α.
ie $f(\alpha) = 0$ and $f'(\alpha) = 0$

$f'(x) = 3x^2 + p \Rightarrow f'(\alpha) = 3\alpha^2 + p = 0$
$\therefore p = -3\alpha^2$
and $\alpha^2 = \frac{p}{-3}$ (1) 1 ✓

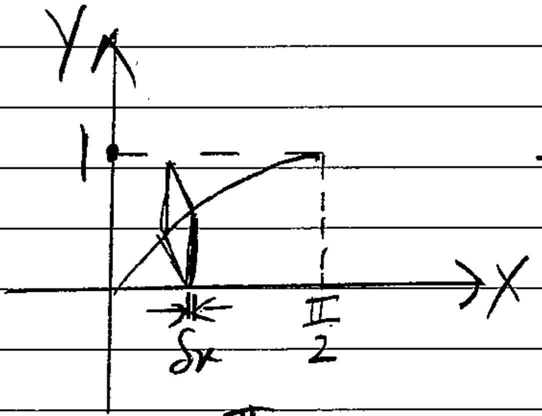
$f(x) = x^3 + px + q$
$= x(x^2 + p) + q$

$$f(\alpha) = \alpha(\alpha^2 + p) + q$$
$$= \alpha\left(\frac{p}{3} + p\right) + q \quad (\text{from } ①)$$
$$= \left(\frac{2p}{3}\right)\alpha + q = 0 \quad \checkmark$$
$$\therefore 2p\alpha + 3q = 0$$
$$\alpha = -\frac{3q}{2p}$$
$$\alpha^2 = \frac{9q^2}{4p^2}$$

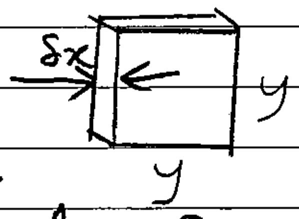
$$\text{So } \left(-\frac{p}{3}\right) = \frac{9q^2}{4p^2} \quad (\text{from } ①)$$

$$-4p^3 = 27q^2 \ \checkmark \quad \text{or} \quad 4p^3 + 27q^2 = 0$$
as required.

(c)



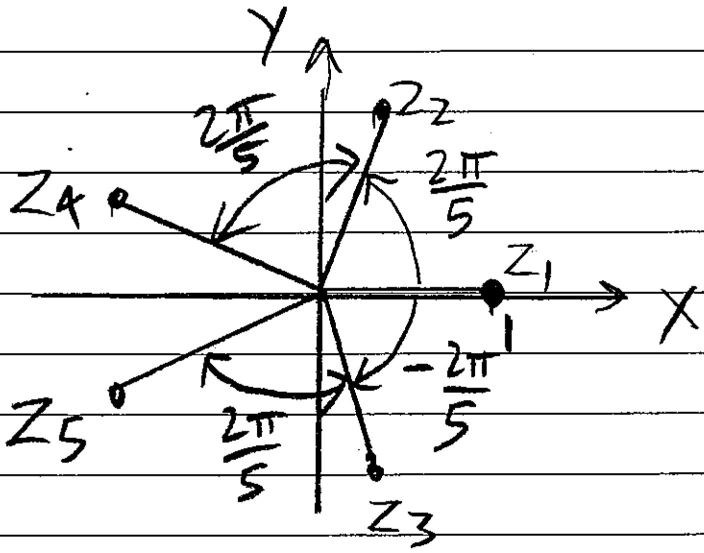
Cross-section



$$A = y^2 = \sin^2 x$$

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{\frac{\pi}{2}} (\sin^2 x)\,\delta x$$
$$= \int_0^{\frac{\pi}{2}} \sin^2 x\, dx \ \checkmark = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2x)\, dx \quad \left(\begin{array}{l}\text{since} \\ \cos 2x = 1 - 2\sin^2 x\end{array}\right)$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}} \checkmark = \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin \pi\right]$$
$$= \frac{\pi}{4} \ u^3. \ \checkmark$$

