Student Number__________________

ASCHAM SCHOOL

2014

YEAR 12

TRIAL

EXAMINATION

Mathematics

Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

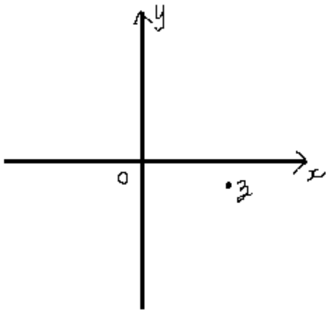
Total marks – 100

- Attempt Sections A and B.
- Section A is worth 10 marks.
- Answer Section A on the multiple choice answer sheet.
- Detach the multiple choice answer sheet from the back of the examination paper.
- Section B contains 6 questions worth 15 marks each.
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher.

SECTION A – 10 MULTIPLE CHOICE QUESTIONS 10 MARKS

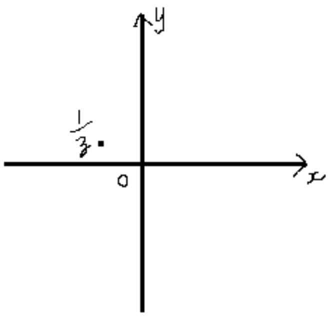
ANSWER ON THE ANSWER SHEET

1 The complex number z is sketched below.

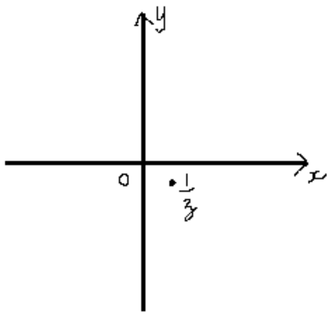


Which of the following sketches could describe $\frac{1}{z}$?

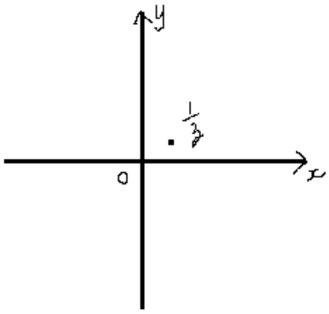
A



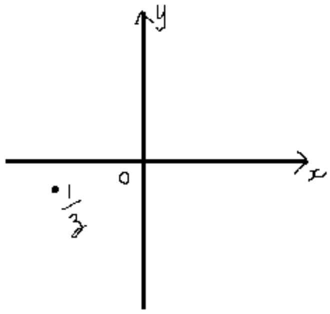
B



C



D



2 If z is complex, a solution to $z^5 = -1$ is:

A $z = cis\dfrac{\pi}{5}$

B $z = cis\dfrac{2\pi}{5}$

C $z = cis\dfrac{8\pi}{5}$

D $z = cis\dfrac{-2\pi}{5}$

3 If a particle of mass m is projected downwards under gravity and undergoes a resistive force of magnitude kv^3 then the acceleration is given by :

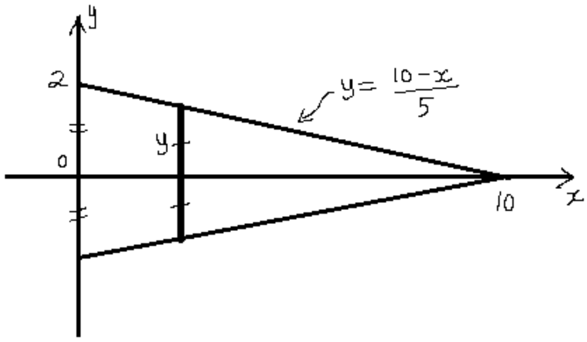
A $\ddot{x} = g - kv^3$

B $\ddot{x} = -g - kv^3$

C $\ddot{x} = g - \dfrac{kv^3}{m}$

D $\ddot{x} = -g - \dfrac{kv^3}{m}$

4 The volume of a kookaburra's beak is modelled by taking cross sections which are rhombuses, the longer diagonal of which is the length parallel to the y-axis between the lines shown. The smaller diagonal is half the length of the longer diagonal.



The volume would be given by:

A $\int_0^{10} \dfrac{100 - 20x + x^2}{5}\, dx$

B $\int_0^{10} \dfrac{100 - 20x + x^2}{10}\, dx$

C $\int_0^{10} 4 - \dfrac{4x}{5} + \dfrac{x^2}{25}\, dx$

D $\int_0^{10} 1 - \dfrac{x}{5} + \dfrac{x^2}{100}\, dx$

5 Let $f\left(x\right)=x^3+x$ be an increasing function. Let $h\left(x\right)$ be the inverse function of $f\left(x\right)$. The point $\left(1,2\right)$ lies on $y=f\left(x\right)$. The value of $h'\left(2\right)=$

A $\frac{1}{4}$

B $\frac{1}{13}$

C 4

D 13

6 Consider the conic $\frac{x^2}{\lambda-9}+\frac{y^2}{\lambda-4}=1$, where λ is a constant. If it is always an ellipse then:

A $\lambda<4$

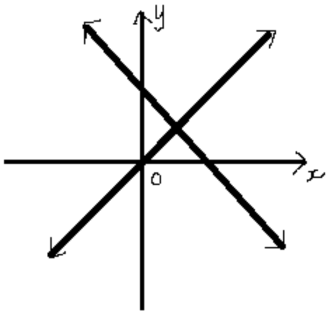
B $4<\lambda<9$

C $\lambda>9$

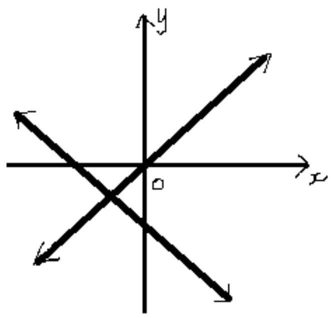
D $\lambda\le 4\ or\ \lambda\ge 9$.

