Student Number__________________

ASCHAM SCHOOL

2012

YEAR 12

TRIAL

EXAMINATION

Mathematics

Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

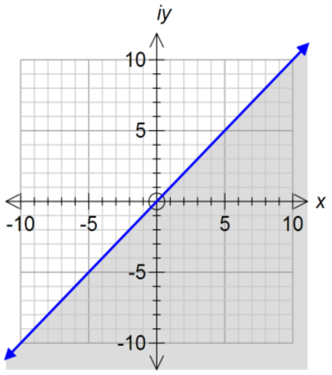
Total marks – 100

- Attempt Sections A and B.
- Section A is worth 10 marks.
- Recommended time on Section A: 15 minutes
- Answer Section A on the multiple choice answer sheet.
- Detach the multiple choice answer sheet from the back of the examination paper.
- Section B contains 6 questions worth 15 marks each.
- Recommended time on Section B: 2 hours 45 minutes
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher.

SECTION A – 10 MULTIPLE CHOICE QUESTIONS 10 MARKS

ANSWER ON THE ANSWER SHEET

1 Which of the following equations describes the graph below?



A $\text{Re}\ z \geq \text{Im}\ z$

B $\text{Re}\ z \leq \text{Im}\ z$

C $\text{Re}\ z^2 \geq 0$

D $\text{Im}\ z^2 \leq 0$

2 Two of the roots of the equation $z^5 + Bz^4 + Cz^3 + Dz^2 + Ez + 15 = 0$, where B, C, D, E are real could be:

A $2-i,\ 2$
B $2+i,\ 3$
C $3+i,\ 2$
D $3-i,\ -2$

3 The asymptote(s) of $y = \dfrac{x^3+4x^2-6}{x^2+4}$ is/are:

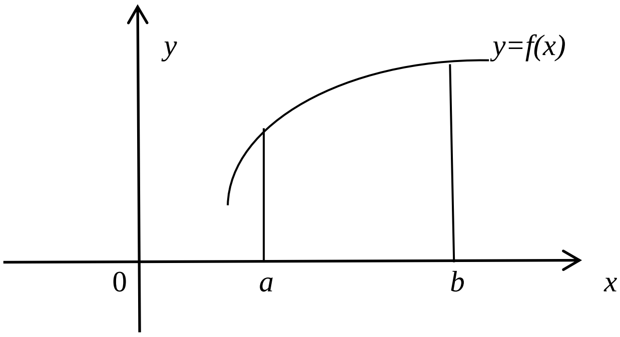
A $y = x^2 + 4$

B $y = x + 4$

C $x = \pm 2$

D $y = x - 4$

4 If f is the continuous, strictly increasing function on the interval $a \le x \le b$, as shown below, which of the following three statements must be true?



$I \quad \int_a^b f(x)\,dx < f(b)(b-a)$

$II \quad \int_a^b f(x)\,dx > f(a)(b-a)$

III there exists a number c where $a < c < b,$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$

A *I* only
B *II* only
C *III* only
D *I*, *II* and *III*

5 The terminal velocity of a particle with displacement given by

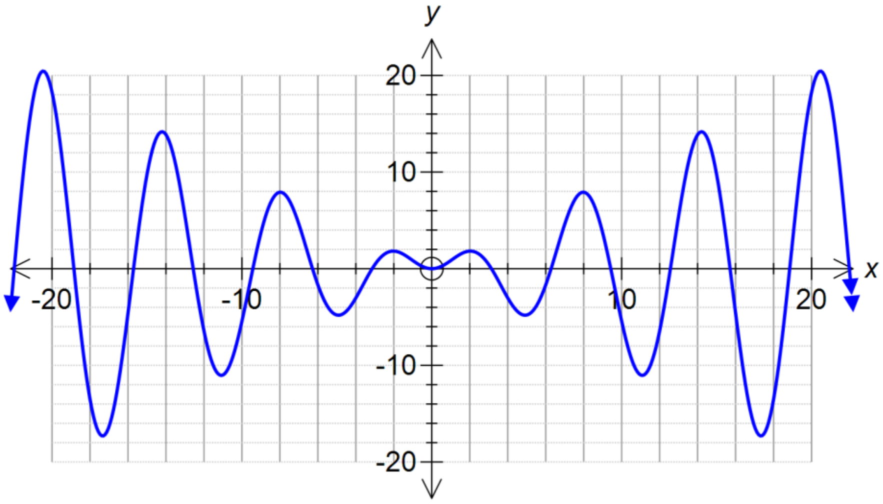
$x = \dfrac{V_o^2(1-e^{-2t})}{1+e^{-2t}}$, where V_o is initial velocity, is:

A 1
B 0
C V_o^2
D ∞

6 The conic $\dfrac{x^2}{a^2-k}+\dfrac{y^2}{b^2-k}=1$, where k is a constant and $a>b$, is always an ellipse for:

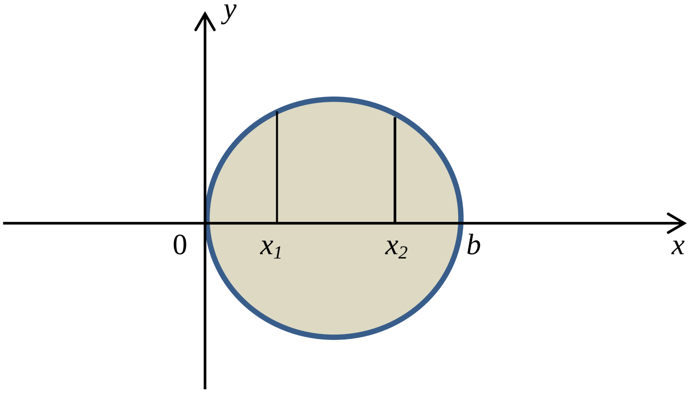
A $a^2 \le k \le b^2$
B $b^2 \le k \le a^2$
C $k < a^2$ *if* $a > b$
D $k < b^2$ *if* $a > b$

7 The equation of the graph below could be:



A $y = e^{\sin x}$
B $y = x\sin x$
C $y = x\cos x$
D $y = e^{\cos x}$

8 The circle shown is rotated about the *y*-axis. The volume is found by summing cylindrical shells with volume:



A $2\pi x 2y\delta x$

B $2\pi x 2y\delta y$

C $2\pi y\ \ x_2 - x_1\ \ \delta y$

D $2\pi y\ \ x_2 - x_1\ \ \delta x$

9 If $x^3 + y^3 x = y^2$ then $\dfrac{dy}{dx} =$

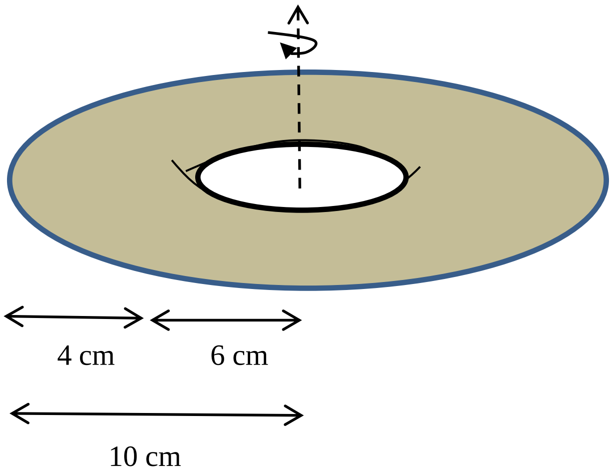
A $\dfrac{3x^2 + y^3}{2y - 3y^2 x}$

B $\dfrac{3x^2 + y^3}{3y^2 x - 2y}$

C $\dfrac{3x^2 + 3y^2 x + y^3}{2y}$

D $\dfrac{3x^2 + 3y^2}{2y}$

10 A torus can be generated by rotating a circle around an axis. Using Pappus's Theorem or otherwise, the volume of a torus with outer radius 10 cm and inner radius 6 cm is:



A $2\pi \times 8 \times \pi \times 2^2$

B $2\pi \times 10 \times \pi \times 2^2$

C $2\pi \times 6 \times \pi \times 2^2$

D $2\pi \times 8 \times \pi \times 4^2$

SECTION 2 – 6 QUESTIONS EACH WORTH 15 MARKS

Question 11 – Begin a new writing booklet

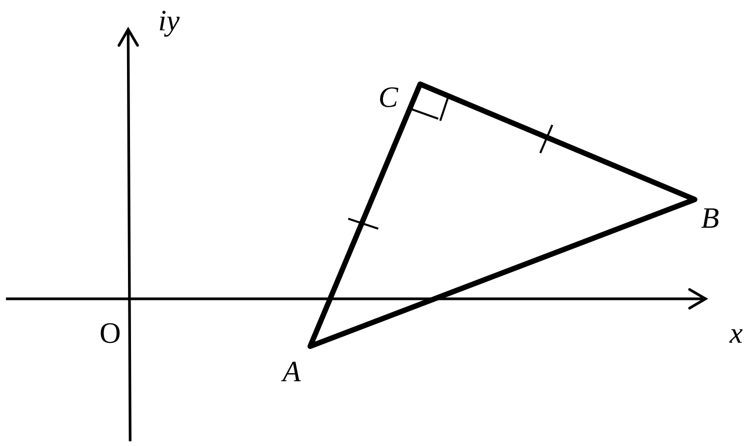
a Find $\int \frac{dx}{x \ln x}$. **2**

b Find $\int \sin^{-1} x \, dx$. **2**

c Find $\int \tan^4 x \, dx$. **3**

d

The points *A, B, C* representing the complex numbers $\tilde{a}, \tilde{b}, \tilde{c}$ form an isosceles triangle as shown in the diagram. $\angle ACB = \frac{\pi}{2}$ and *AC* = *CB*.



i Express the vector $\overrightarrow{CA}$ in terms of $\tilde{a}$ and $\tilde{c}$. **1**

ii Hence express the vector $\overrightarrow{CB}$ in terms of $\tilde{a}$ and $\tilde{c}$. **1**

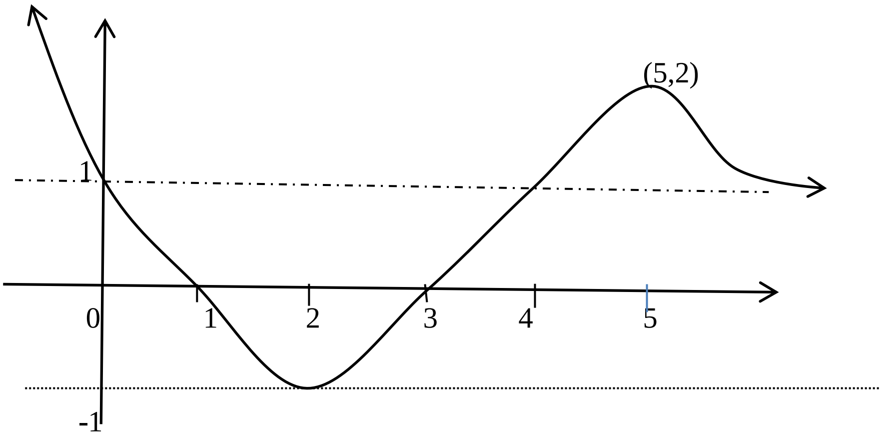
iii Find an expression for $\tilde{b}$ in terms of $\tilde{a}$ and $\tilde{c}$. **1**

e Find the locus of z if $\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{2}$. Draw a sketch. **2**

f Find the maximum value of $\arg z$ if $|z-2| = 1$. **3**

Question 12 – Begin a new writing booklet

a Consider the function $y = g(x)$ as shown in the diagram.



Sketch, showing essential features:

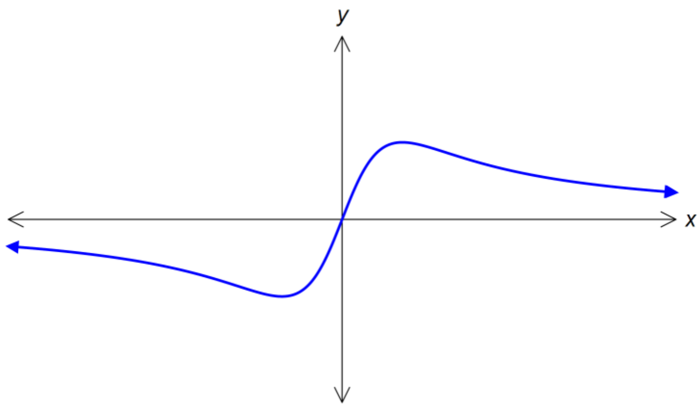
i $y = [g(x)]^3$, **2**

ii $y = \ln g(x)$, **2**

iii $y = e^{g(x)}$, **2**

iv $y = xg(x)$. **2**

b **3**



Write a possible equation for the graph above in the form $y = \dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials.