(d) (i) $z^5 = 1$



roots will be $z_1 = 1$

$$z_2 = 1 \operatorname{cis} \frac{2\pi}{5}$$

$$z_3 = 1 \operatorname{cis}\left(-\frac{2\pi}{5}\right)$$

$$z_4 = 1 \operatorname{cis}\left(\frac{4\pi}{5}\right)$$

$$z_5 = 1 \operatorname{cis}\left(-\frac{4\pi}{5}\right)$$

$$z^5 - 1 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)$$

$$= (z-1)\left(z - \operatorname{cis}\frac{2\pi}{5}\right)\left(z - \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)\left(z - \operatorname{cis}\left(\frac{4\pi}{5}\right)\right) \times \left(z - \operatorname{cis}\left(-\frac{4\pi}{5}\right)\right)$$

now $\left(z - \operatorname{cis}\left(\frac{2\pi}{5}\right)\right)\left(z - \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)$

$$= \left(z^2 - z\left(\operatorname{cis}\frac{2\pi}{5} + \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right) + \operatorname{cis}\left(\frac{2\pi}{5}\right)\operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)$$

$$= \left(z^2 - z\left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)\right) + \left(\cos\left(\frac{2\pi}{5}\right)\cos\left(-\frac{2\pi}{5}\right) + i^2\left(\sin\left(\frac{2\pi}{5}\right)\right)\left(\sin\left(-\frac{2\pi}{5}\right)\right)\right)\right)$$

$$= z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right) + \sin^2\left(\frac{2\pi}{5}\right)$$

$$= z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + 1$$

since $\cos\left(-\frac{2\pi}{5}\right) = \cos\frac{2\pi}{5}$ and $\sin\left(-\frac{2\pi}{5}\right) = -\sin\left(\frac{2\pi}{5}\right)$

similarly $\left(z - \text{cis}\left(\frac{4\pi}{5}\right)\right)\left(z - \text{cis}\left(-\frac{4\pi}{5}\right)\right)$

$$= z^2 - 2z\,\text{cis}\left(\frac{4\pi}{5}\right) + 1$$

so $z^5 - 1 = (z-1)\left(z^2 - 2z\,\text{cis}\left(\frac{2\pi}{5}\right) + 1\right)\left(z^2 - 2z\,\text{cis}\left(\frac{4\pi}{5}\right) + 1\right)$ (1)

as required.

(iii) in the expansion of (1) above equate coefficients of z from both sides:

$LHS = 0z$

$RHS = z + 2z\,\text{cis}\left(\frac{2\pi}{5}\right) + 2z\,\text{cis}\left(\frac{4\pi}{5}\right)$

$= z\left(1 + 2\,\text{cis}\left(\frac{2\pi}{5}\right) + 2\,\text{cis}\left(\frac{4\pi}{5}\right)\right)$

$\therefore 1 + 2\,\text{cis}\,\frac{2\pi}{5} + 2\,\text{cis}\left(\frac{4\pi}{5}\right) = 0$ ✓

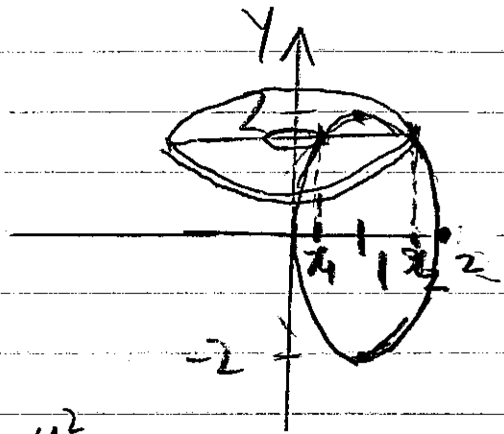
$2\,\text{cis}\,\frac{2\pi}{5} + 2\,\text{cis}\,\frac{4\pi}{5} = -1$