7 The graph of $\left|x-1\right|=\left|y-1\right|$ could be:

A



B



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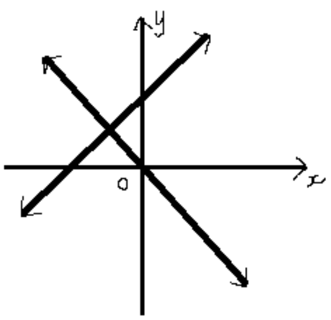
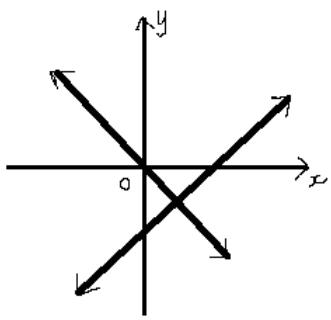


D



8 The solution to the inequality $\frac{1}{x}+\frac{1}{1-x}>0$ is:

A $x<0$
B $x>1$
C $0<x<1$
D $x<0 \ \ or \ \ x>1$.

9 A possible factor of $15x^7+10x^5-2x^3+14$ would be:

A $2x-7$
B $3x-7$
C $5x+3$
D $3x-5$.

10 The value of $\lim_{x\to\infty} e^{-\frac{1}{x^2}} =$

A ∞
B 1
C – 1
D 0.

SECTION 2 – 6 QUESTIONS EACH WORTH 15 MARKS

Question 11 – Begin a new writing booklet

a Find $\int \cot x \; \cos ec^2 x \, dx$. **2**

b Evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}} \, dx$. **3**

c Use the substitution $u = \sqrt{e^x + 1}$ to find $\int \frac{e^{2x} \, dx}{\sqrt{e^x + 1}}$. **3**

d

(i) Find constants A and B such that $\frac{\cos x}{1-\sin^2 x} = \frac{A\cos x}{1+\sin x} + \frac{B\cos x}{1-\sin x}$. **2**

(ii) Hence or otherwise find $\int \sec x \; dx$. **2**

e Use the t – results to find $\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\cos\theta}$. **3**

Question 12 – Begin a new writing booklet

a Find a square root of $15+8i$. **2**

b Let $\alpha = 1-i$ and $\beta = \sqrt{3}+i$.

i Find $\alpha\beta$ in Cartesian form. **1**

ii Express $\beta = \sqrt{3}+i$ in mod-arg form. **2**

iii If $\alpha = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$, find $\alpha\beta$ in mod-arg form. **1**

iv Hence find the exact value of $\cos\left(-\frac{\pi}{12}\right)$. **1**

c Sketch the locus of z if $\arg(z+2) = \arg(z-2i)$. **2**

d Consider the quadrilateral *ABCD* representing the complex numbers α,β,γ and δ, respectively.

Given that $\beta = \frac{1}{2}i\alpha$, $\alpha = -\gamma$, $|\beta| = |\delta|$ and $\arg\left(\frac{\beta}{\delta}\right) = \pi$,

(i) sketch the information on an Argand diagram, **1**

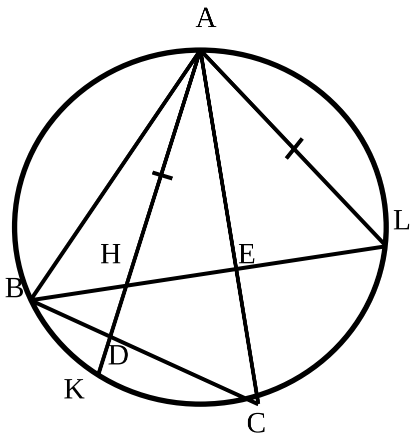
(ii) determine which type of quadrilateral is *ABCD*, giving reasons. **2**

e **i** Sketch $y = e^{2x}+1$. **1**

ii Hence sketch $y = \dfrac{e^{2x}+1}{x}$. **2**

Question 13 – Begin a new writing booklet

a Given that $ALCK$ is a cyclic quadrilateral and H is a point on AK such that $AH = AL$. LH produced meets the circle again at B and meets AC at E. BC meets AK at D.



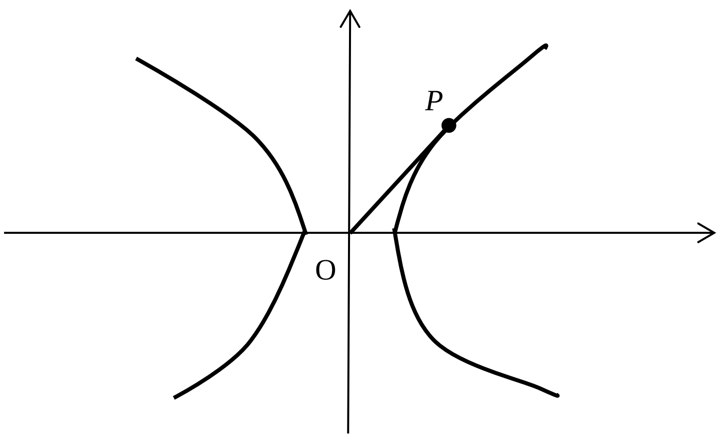
i Prove that $\angle AHL = \angle ACB$. **2**

ii Hence state why that $HECD$ is a cyclic quadrilateral. **1**

iii Given arc KC = arc CL, prove that HC is a diameter of $HECD$. **3**

b **i** Show $\sqrt{\left(e^{x}+e^{-x}\right)^{2}-4}=e^{x}-e^{-x}$ (Assume $x > 0$.) **2**

