ASCHAM SCHOOL
MATHEMATICS EXTENSION 2

TRIAL EXAMINATION
2013

Time : 3 hours + 5 minutes reading time

General Instructions:

Write using a black or blue pen. Black pen is preferred.

Board-approved calculators may be used

A Table of Standard Integrals is provided on page 2.

In questions 11-16, show relevant mathematical reasoning and/or calculations.

Total marks – 100

Section I Pages 3-4. 10 marks.

Attempt questions 1-10. Allow about 15 minutes for this section.

Use the multiple choice answer sheet for Questions 1-10

Section II Pages 5-10. 90 marks

Attempt questions 11-16. Allow about 2 hours and 45 minutes for this section.

Each question from this section should be answered in a separate booklet.

Extra booklets are available.

Full marks may not be awarded for careless or badly arranged work.

Do not use whiteout, marks may be awarded for scored out work if it is legible.

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

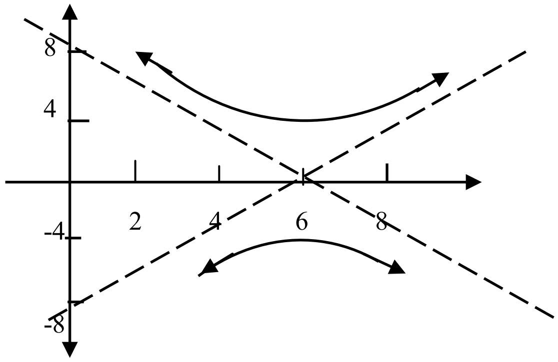
$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I : Multiple Choice – 1 mark each.
Use the multiple choice answer sheet for Questions 1-10

1.



The graph shown above could have the equation

A. $\dfrac{(x-6)^2}{16}-\dfrac{y^2}{9}=-1$ B. $\dfrac{(x-6)^2}{4}-\dfrac{y^2}{36}=-1$

C. $\dfrac{(x-6)^2}{4}-\dfrac{y^2}{9}=-1$ D. $\dfrac{(x-6)^2}{16}-\dfrac{y^2}{36}=-1$

2. If $\cos x=-\dfrac{1}{5}$ and $\dfrac{\pi}{2}\le x\le\pi$, then $\cot x$ is equal to

A. $-2\sqrt{6}$ B. $2\sqrt{6}$ C. $\dfrac{-1}{2\sqrt{6}}$ D. $\dfrac{1}{2\sqrt{6}}$

3. Which one of the following is a polar form of $2-2i$

A. $2\sqrt{2}cis\left(\dfrac{\pi}{4}\right)$ B. $2\sqrt{2}cis\left(-\dfrac{\pi}{4}\right)$

C. $2cis\left(\dfrac{7\pi}{4}\right)$ D. $2cis\left(-\dfrac{7\pi}{4}\right)$

4.If $z^2=16cis\left(\dfrac{2\pi}{3}\right)$ then z is equal to

A. $2-2\sqrt{3}i$ or $-2+2\sqrt{3}i$ B. $2+2\sqrt{3}i$ or $-2-2\sqrt{3}i$

C. $2\sqrt{3}+2i$ or $-2\sqrt{3}-2i$ D. $2\sqrt{3}-2i$ or $-2\sqrt{3}+2i$

5. If $P(z) = z^3 + 2iz^2 + 3i \quad z \in C$, then a linear factor of $P(z)$ is

A. -1 B. i C. $z - i$ D. $z + 2i$

6. Which one of the following is an antiderivative of $\dfrac{5x+1}{(x-1)(x+2)}$ for $1 < x$?

A. $2\ln(x-1) + 3\ln(x+2)$ B. $2\ln(x-1) - 3\ln(x+2)$

C. $3\ln(x-1) + 2\ln(x+2)$ D. $3\ln(x-1) - 2\ln(x+2)$

7. If $f'(x) = 2\cos^2(x) - 1$ and $f\left(\dfrac{\pi}{6}\right) = 0$, then $f(x)$ is equal to

A. $\dfrac{\sin 2x}{2}$ B. $\dfrac{\sqrt{3}}{2} - \sin 2x$ C. $\dfrac{\sin 2x}{2} + \dfrac{\sqrt{3}}{4}$ D. $\dfrac{2\sin 2x - \sqrt{3}}{4}$

8. A particle moves with a constant acceleration in a straight line so that at time t, $t \geq 0$ its velocity is v and its displacement from a fixed point on the line is x. Which one of the following equations could not be true?