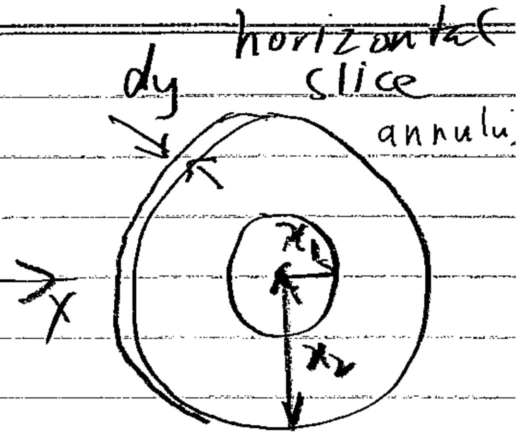
$\text{cis}\,\frac{2\pi}{5} + \text{cis}\,\frac{4\pi}{5} = -\frac{1}{2}$

as required.

Question 5

(a)





$(x-1)^2 = 1 - \frac{y^2}{4}$

$x - 1 = \pm\sqrt{1 - \frac{y^2}{4}}$

$\delta V = \pi(x_2^2 - x_1^2)\delta y$

$\delta V = \pi(x_2 + x_1)(x_2 - x_1)\delta y$

$x_1 = 1 - \sqrt{1 - \frac{y^2}{4}}$

$x_2 = 1 + \sqrt{1 - \frac{y^2}{4}}$ 1 ✓

$x_1 + x_2 = 2$

$x_2 - x_1 = 2\sqrt{1 - \frac{y^2}{4}}$

$\therefore \delta V = \pi(2)\left(2\sqrt{1 - \frac{y^2}{4}}\right)\delta y$ 1 ✓

$V = \lim_{\delta y \to 0} \sum_{y=-2}^{2} \delta V$

$= 2\pi \int_0^2 4\sqrt{1 - \frac{y^2}{4}}\, dy = 4\pi \int_0^2 \sqrt{4 - y^2}\, dy$ 1 ✓

let $y = 2\sin\theta$ $\frac{dy}{d\theta} = 2\cos\theta$ $\therefore dy = 2\cos\theta\, d\theta$

when $y = 0$, $\theta = 0$ when $y = 2$, $\theta = \frac{\pi}{2}$

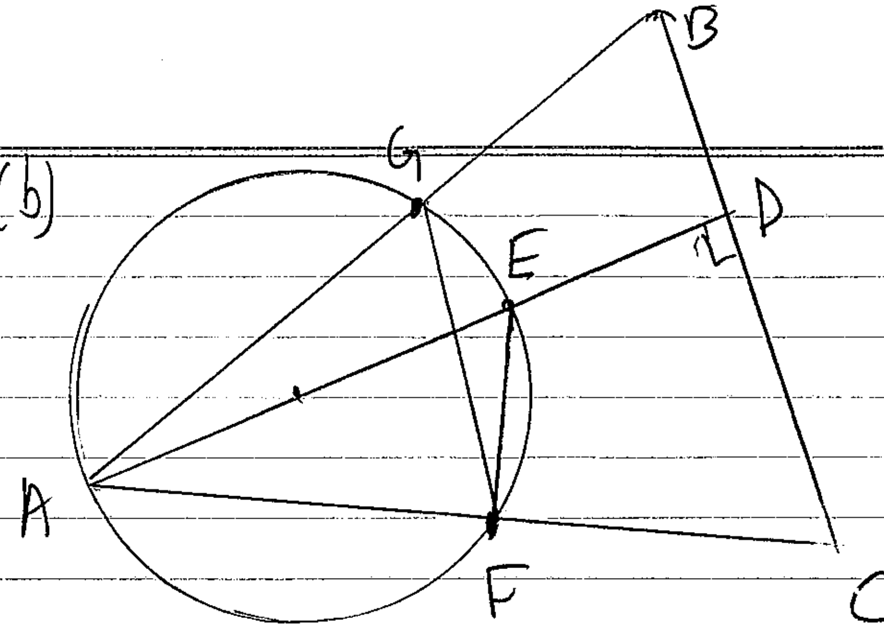
$V = 4\pi \int_0^{\frac{\pi}{2}} \sqrt{4 - (2\sin\theta)^2}\,(2\cos\theta\, d\theta)$