Consider the rectangular hyperbola $x^2 - y^2 = 1$ and the line segment OP.



ii Show that the point $P\left(\dfrac{e^{x}+e^{-x}}{2},\dfrac{e^{x}-e^{-x}}{2}\right)$ lies on the hyperbola. **1**

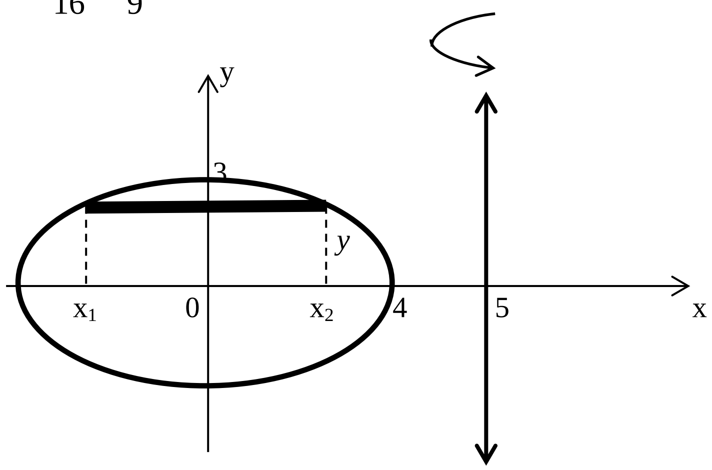
iii Using integration by parts (and then the table of standard integrals), show that **3**

$$\int \sqrt{x^2-1}\, dx = \frac{x\sqrt{x^2-1}}{2} - \frac{\ln\left(x+\sqrt{x^2-1}\right)}{2} + C$$

iv Show that the area bounded by the hyperbola, OP and the x-axis is $\dfrac{x}{2}$ units^2. **3**

Question 14 – Begin a new writing booklet

a The ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ is rotated about the line $x=5$.



(i) Show that the area of one annular type slice taken perpendicular to the axis of rotation is given by $A\approx\dfrac{80\pi}{3}\sqrt{9-y^2}$. **2**

(ii) Hence, by summing slices or otherwise, show that the volume of the resulting solid is $120\pi^2$ cubic units. **1**

b A particle of mass *M* is projected vertically upwards from *O* with initial speed *I* m/s. The particle is subjected to a constant gravitational force *g* m/s^2 downwards and a resistance of Mkv^2, $k>0$, where *v* is the speed at time *t*. Let *x* m be the displacement above *O* at time *t* seconds.

i Show that the greatest height reached, *H* metres is given by $H=\dfrac{1}{2k}\ln\left(\dfrac{g+kI^2}{g}\right)$. **3**

ii The particle then begins to fall. Write down an equation for the acceleration $\ddot{x}$ on its downward journey and find the maximum speed the particle reaches on the downward journey, giving reasons. **2**

iii The particle returns to its point of projection with speed *V* m/s. Derive the equation for the distance travelled downwards and hence show that $\left(g+kI^2\right)\left(g-kV^2\right)=g^2$. **3**

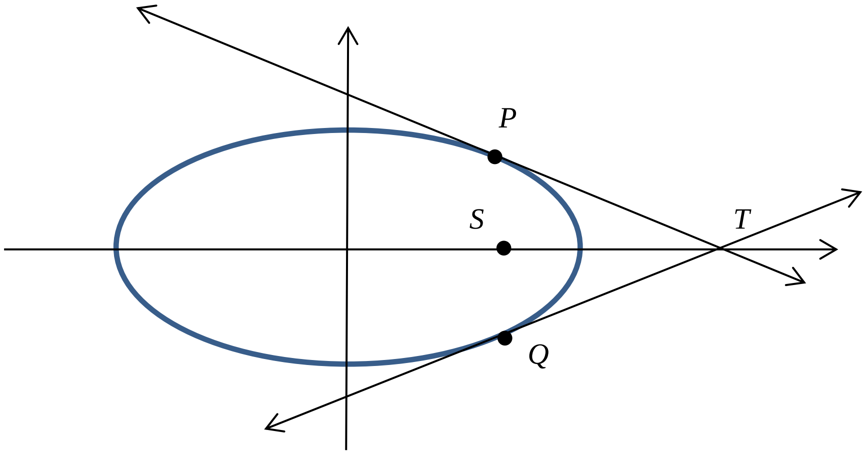
c We know that $A^2>0, B^2>0, \left(A\pm B\right)^2>0\ \ for\ \ A,B\neq 0.$ Prove for $x,y\neq 0$:

i $4x^2+6xy+4y^2>0$, **2**

ii $3x^2+5xy+3y^2>0$. **2**

Question 15 – Begin a new writing booklet

a Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Tangents to the ellipse are drawn at $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$.



i Derive the equation of the tangent at P and state a similar result for the tangent at Q. **2**

ii State the equation of the chord of contact from an external point $T\left(x_0, y_0\right)$. **1**

The tangents meet at the point $T\left(\frac{a}{e},0\right)$.

iii Prove that the chord PQ passes through the focus S. **1**

b It is given that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

i Hence solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$. **2**

ii Show that $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$ and $\cos^2\frac{2\pi}{5}\cos^2\frac{4\pi}{5}=\frac{1}{16}$. **2**

iii Hence deduce that the exact values are $\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$ and $\cos\frac{4\pi}{5}=\frac{-1-\sqrt{5}}{4}$. **2**

Question 15 continues on the next page.