A. $t^3 = x - 1$ B. $2x + 4 = v^2$ C. $4t = v - 9$ D. $x = t^2 - t + 4$

9. At time t seconds, $t \geq 0$, the velocity v m/s of a particle moving in a straight line is given by

$$v = \sqrt{3}\cos(t) + \sin(t) - 2.$$

For what value of t does the particle first attain its maximum speed of 4m/s?

A. $t = \dfrac{\pi}{6}$ B. $t = \dfrac{7\pi}{6}$

C. $x = \dfrac{4\pi}{3}$ D. The particle never attains a speed of 4 m/s.

10. The graph of $y = \dfrac{x^2}{x^2 - 4}$ has

A. a single vertical asymptote, two horizontal asymptotes and no turning points
B. a single horizontal asymptote, two vertical asymptotes and no turning points
C. a single vertical asymptote, two horizontal asymptotes and one turning points
D. a single horizontal asymptote, two vertical asymptotes and one turning points

Question 11

(a) Find the exact value of

(i) $\int_0^{3\pi} x\cos x\,dx$ [2]

(ii) $\int_0^1 \frac{dx}{x^2+4x+5}$ [3]

(b) (i) write $\frac{x}{(x-1)^2(x-2)}$ in the form $\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$ [2]

(ii) Hence show that $\int_0^{\frac{1}{2}} \frac{x}{(x-1)^2(x-2)}dx = 2\ln\left(\frac{3}{2}\right)-1$ [3]

(c) (i) Use the substitution $x=a-t$, where a is a constant, to prove that

$$\int_0^a f(x)dx = \int_0^a f(a-t)dt$$ [2]

(ii) Hence, or otherwise, show that $\int_0^1 x(1-x)^{99}\,dx = \frac{1}{10100}$ [3]

Question 12 (Start a new booklet)

(a) (i) Write down the 3 relations which hold between the roots α,β,γ of the equation

$ax^3+bx^2+cx+d=0$ where $a\neq 0$ and the coefficients $a,b,c,d.$ [2]

(ii) Consider the equation $36x^3-12x^2-11x+2=0$. You are given that the roots of this equation satisfy $\alpha=\beta+\gamma$. Use part (i) to find α . [1]

(iii) Suppose the equation $x^3+px^2+qx+r=0$ has roots λ,μ,ν which satisfy $\lambda=\mu+\nu$
Show that $p^3-4pq+8r=0$ [3]

(b) (i) Find the turning points of the cubic polynomial $P(x)=x^3-x^2-5x-1$ [3]
and without attempting to solve the equation, show that the equation $P(x)=0$ has three distinct real roots, two of which are negative

(ii) Sketch the graph of $P(x)$ [2]

(iii) Starting with the approximation $x=0$ use one application of Newton's method to estimate a root of the equation $P(x)=0$ [2]

(iv) What initial approximation would you use to estimate the positive root of $P(x)=0$ by Newton's method? State briefly your reasons for this choice. [2]

Question 13 (Start a new booklet)

(a) Reduce the complex expression $\frac{(2-i)(8+3i)}{(3+i)}$ to the form $a+ib$ where a and b are real numbers [3]

(b) The complex number z is given by $z=-\sqrt{3}+i$.

(i) Write down the values of $\arg z$ and $|z|$. [3]

(ii) Hence or otherwise, show that $z^7+64z=0$ [3]

(c) On an Argand diagram let $A=3+4i$ and $B=9+4i$

(i) Draw a clear sketch to show the important features of the curve defined by

$$|z-A|=5$$ [2]

(ii) Also for z on this curve, find the maximum value of $|z|$ [1]

(d) On a separate diagram draw a neat sketch to show the important features of the curve defined by:

$$|z-A|+|z-B|=12$$ [2]

For z on this curve, find the greatest value of $\arg z$ [1]

Question 14 (Start a new booklet)

(a) Let $f(x)=\frac{1-x}{x}$. On separate diagrams sketch the following graphs. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y=f(x)$ [2]

(ii) $y=f(|x|)$ [2]

(iii) $y=e^{f(x)}$ [2]

(iv) $y^2=f(x)$ [2]

Also discuss the behaviour of the curve of (iv) at x =1 [1]

(b) Suppose k is a constant greater than 1. Let $f(x)=\frac{1}{1+(\tan x)^k}$ where $0\le x\le\frac{\pi}{2}$

(You may assume $f\left(\frac{\pi}{2}\right)=0$)

Question 4b) continues on the next page.