$= 4\pi \int_0^{\frac{\pi}{2}} (2\cos\theta)^2\, d\theta = 8\pi \int_0^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$

$= 8\pi \int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)\, d\theta = 8\pi\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{2}}$

$= 4\pi^2 \ u^3$ 1 ✓

(b)



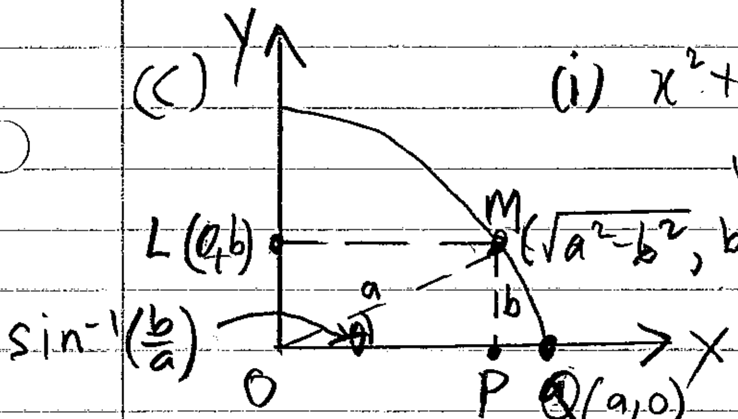
$\angle AEF = 90^\circ$ ($\angle$ in a semicircle)
$\angle AEF = \angle EDC$ hence EDFC is a cyclic quadrilateral since the external $\angle$ = internal opposite $\angle$. ✓

In EDFC, $\angle AEF = \angle DCF$ (external $\angle$ = opposite internal $\angle$)
$= \angle BCF$ ✓

Also $\angle AGF = \angle AEF$ ($\angle$s in the same segment) ✓

So $\angle BCF = \angle AGF$ and GBFC is cyclic since the exterior $\angle$ = interior opposite. ✓

(c)



(i) $x^2 + y^2 = a^2$
when $y = b$, $x = \sqrt{(a^2-b^2)}$

area of LMQO
$= \frac{1}{2}\left(\frac{\pi r^2}{4}\right)$
$= \frac{1}{2}\left(\frac{\pi a^2}{4}\right)$

area of LMQO = area of LMPO + area of $\frac{1}{2}$ segment MQP

$\frac{1}{2}\left(\frac{\pi a^2}{4}\right) = b\sqrt{a^2-b^2} + \frac{1}{2}\left(\frac{1}{2}r^2(2\theta - \sin 2\theta)\right)$ ✓

$a^2\left(\frac{\pi}{4}\right) = 2b\sqrt{a^2-b^2} + \frac{1}{2}a^2\left(2\sin^{-1}\left(\frac{b}{a}\right) - 2\sin\left(\sin^{-1}\frac{b}{a}\right) \times \cos\left(\sin^{-1}\frac{b}{a}\right)\right)$ ✓

$\frac{\pi}{4} = \frac{2b\sqrt{a^2-b^2}}{a^2} + \sin^{-1}\left(\frac{b}{a}\right) - \frac{b}{a}\left(\frac{\sqrt{a^2-b^2}}{a}\right)$ ✓

$= \frac{b\sqrt{a^2-b^2}}{a^2} + \sin^{-1}\left(\frac{b}{a}\right)$ as required.