Question 15 continued

c

i Prove the identity $\dfrac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}+\cos\left(n+1\right)x=\dfrac{\sin\left(n+\frac{3}{2}\right)x}{2\sin\frac{x}{2}}$. **2**

[Hint: $\cos\left(n+1\right)x=\cos\left(\left(n+\frac{1}{2}\right)x+\frac{1}{2}x\right)$]

ii Hence prove by Mathematical Induction (if $\sin\frac{x}{2}\neq 0$) that for n = 1, 2, 3,… **3**

$$\frac{1}{2}+\cos x+\cos 2x+\cos 3x+...+\cos nx=\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

Question 16 – Begin a new writing booklet

a Solve $\cos 5x = \sin 9x$. **2**

b The black-winged petrel from Lord Howe Island produces chicks of mass 30 grams. The approximate growth pattern of the chick then follows the equation $M = 4t + 30 + 8\sin\dfrac{\pi t}{2}$, where M is the mass of the chick in grams after t days, $0 \le t \le 70$. After 70 days the parents stop feeding the chick and it must then fend for itself.

i Between which two lines does this function lie? **2**

ii What is the approximate maximum mass the chick can reach in this range? Give reasons. **2**

iii Sketch the function and determine approximately how often the chicks get food. **2**

c Consider the function $G_n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$, where $n = 1, 2, 3, \ldots$

i State an expression for G_{n+1}. **1**

ii Use integration by parts to show that $G_{n+1} = nG_n$. **2**

iii Show $G_1 = 1$. **2**

iv Show $G_n = (n-1)!$ for all $n = 1, 2, 3, \ldots$ **2**

The end! ☺

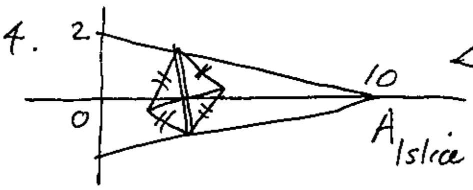
Solutions to
Title: Ascham 2014 Y12 Trial Math Ext 2 Exam

MC: 1. $z = r\,\text{cis}\,\theta \therefore \frac{1}{z} = \frac{1}{r}\text{cis}(-\theta)$
reflect angle $\therefore$ (C)

2. $z^5 = -1 \therefore z_1 = \text{cis}\frac{\pi}{5}$ then equally spaced, $\frac{2\pi}{5}$
$= \text{cis}\,\pi \therefore$ (A)

3. $\downarrow g$ $R = ma = -kv^3$
$\therefore \ddot{x} = g - \frac{kv^3}{m} \therefore$ (C)

4.



$A_{1\,slice} = \frac{1}{2} \times 2y \times y$
$= y^2$

$V = \int_0^{10} y^2\,dx$ $\quad y^2 = \left(\frac{10-x}{5}\right)^2$
$= \frac{100 - 20x + x^2}{25}$

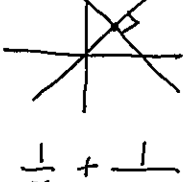
$\therefore$ (C)

5. $f(x) = x^3 + x \therefore$ let $y = x^3 + x$
$\therefore$ Inverse of $y = x^3 + x$ is $x = y^3 + y$
$\frac{dy}{dx} = 3x^2 + 1$
$\therefore \frac{dx}{dy} = \frac{1}{3x^2+1}$
$\therefore y = h(x)$ so $h'(2)$ means when $x = 2$ is inverse in $x = y^3 + y \therefore y = 1$
So in original D & R swap so
$x = 1 \therefore \frac{dx}{dy} = \frac{1}{3(1)^2+1} = \frac{1}{4} \therefore$ (A)

6. $\frac{x^2}{\lambda - 9} + \frac{y^2}{\lambda - 4} = 1$ ellipse then
$\lambda - 9 > 0$ and $\lambda - 4 > 0$
$\therefore \lambda > 9$ and $\lambda > 4$
$\therefore \lambda > 9 \therefore$ (C)

7. $|x-1| = |y-1|$ $\quad |x| = |y|$

Move centre to $(1,1)$.



$\therefore$ (A)

8. $\frac{1}{x} + \frac{1}{1-x} > 0,\ x \neq 0,\ x \neq 1$
Test $\quad - \ \ 0 \ \ + \ \ 1 \ \ -$
$\frac{1 - x + x}{x(1-x)} > 0$ $\quad \therefore 0 < x < 1$
$\frac{1}{x(1-x)} > 0$ $\quad \therefore$ (C)

MC cont'd:
9. Factor $ax - b$, "a" factor of $15x^7$, b factor of 14 $\therefore 3x - 7 \therefore$ (B)

10. $\lim_{x \to \infty} e^{-\frac{1}{x^2}}$ $\quad \frac{1}{x^2} \to 0$ as $x \to \infty$
$\therefore e^{-0} = e^0 = 1 \therefore$ (B)