(i) Show that $f(x)+f\left(\frac{\pi}{2}-x\right)=1$ for $0 \le x \le \frac{\pi}{2}$ [2]

(ii) Sketch $y=f(x)$ for $0 \le x \le \frac{\pi}{2}$ [2]

[Note: There is no need to find $f'(x)$ but assume $y=f(x)$ has a horizontal tangent at $x=0$. Your graph should exhibit the property of (b)(i)]

(iii) Hence or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+(\tan x)^k}$ [2]

Question 15 (Start a new booklet)

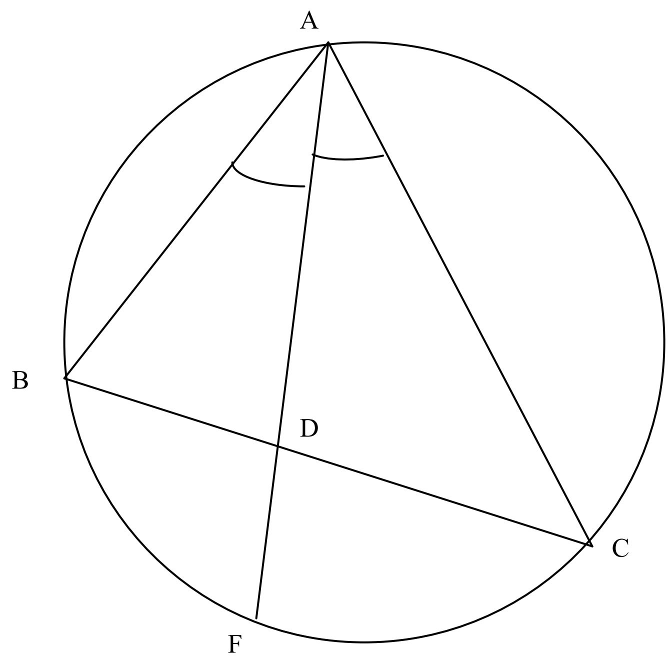
(a) The ellipse E has equation $\frac{x^2}{100}+\frac{y^2}{75}=1$

(i) Sketch E showing on your diagram the coordinates of the foci and the equation of each directrix. [3]

(ii) Find the equation of the normal to the ellipse at the point P(5, 7.5) [2]

(iii) Find the equation of the circle that is tangential to the ellipse at P and also at Q(5, -7.5) [2]

(b)



In the diagram the bisector AD of $\angle$BAC has been extended to intersect the circle ABC at F

(i) Prove that the triangles ABF and ADC are similar [3]

(ii) Show that $AB.AC = AD.AF$ [2]

(iii) Prove that $AB.AC - BD.DC = AD^2$ [3]

Question 16 (Start a new booklet)

a) A mass of m kilograms falls from a stationary balloon at height h metres above the ground. It experiences air resistance during its fall equal to mkv^2, where v is the speed in metres per second and k is a positive constant.
Let x be the distance in metres of the mass from the balloon, measured positively as it falls. Let g be the acceleration due to gravity.

(i) Show that the equation of motion of the mass is $\ddot{x} = g - kv^2$ [1]

(ii) Find v^2 as a function of x. Hint $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2) = v\frac{dv}{dx}$ [3]

(iii) Find the velocity V as the mass hits the ground in terms of g, k and h. [1]

(iv) Find the velocity of the mass as it hits the ground if air resistance is neglected. [2]

(b) (i) On a number plane shade in the region representing the inequality

$$\left(x-2R\right)^2 + y^2 \le R^2$$ [1]

(ii) Show that the volume of the of a right circular cylindrical shell of height h with inner and outer radii x and $x+\delta x$ respectively is $2\pi xh\delta x$ when δx is sufficiently small for terms involving $(\delta x)^2$ to be neglected. [2]

(iii) The region $\left(x-2R\right)^2 + y^2 \le R^2$ is rotated about the y axis forming a solid of evolution called a torus. By summing volumes of cylindrical shells show that the volume V of the torus is given by,

$$V = 4\pi^2 R^3$$ [5]

Hint: When evaluating the integral you may find it useful to make the substitution

$$x - 2R = R\cos\theta$$

End of Examination