(ii) $a = 1$ (1), $\theta = \sin^{-1}(b)$ (2)

hence $b = \sin\theta$ (3)

sub (1), (2) and (3) into part (i):

$$\sin^{-1}\left(\frac{b}{1}\right) + \frac{b\sqrt{1^2 - b^2}}{1^2} = \frac{\pi}{4}$$ 1 ✓

$$\theta + \sin\theta\sqrt{1 - \sin^2\theta} = \frac{\pi}{4}$$ 1 ✓

$$\theta + \sin\theta\cos\theta = \frac{\pi}{4}$$ 1 ✓

$$2\theta + 2\sin\theta\cos\theta = \frac{\pi}{2}$$

$$2\theta + \sin 2\theta = \frac{\pi}{2}$$ as required.

(iii) use Newton's method to approximate it (or halving the interval) 1 ✓

Question 6

(a) (i) $\frac{x^2}{4}+\frac{y^2}{3}=1=\frac{x^2}{2^2}+\frac{y^2}{(\sqrt{3})^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

$b^2=a^2(1-e^2) \Rightarrow 1-\frac{b^2}{a^2}=e^2$

$1-\frac{3}{4}=\frac{1}{4}=e^2=\left(\frac{1}{2}\right)^2 \Rightarrow e=\frac{1}{2}$ 1✓

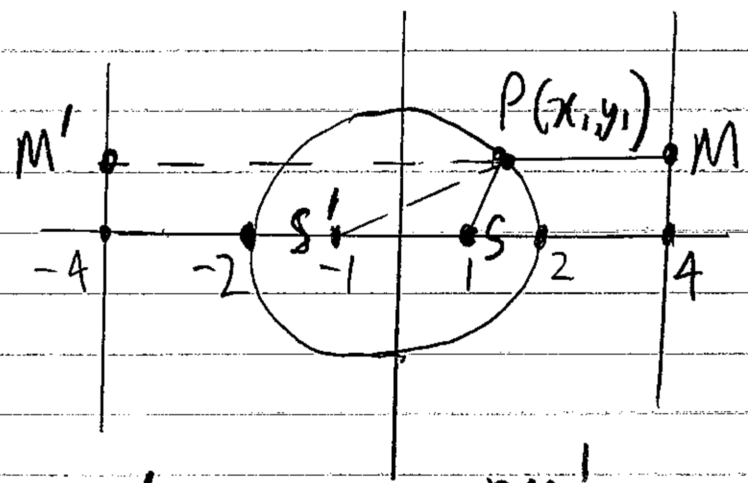
$S(ae,0)=S(2\times\frac{1}{2},0)=S(1,0)$ 1✓

$S'(-ae,0)=S'(-1,0)$

$d: \ x=\frac{a}{e}=\frac{2}{\frac{1}{2}}=4$ so $\underline{x=4}$

$d': \ \underline{x=-4}$ 1✓

(ii)



$PS=ePM$

$PS'=ePM'$

$PS+PS'=ePM+ePM'$ 1✓

$=e(PM+PM')$ 1✓

$=e(MM')$

$=\frac{1}{2}\times 8=4$, 1✓ (independent of $P(x_1,y_1)$)

as required.

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \quad (1)$$

(iii) $\frac{d}{dx}\left(\frac{x^2}{4}\right)+\frac{d}{dx}\left(\frac{y^2}{3}\right)=\frac{d}{dx}(1)$

$$\frac{2x}{4}+\frac{2y}{3}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{2x}{4}\times\frac{3}{2y}=\frac{-3x}{4y}$$

at (x_1, y_1) $m=\frac{-3x_1}{4y_1}$

so tangent has eqn: $y-y_1=m(x-x_1)$

$$y-y_1=\frac{-3x_1}{4y_1}(x-x_1)$$

$$4y_1y-4y_1^2=-3x_1x+3x_1^2$$

$$\frac{3x_1x}{12}+\frac{4y_1y}{12}=\frac{3x_1^2}{12}+\frac{4y_1^2}{12}$$

$$\frac{x_1x}{4}+\frac{y_1y}{3}=\frac{x_1^2}{4}+\frac{y_1^2}{3}=1$$

as required. (from (1))