Q11 a) $\int \cot x \,\text{cosec}^2 x\,dx$
$= -\int -\cot x\,\text{cosec}^2 x\,dx$ $\quad$ (2)
$= -\frac{\cot^2 x}{2} + C$

b) $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1 - x^4}}\,dx = \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} \frac{2x}{\sqrt{1 - (x^2)^2}}\,dx$
$= \frac{1}{2}\left[\sin^{-1}(x^2)\right]_0^{\frac{1}{\sqrt{2}}}$
(3) $\quad = \frac{1}{2}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)^2 - \sin^{-1}(0^2)\right]$
$= \frac{1}{2}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right]$
$= \frac{1}{2}\left[\frac{\pi}{6} - 0\right]$
$= \frac{\pi}{12}$

c) $\int \frac{e^{2x}\,dx}{\sqrt{e^x + 1}} = \int \frac{e^x \cdot e^x\,dx}{\sqrt{e^x + 1}}$
let $u = \sqrt{e^x + 1}$
$du = \frac{1}{2}(e^x + 1)^{-\frac{1}{2}} \times e^x\,dx$
$u^2 = e^x + 1$
$e^x = u^2 - 1$
$2du = \frac{e^x\,dx}{\sqrt{e^x + 1}}$
$= \int (u^2 - 1) \cdot 2\,du$ $\quad$ (3)
$= \frac{2u^3}{3} - 2u + C$
$= \frac{2\sqrt{(e^x + 1)^3}}{3} - 2\sqrt{e^x + 1} + C$

d) i) Find A & B: $\quad$ (2)
$\frac{\cos x}{1 - \sin^2 x} = \frac{A\cos x}{1 + \sin x} + \frac{B\cos x}{1 - \sin x}$
$\frac{\cos x}{1 - \sin^2 x} = \frac{A(1 - \sin x)\cos x + B(1 + \sin x)\cos x}{(1 + \sin x)(1 - \sin x)}$
$\therefore 1\cos x \equiv \cos x\,(A - A\sin x + B + B\sin x)$
$\therefore A + B = 1, \quad B - A = 0 \quad \therefore A = B = \frac{1}{2}$

Solutions to
Title: Ascham 2014 Y12 Trial Math Ext 2 Exam

Q11 cont'd:

d i) $\therefore \dfrac{\cos x}{1-\sin^2 x} = \dfrac{1}{2}\left(\dfrac{\cos x}{1+\sin x} + \dfrac{\cos x}{1-\sin x}\right)$

$\therefore$ ii) $\int \sec x\, dx = \int \dfrac{\cos x}{\cos^2 x}\, dx$

$= \int \dfrac{\cos x}{1-\sin^2 x}\, dx$ (2)

$= \dfrac{1}{2}\left[\ln(1+\sin x) - \ln(1-\sin x)\right] + C$

$= \ln\sqrt{\dfrac{1+\sin x}{1-\sin x}} + C$

e) Let $t = \tan\frac{\theta}{2}$ $\quad dt = \frac{1}{2}\sec^2\frac{\theta}{2}\, d\theta$

$\therefore 2dt = (t^2+1)d\theta$

$\therefore d\theta = \dfrac{2dt}{t^2+1}$

$\theta = \frac{\pi}{3}$, $t = \frac{1}{\sqrt{3}}$

$\theta = 0$, $t = 0$

$\therefore \int_0^{\frac{\pi}{3}} \dfrac{d\theta}{1+\cos\theta}$

$= \int_0^{\frac{1}{\sqrt{3}}} \dfrac{\frac{2dt}{t^2+1}}{1+\frac{1-t^2}{1+t^2}}$

$= \int_0^{\frac{1}{\sqrt{3}}} \dfrac{2\,dt}{t^2+1} \div \left(\dfrac{1+t^2+1-t^2}{1+t^2}\right)$

$= \int_0^{\frac{1}{\sqrt{3}}} \dfrac{2\,dt}{2}$ (3)

$= \int_0^{\frac{1}{\sqrt{3}}} 1\, dt$

$= \left[t\right]_0^{\frac{1}{\sqrt{3}}}$

$= \dfrac{1}{\sqrt{3}}$.

Q12

a) If $\sqrt{15+8i} = a+ib$ where $a, b \in \mathbb{R}$

then $15+8i = (a+ib)^2 = a^2-b^2+2abi$

Equating real & imaginary parts: (2)

$a^2-b^2 = 15$, $8 = 2ab \Rightarrow 4 = ab$

By inspection, $a=4, b=1$ or $a=-4, b=-1$

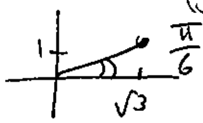
$\therefore$ A root is $4+i$ (or $-4-i$).