c Solve for x: $\cos 3x = \sin 5x$. **2**

d If $(x+iy)^n = a+ib$, where x, y, a, b are real, find a^2+b^2. **2**

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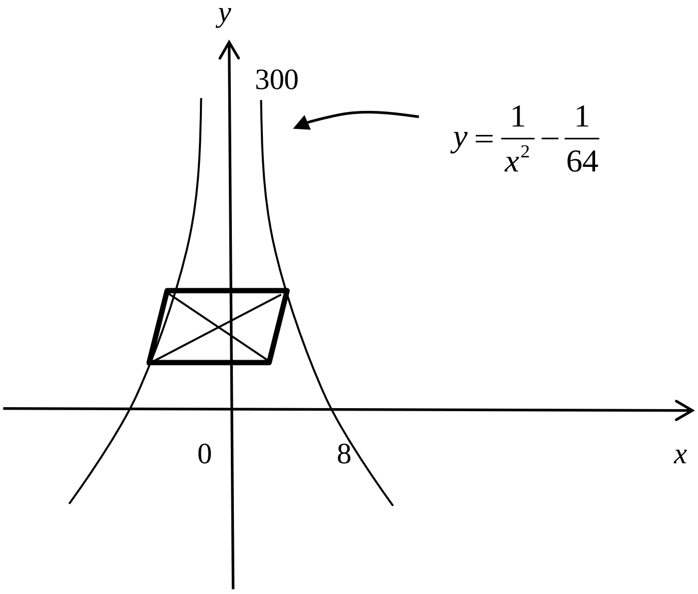
Question 13 – Begin a new writing booklet

a A solid model emulating the exterior shape of the Eiffel tower is generated by cutting cross-sectional slices perpendicular to the axis of symmetry of the curve $y = \frac{1}{x^2} - \frac{1}{64}$. The cross-sections are in the shape of a square with opposite vertices on the curve as shown. Taking the x-axis as ground level, the tower model is 300 cm high. Find the volume enclosed by the model.

Diagram not to scale



i Show that the volume of one slice is given by $\delta V = \frac{128}{64y+1}\delta y$. **2**

ii Hence find the total volume of the model. **2**

Question 13 continues on the next page.

b The lines RT and AC intersect at S. RA is produced and CT is produced to meet at B. RD is perpendicular to RA. $\angle DSC = 90°$ and $\angle DTB = 90°$.

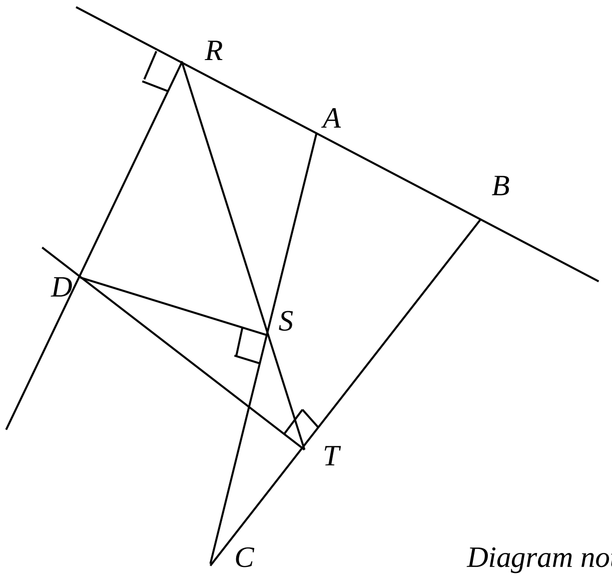


Diagram not to scale.

i Prove $\angle DST = 180° - \angle DCT$ **2**

ii Prove $\angle DCB = \angle DAR$. **2**

iii Prove $DABC$ forms a cyclic quadrilateral. **2**

c

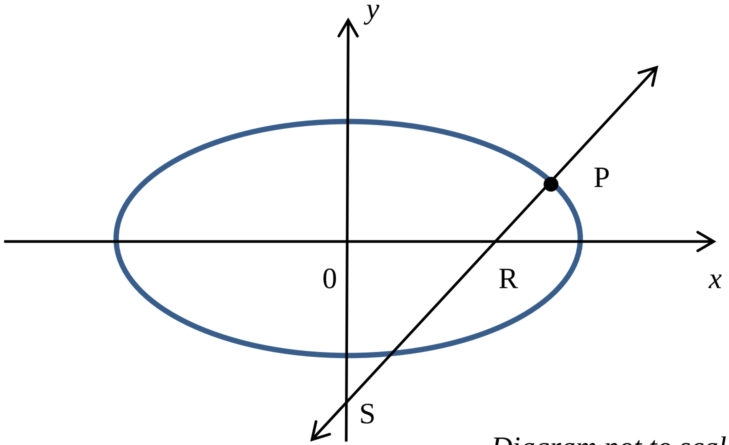


Diagram not to scale.

The normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point $P(a\cos\theta, b\sin\theta)$ meets the x- and y- axes in R and S respectively. You may assume that the equation of the normal is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$.

i If O is the centre, find the area of triangle ORS in terms of a, b and θ. **3**

ii Hence, find the values of θ for which triangle ORS has the largest area. **2**
[Note: $0 \le \theta \le 2\pi$.]

Question 14 – Begin a new writing booklet

a **i** If *a*, *b* are real and unequal, show that $a^2+b^2>2ab$. **1**

ii Hence show that if *a*, *b*, *c* are real and unequal $a^2+b^2+c^2>ab+bc+ca$. **2**

iii If $a+b+c=6$, show that $ab+bc+ca<12$. **2**

b Let $I_n=\int_1^2 (\ln x)^n\,dx$ for $n\in\mathbb{Z}^+$.

i Prove that $I_n=2(\ln 2)^n-nI_{n-1}$. **2**

ii Hence evaluate $\int_1^2 (\ln x)^4 dx$ as a polynomial in ln2. **2**

c Consider a right-angled triangle *ABC*, where $\angle BAC=90°$. The bisector of $\angle BAC$ meets *BC* in *D*. **3**

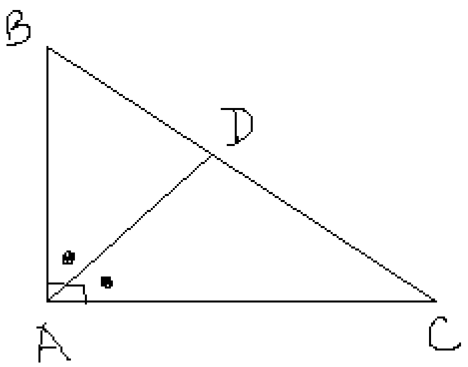


Diagram not to scale.

Prove $\dfrac{\sqrt{2}}{AD}=\dfrac{1}{AB}+\dfrac{1}{AC}$.

d **i** If $f(x)$ is a continuous function, show with the aid of a diagram the meaning of **1**

$$\lim_{n\to\infty}\frac{1}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right)=\int_0^1 f(x)\,dx.$$

ii Hence evaluate **2**

$$\lim_{n\to\infty}\frac{1}{n}\left(\sin\left(\frac{\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)+\sin\left(\frac{3\pi}{n}\right)+\ldots+\sin\left(\frac{n\pi}{n}\right)\right).$$

Question 15 – Begin a new writing booklet

a Given that $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a triple root, find all roots of $P(x) = 0$. **3**

b **i** Find the seventh roots of unity in $\cos\theta + i\sin\theta$ form and represent them on an Argand diagram. **2**

ii If one of the complex roots is ψ, show that the quadratic equation whose roots are $\psi + \psi^2 + \psi^4$ and $\psi^3 + \psi^5 + \psi^6$ is $x^2 + x + 2 = 0$. **2**

c A chicken, P_1, of mass M falls vertically from rest from O, in a resisted medium with resistance Mkv, $k > 0$, where v is velocity in m/s at time t seconds. Let acceleration due to gravity be g m/s^2. **3**

i Explain why $\ddot{x}_1 = g - kv$. **1**

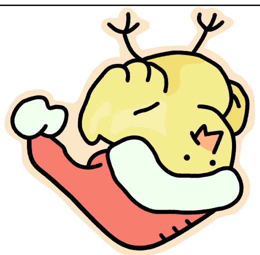
ii Obtain an expression for v after t seconds. **2**

A second chicken, P_2, of mass M is projected vertically up from O with initial velocity U in the same medium, simultaneously as P_1.

iii Show that for P_2, $t = -\dfrac{1}{k}\ln\left(\dfrac{kv + g}{kU + g}\right)$. **3**

iv Show that when P_2 is momentarily at rest, the velocity of P_1 is given by $\dfrac{VU}{V + U}$ where V is the terminal velocity of P_1. **2**

Chicken chickened out of sitting the Extension 2 trial. Preferred to go sky-diving instead....