(iv) $P(x_1, y_1)$ $S(1,0)$

for T: sub $x=4$ into eqn. of tangent

$$\frac{4x_1}{4}+\frac{y_1y}{3}=1 \Rightarrow y=\frac{3(1-x_1)}{y_1}$$

$T\left(4, \frac{3}{y_1}(1-x_1)\right)$ ✓

$m_{PS}=\frac{y_1}{x_1-1}$ ✓ $m_{ST}=\frac{\frac{3}{y_1}(1-x_1)}{(4-1)}=\frac{1-x_1}{y_1}$ ✓

$$m_{PS}\times m_{ST}=\frac{y_1}{-(1-x_1)}\times\frac{(1-x_1)}{y_1}=-1$$

$\therefore \angle PST=90^\circ$ as required.

(b) (i) for all a, b

$$(a-b)^2 \geqslant 0$$
$$a^2 - 2ab + b^2 \geqslant 0$$
$$a^2 + b^2 \geqslant 2ab \quad \text{as required.}$$

(ii) $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ac}{abc}$$
$$= \frac{(ab+bc+ac)(a+b+c)}{abc}$$
$$= \frac{(a^2b + ab^2 + abc + abc + b^2c + bc^2 + a^2c + abc + ac^2)}{abc}$$
$$= \frac{(3abc + a(b^2+c^2) + b(a^2+c^2) + c(b^2+a^2))}{abc}$$
$$\geqslant \frac{3abc + a(2bc) + b(2ac) + c(2ba)}{abc}$$
$$\geqslant \frac{9abc}{abc}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geqslant 9$$

Question 7

(a)(i) $x + t^2 y = 2ct$ or $y = -\frac{1}{t^2}x + \frac{2c}{t}$

at Q: when $x = 0$, $t^2 y = 2ct$ $\therefore Q\left(0, \frac{2c}{t}\right)$ 1 ✓

$y = \frac{2c}{t}$

at P: when $y = 0$, $x = 2ct$ $\therefore P(2ct, 0)$ 1 ✓

(ii) $m_{tangent} = -\frac{1}{t^2} \Rightarrow m_{normal} = t^2$ 1 ✓

$T\left(ct, \frac{c}{t}\right)$

eqn. of normal: $y - \frac{c}{t} = m_{normal}(x - ct)$

$y - \frac{c}{t} = t^2(x - ct)$

$ty - c = t^3x - ct^4$ 1 ✓

$t^3x - ty = ct^4 - c$

(iii) for R: solve $t^3x - ty = ct^4 - c$ ①

and $y = x$ ②

simultaneously

$t^3x - tx = c(t^4 - 1)$ 1 ✓

$x(t^3 - t) = c(t^2 - 1)(t^2 + 1)$

$x = \frac{c(t^2-1)(t^2+1)}{t(t^2-1)}$ 1 ✓

$x = \frac{c}{t}(t^2+1)$ as required.

(b) (i) $I_n = \int \frac{1\,dx}{(x^2+1)^n} = \int u\,dv$

where $u = (x^2+1)^{-n}$ $\quad v = x$

$\frac{du}{dx} = -2nx(x^2+1)^{-n-1}$ $\quad \frac{dv}{dx} = 1$

$$I_n = uv - \int v\,du$$

$$= x(x^2+1)^{-n} - \int x(-2nx)(x^2+1)^{-(n+1)}dx$$

$$= \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2}{(x^2+1)^{(n+1)}}dx$$

$$= \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2+1-1}{(x^2+1)^{n+1}}dx$$

$$I_n = \frac{x}{(x^2+1)^n} + 2n\int \frac{dx}{(x^2+1)^n} - 2n\int \frac{dx}{(x^2+1)^{n+1}}$$

$$I_n = \frac{x}{(x^2+1)^n} + 2n\,I_n - 2n\,I_{n+1}$$

$$2n\,I_{n+1} = \frac{x}{(x^2+1)^n} + I_n(2n-1)$$

replace n with $n-1$

$$2(n-1)\,I_n = \frac{x}{(x^2+1)^{n-1}} + I_{n-1}(2(n-1)-1)$$

$$I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3)\,I_{n-1}\right]$$

as required.