Q12 cont'd

b) $\alpha = 1-i$, $\beta = \sqrt{3}+i$

i) $\alpha\beta = (1-i)(\sqrt{3}+i) = \sqrt{3}+1+i(1-\sqrt{3})$ (1)

ii) $\beta = \sqrt{3}+i = 2\,\text{cis}\frac{\pi}{6}$ $\quad |\beta| = \sqrt{\sqrt{3}^2+1^2} = 2$ (2)



iii) $\alpha = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$\alpha\beta = 2\,\text{cis}\frac{\pi}{6} \times \sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)$ (1)

$= 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{6}-\frac{\pi}{4}\right)$

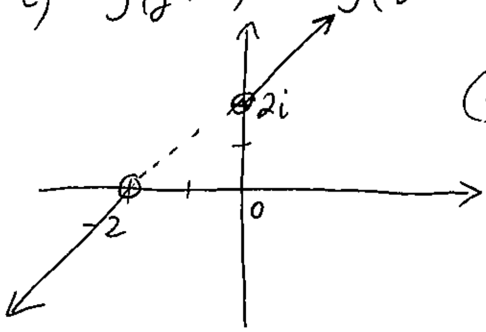
$= 2\sqrt{2}\,\text{cis}\left(\frac{-\pi}{12}\right) = 2\sqrt{2}\left(\cos\left(\frac{-\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)\right)$

iv) $= \sqrt{3}+1+i(1-\sqrt{3})$ Equating reals:

$2\sqrt{2}\cos\left(\frac{-\pi}{12}\right) = \sqrt{3}+1$ (1)

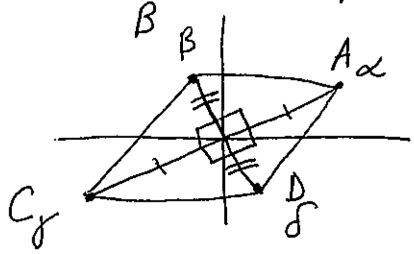
$\therefore \cos\left(\frac{-\pi}{12}\right) = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

c) $\arg(z+2) = \arg(z-2i)$



(2)

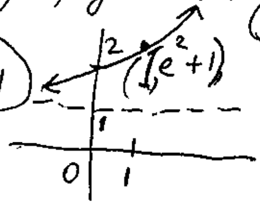
d) $\beta = \frac{1}{2}i\alpha$, $\alpha = -\gamma$, $|\beta| = |\delta|$, $\arg\left(\frac{\beta}{\delta}\right) = \pi$



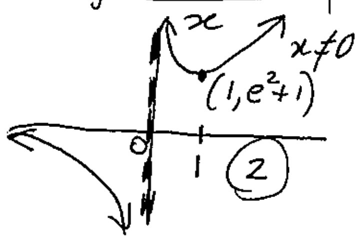
(1)

ABCD is a rhombus since diagonals bisect at ⊥ and that's it! (2)

e) i) $y = e^{2x} + 1$ (1)



$y = \dfrac{e^{2x}+1}{x}$

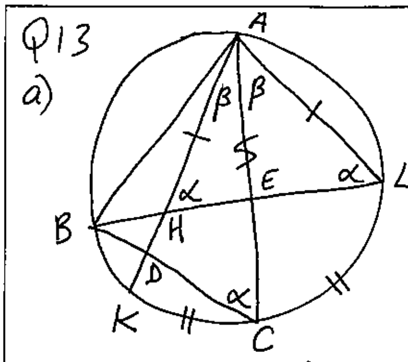


(2)

Solutions to
Title: Ascham 2014 Y12 Trial Math Ext 2 Exam

Q13

a)



Let $\angle AHL = \alpha$

i) $\angle ALH = \alpha$ (base $\angle$s of isosceles $\triangle AHL$ are equal)

Also $\angle ACB = \alpha$ ($\angle$s standing at circumference on arc AB equal) (2)

$\therefore \angle AHL = \angle ACB$ (both equal $\angle ALH$)

ii) $\therefore$ HECD is a cyclic quad (1)
($\angle AHL = \angle ACB$, exterior $\angle$ of quad HECD = interior opposite $\angle$)

iii) In $\triangle AHE$ and $\triangle ALE$, (3)
1. AE common
2. $\angle HAE = \angle LAE$ (equal arcs subtend equal $\angle$s at circumference)
3. $AH = AL$ (given)

$\therefore \triangle AHE \equiv \triangle ALE$ (SAS)

$\therefore \angle AEH = \angle AEL$ (matching $\angle$s in congruent $\triangle$s)

but $\angle AEH + \angle AEL = 180^\circ$ (straight line)

$\therefore 2x = 180$ $(x = \angle AEH)$

$x = 90$

$\therefore \angle AEH = 90^\circ$ then $\angle HEC = 90^\circ$ (vertically opposite $\angle AEH$)

$\therefore$ HC is diameter ($\angle$ in semicircle is 90°)

b) i) (2)
$$\sqrt{(e^x+e^{-x})^2-4} = \sqrt{e^{2x}+2+e^{-2x}-4}$$
$$= \sqrt{e^{2x}-2+e^{-2x}}$$
$$= \sqrt{(e^x-e^{-x})^2}$$
$$= e^x-e^{-x} \text{ if } x>0$$

ii) Sub: $\left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$ (1)
$$= \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4}$$
$$= \frac{4}{4} = 1. \quad \therefore \text{lies on } x^2-y^2=1.$$

Q13 cont'd:

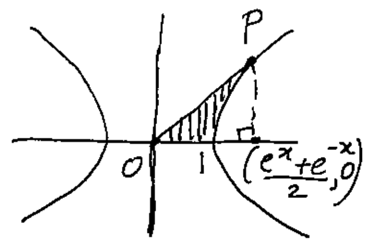
iii) $\int \sqrt{x^2-1}\,dx$

let $u = \sqrt{x^2-1}$, $du = \frac{1}{2}(x^2-1)^{-\frac{1}{2}} \cdot 2x\,dx$

$dv = 1\,dx$, $v = x$

$$= uv - \int v\,du$$
$$= x\sqrt{x^2-1} - \int \frac{x \cdot x\,dx}{\sqrt{x^2-1}}$$
$$= x\sqrt{x^2-1} - \int \frac{x^2}{\sqrt{x^2-1}}\,dx \quad (3)$$
$$= x\sqrt{x^2-1} - \int \frac{x^2-1}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}}\,dx$$
$$= x\sqrt{x^2-1} - \int\sqrt{x^2-1}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx$$
$$\therefore 2\int\sqrt{x^2-1}\,dx = x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right) + C$$
$$\therefore \int\sqrt{x^2-1}\,dx = \frac{x\sqrt{x^2-1}}{2} - \frac{\ln\left(x+\sqrt{x^2-1}\right)}{2} + C$$

iv)