Question 16 – Begin a new writing booklet

a

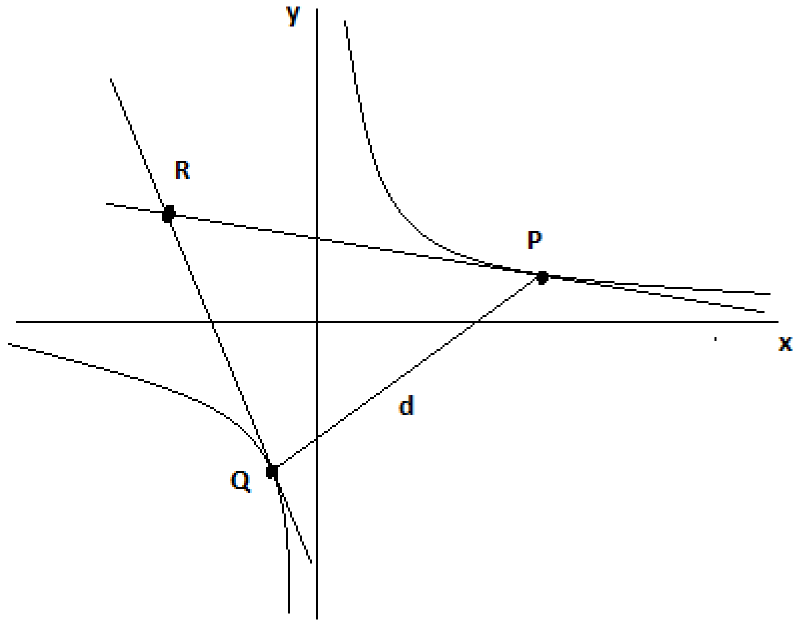


Diagram not to scale.

The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the hyperbola $xy = c^2$. Tangents are drawn from P and Q to meet at $R\left(x_0, y_0\right)$. The equation of the tangent at P is $x + p^2 y = 2cp$. Let the distance PQ be d units.

i Prove that $pq = \frac{x_0}{y_0}$ and $p + q = \frac{2c}{y_0}$. **2**

ii Find an expression for d^2 in terms of *c, p* and *q*. Give your answer in factorised form. **2**

iii If *d* is fixed, deduce the locus of *R* is $4c^2\left(x^2 + y^2\right)\left(c^2 - xy\right) = x^2 y^2 d^2$. **2**

b Given $2\cos A \sin B = \sin\left(A+B\right) - \sin\left(A-B\right)$, prove by induction for integers $n \geq 1$, that **3**

$$\cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos\left(2n-1\right)\theta = \frac{\sin 2n\theta}{2\sin\theta}.$$

Question 16 continues on the next page.

c

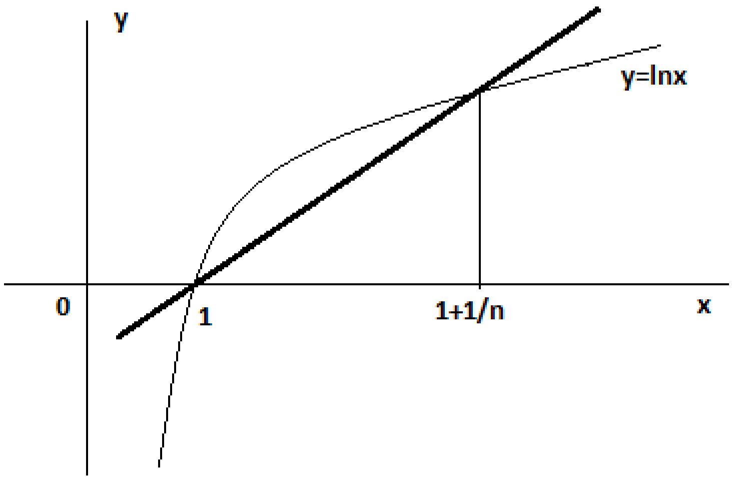


Diagram not to scale.

Consider the secant drawn on $y = \ln x$ between $x = 1$ and $x = 1 + \frac{1}{n}$.

i Find an expression for the gradient of the secant. **1**

ii Using part (i) and the fact that $\frac{d}{dx} \ln x = \frac{1}{x}$, show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. **2**

iii Also using the method in part (i) show that **2**

$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$.

Explain with the aid of a sketch.

iv What implication does this have for compound interest? **1**

The end! ☺