(iv) $R\left(\frac{c}{t}(t^2+1),\ \frac{c}{t}(t^2+1)\right)$ ✓

$Q\left(0, \frac{2c}{t}\right) \qquad P(2ct, 0)$

$$QR^2 = \left(\frac{c}{t}(t^2+1) - \frac{2c}{t}\right)^2 + \left(\frac{c}{t}(t^2+1)\right)^2$$

$$= \frac{c^2}{t^2}(t^2+1-2)^2 + \frac{c^2}{t^2}(t^2+1)^2$$

$$= \frac{c^2}{t^2}\left((t^2-1)^2 + (t^2+1)^2\right)$$

$$= \frac{c^2}{t^2}\left(t^4-2t^2+1 + t^4+2t^2+1\right)$$

$$= \frac{2c^2}{t^2}(t^4+1) \quad ✓$$

$$PR^2 = \left(\frac{c}{t}(t^2+1) - 2ct\right)^2 + \left(\frac{c}{t}(t^2+1)\right)^2$$

$$= \frac{c^2}{t^2}\left((t^2+1) - 2t^2\right)^2 + \frac{c^2}{t^2}(t^2+1)^2$$

$$= \frac{c^2}{t^2}\left((1-t^2)^2 + (t^2+1)^2\right)$$

$$= \frac{c^2}{t^2}\left(t^4-2t^2+1 + t^4+2t^2+1\right)$$

$$= \frac{c^2}{t^2}(2t^4+2)$$

$$= \frac{2c^2}{t^2}(t^4+1) = QR^2 \quad ✓$$

$\therefore PR = QR$ and $\triangle PQR$ is isosceles.

(ii) $\int_0^1 \frac{dx}{(x^2+1)^2} = \frac{1}{2(2-1)}\left[\left[\frac{x}{x^2+1}\right]_0^1 + (2\times2-3)I_1\right]$

$= \frac{1}{2}\left[\frac{1}{2} + \int_0^1 \frac{1}{x^2+1}dx\right]$ ✓ 1

$= \frac{1}{4} + \frac{1}{2}\left[\tan^{-1}x\right]_0^1$

$= \frac{1}{4} + \frac{1}{2}\left[\frac{\pi}{4}\right]$

$= \frac{1}{4}\left(1+\frac{\pi}{2}\right)$ ✓ 1

Question 8

(a) (i) $M\ddot{x} = -Bv^2$

$\ddot{x} = -\frac{B}{M}v^2$

$v.\frac{dv}{dx} = -\frac{B}{M}v^2$ 1 ✓

$\frac{dx}{dV} = -\frac{M}{BV}$ 1 ✓

$x = -\frac{M}{B}\int_V^U \frac{1}{V}\,dV$ 1 ✓

$= -\frac{M}{B}\Big[\ln V\Big]_V^U$

$= -\frac{M}{B}\Big[\ln(U) - \ln(V)\Big]$

$x = \frac{M}{B}\ln\left(\frac{U}{V}\right)^{-1}$ 1 ✓

$\therefore D_1 = \frac{M}{B}\ln\frac{V}{U}$ as required.

(ii) $M\ddot{x} = -(A+BV^2)$.