$x^2-y^2=1$

$y^2 = x^2-1$

$y = \pm\sqrt{x^2-1}$

$$\text{Area} = \text{Triangle} - \int_1^{\frac{e^x+e^{-x}}{2}} \sqrt{x^2-1}\,dx$$
$$= \frac{1}{2}bh - \int_1^{\frac{e^x+e^{-x}}{2}} \sqrt{x^2-1}\,dx$$
$$= \frac{1}{2}\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right) - \left[\frac{x\sqrt{x^2-1}}{2} - \frac{\ln\left(x+\sqrt{x^2-1}\right)}{2}\right]_1^{\frac{e^x+e^{-x}}{2}}$$
$$= \frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{4}\right) - \left(\left[\frac{e^x+e^{-x}}{4}\sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2-1} - \frac{\ln\left(\frac{e^x+e^{-x}}{2} + \sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2-1}\right)}{2}\right] - \left[\frac{\sqrt{1^2-1}}{2} - \frac{\ln\left(1+\sqrt{1^2-1}\right)}{2}\right]\right) \quad (3)$$
$$= \frac{e^{2x}-e^{-2x}}{8} - \left(\frac{e^x+e^{-x}}{4}\sqrt{\frac{e^{2x}+2+e^{-2x}-4}{4}} - \frac{\ln\left(\frac{e^x+e^{-x}}{2} + \sqrt{\frac{e^{2x}+2+e^{-2x}-4}{4}}\right)}{2}\right) - 0$$

Using (i)
$$= \frac{e^{2x}-e^{-2x}}{8} - \left(\frac{e^x+e^{-x}}{4} \times \frac{e^x-e^{-x}}{2} - \frac{\ln\left(\frac{e^x+e^{-x}+e^x-e^{-x}}{2}\right)}{2}\right)$$
$$= \frac{e^{2x}-e^{-2x}}{8} - \frac{e^{2x}-e^{-2x}}{8} + \frac{\ln e^x}{2}$$
$$= 0 + \frac{\ln e^x}{2} = \frac{x\ln e}{2} = \frac{x}{2} \text{ unit}^2$$

Solutions to
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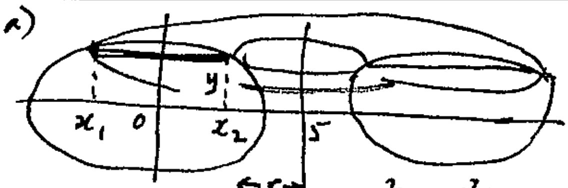
Q14 a)



$\frac{x^2}{16}+\frac{y^2}{9}=1$

i) Area of 1 slice $\doteqdot \pi R^2-\pi r^2$ (2)

$= \pi(R+r)(R-r)$

$= \pi((5-x_1)+(5-x_2))((5-x_1)-(5-x_2))$

$= \pi(10-(x_1+x_2))(x_2-x_1)$

Now make x subject to find x_1, x_2:

$\frac{x^2}{16} = 1-\frac{y^2}{9} \Rightarrow x = \pm\sqrt{16\left(\frac{9-y^2}{9}\right)}$

$= \pm\frac{4}{3}\sqrt{9-y^2}$

so $x_1 = -\frac{4}{3}\sqrt{9-y^2}$ $x_2 = +\frac{4}{3}\sqrt{9-y^2}$

$\therefore A \doteqdot \pi(10-(0))(2\times\frac{4}{3}\sqrt{9-y^2})$

$\doteqdot \pi\times\frac{80}{3}\sqrt{9-y^2}$ #

ii) $V \doteqdot \sum_{y=-3}^{3} \frac{80\pi}{3}\sqrt{9-y^2}\,\Delta y$

$= 2\int_0^3 \frac{80\pi}{3}\sqrt{9-y^2}\,dy$

$\frac{1}{4}\pi r^2$

$= 2\times\frac{80\pi}{3}\times\frac{1}{4}\pi 3^2$

$= 120\pi^2$ unit^3 (1)

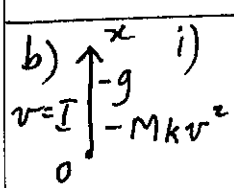
OR Using Pappus' Theorem:

$V =$ Area of Cross-section × Average Radius Circumference

$= \pi\times3\times4\times2\pi\times5$

$= 120\pi^2$ u^3

b) i) $F=ma=-Mg-Mkv^2$



$\therefore \ddot{x} = -g-kv^2$

$v\frac{dv}{dx} = \frac{-g-kv^2}{v}$

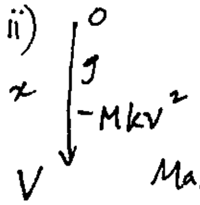
$\therefore \int 2k\frac{dx}{dv}dv = \int_I^0 -\frac{2kv}{g+kv^2}\,dv$ (3)

$\therefore 2k\,x = -\left[\ln(g+kv^2)\right]_I^0$

Q14 b) i) cont'd $x = -\frac{1}{2k}\left[\ln(g+k0^2)-\ln(g+kI^2)\right]$

$\therefore H = \frac{1}{2k}\ln\left(\frac{g+kI^2}{g}\right)$

ii) $F=ma=Mg-Mkv^2$



$\therefore \ddot{x} = g-kv^2$ (2)

Max speed should be terminal speed as starting from rest $\therefore\ \ddot{x}=0=g-kv^2$

$\therefore v = \sqrt{\frac{g}{k}}$ since $v>0$ (speed)

iii) $v=V$ when $x=H$ down.