$V.\frac{dV}{dx} = \frac{-(A+BV^2)}{M}$

$\frac{dV}{dx} = \frac{-(A+BV^2)}{MV}$

$\frac{dx}{dV} = -\frac{MV}{A+BV^2}$ 1 ✓

$$x = -\frac{M}{2B}\int_{U}^{0} \frac{2BV}{A+BV^2}\,dV$$ 1 ✓

$$= -\frac{M}{2B}\left[\ln(A+BV^2)\right]_{U}^{0}$$

$$= -\frac{m}{2B}\left[\ln(A) - \ln(A+BU^2)\right]$$ 1 ✓

$$= \frac{m}{2B}\ln\left(\frac{A}{A+BU^2}\right)^{-1}$$ 1 ✓

$$= \frac{m}{2B}\ln\left(\frac{A+BU^2}{A}\right) = \frac{M}{2B}\ln\left(1+\frac{B}{A}U^2\right)$$

as required

(iii) $M = 100$ tonnes $= 100\,000$ kg

$V = 90$

$U = 60$

$Bv^2 = 125v^2$ so $B = 125$

$A = 75000$ N

$$D = D_1 + D_2$$

$$= \frac{M}{B}\ln\left(\frac{V}{U}\right) + \frac{M}{2B}\ln\left(1+\frac{B}{A}U^2\right)$$ 1 ✓

$$= \frac{100000}{125}\ln\left(\frac{90}{60}\right) + \frac{100000}{2\times125}\ln\left(1+\frac{125}{75000}\times 60^2\right)$$

$$= 800\ln(3/2) + 400\ln(7)$$

$= 1102.74$ m (6 sig figs). 1 ✓

(b) (i)

x	1	2	3	$n-1$
$y = n\sin\frac{\pi x}{2n}$	$n\sin\frac{\pi}{2n}$	$n\sin\frac{2\pi}{2n}$	$n\sin\frac{3\pi}{2n}$	$n\sin\left(\frac{(n-1)\pi}{2n}\right)$

Areas of rectangles $= 1\times n\sin\left(\frac{\pi}{2n}\right) + 1\times n\sin\left(\frac{2\pi}{2n}\right) + 1\times n\sin\left(\frac{3\pi}{2n}\right) + \dots + 1\times n\sin\left(\frac{(n-1)\pi}{2n}\right)$ 1 ✓

Area under curve $= \int_0^n n.\sin\left(\frac{\pi x}{2n}\right)dx$

$= -\left[n\left(\frac{2n}{\pi}\right)\cos\left(\frac{\pi x}{2n}\right)\right]_0^n$

$= -\frac{2n^2}{\pi}\left[\cos\left(\frac{\cancel{n}\pi}{2\cancel{n}}\right) - \cos(0)\right]$

$= \frac{2n^2}{\pi}$ 1 ✓

Areas of rectangles < Area under curve

$n\sin\left(\frac{\pi}{2n}\right) + n\sin\left(\frac{2\pi}{2n}\right) + n\sin\left(\frac{3\pi}{2n}\right) + \dots + n\sin\left(\frac{(n-1)\pi}{2n}\right) < \frac{2n^2}{\pi}$

$n\left(\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \dots + \sin\frac{(n-1)\pi}{2n}\right) < n\left(\frac{2n}{\pi}\right)$ 1 ✓

$\therefore \sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \dots + \sin\frac{(n-1)\pi}{2n} < \frac{2n}{\pi}$. (1)

as required.

(ii) From (1): $\sum_{r=1}^{n-1} \sin\left(\frac{\pi r}{2n}\right) < \frac{2n}{\pi}$

$2n\sum_{r=1}^{n-1} \sin\left(\frac{\pi r}{2n}\right) < \frac{4n^2}{\pi} = \frac{4\pi n^2}{\pi^2}$ (2) 1 ✓

but $\frac{4}{\pi} < \frac{\pi}{2}$ (since $8 < \pi^3$)

so $\frac{4\pi n^2}{\pi^2} < \frac{\pi^2 n^2}{2\pi} = \frac{\pi n^2}{2}$ ✓

$\therefore 2n \sum_{r=1}^{n-1} \sin\left(\frac{\pi r}{2n}\right) < \frac{4\pi n^2}{\pi^2} < \frac{\pi n^2}{2}$

as required.