$\therefore \ddot{x} = g-kv^2$

$v\frac{dv}{dx} = g-kv^2$

$\frac{dv}{dx} = \frac{g-kv^2}{v}$ (3)

$\therefore \int_0^V -2k\frac{dx}{dv} = \int_0^V \frac{-2kv}{g-kv^2}$

$x = \frac{-1}{2k}\left[\ln(g-kv^2)\right]_0^V$

$= -\frac{1}{2k}\left[\ln(g-kV^2)-\ln(g-k0^2)\right]$

$H = \frac{1}{2k}\ln\left(\frac{g}{g-kV^2}\right)$

Distance up = distance down

$\therefore \frac{1}{2k}\ln\left(\frac{g+kI^2}{g}\right) = \frac{1}{2k}\ln\left(\frac{g}{g-kV^2}\right)$

$(g+kI^2)(g-kV^2) = g^2$ #.

c) i) $4x^2+6xy+4y^2 = 3x^2+3y^2+6xy+x^2+y^2$

$= 3(x^2+2xy+y^2)+x^2+y^2$ (2)

$= 3(x+y)^2+x^2+y^2$

>0 if $x,y\neq0$ since $(x+y)^2>0$ and $x^2,y^2>0$

ii) $3x^2+5xy+3y^2 = 2x^2+4xy+2y^2+x^2+xy+y^2$

$= 2(x+y)^2+\frac{x^2}{2}+\frac{y^2}{2}+\frac{x^2+2xy+y^2}{2}$

$= 2(x+y)^2+\frac{x^2}{2}+\frac{y^2}{2}+\frac{(x+y)^2}{2}$ (2)

>0 since all terms >0 for $x,y\neq0$

QED.

Solutions to
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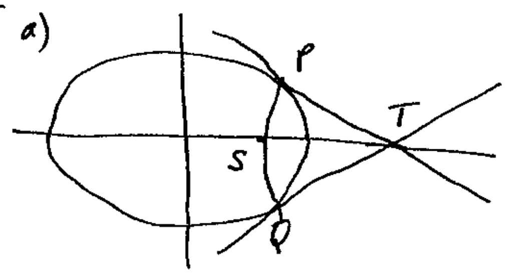
Q15 a)



i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx}=\frac{-b^2x}{a^2y}$ at $x=x_1$, $m=\frac{-b^2x_1}{a^2y_1}$

$\therefore$ Eq'n of tangent is:

$y-y_1=m(x-x_1)$

$y-y_1=\frac{-b^2x_1}{a^2y_1}(x-x_1)$ (2)

$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=-\frac{xx_1}{a^2}+\frac{x_1^2}{a^2}$

$\therefore \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

since (x_1,y_1) lies on ellipse

$\therefore$ Eq'n of tangent at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Similarly tangent at Q is

$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1.$

ii) Eq'n of chord of contact is (1)

$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1.$

iii) chord of contact from $T\left(\frac{a}{e},0\right)$ is chord PQ — sub $\left(\frac{a}{e},0\right) \Rightarrow$ (1)

$\frac{x\left(\frac{a}{e}\right)}{a^2}+\frac{y(0)}{b^2}=1$

$\therefore \frac{x}{ae}=1 \Rightarrow x=ae$ i.e. PQ is line $x=ae$ $\therefore S(ae,0)$ lies on vertical PQ. QED.

b) $\cos 5\theta = 16\cos^5\theta-20\cos^3\theta+5\cos\theta.$

i) $16x^5-20x^3+5x-1=0$ if $x=\cos\theta$ then equivalent to $\cos 5\theta=1$

$\therefore$ $5\theta=0+2k\pi$ where $k\in\mathbb{Z}$

$\theta=\frac{2k\pi}{5}$ i.e. $\theta=0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}\ldots$

etc. so $x=\cos\left(\frac{2k\pi}{5}\right)$ where values are 5 (distinct) roots of equation

$\therefore \cos 0, \cos\frac{2\pi}{5}, \cos\left(-\frac{2\pi}{5}\right), \cos\left(\frac{4\pi}{5}\right), \cos\left(-\frac{4\pi}{5}\right),$

$\cos\frac{6\pi}{5}, \cos\frac{-6\pi}{5}, \cos\frac{8\pi}{5}, \cos\left(-\frac{8\pi}{5}\right)\ldots$

(Distinct) roots are $\cos 0$ (1), $\cos\frac{2\pi}{5}, \cos\frac{4\pi}{5},$ (2)

$\cos\frac{6\pi}{5}\left(\cos\left(-\frac{4\pi}{5}\right)\right), \cos\frac{8\pi}{5}\left(\cos\left(-\frac{2\pi}{5}\right)\right)$

but (ii) $\cos\left(-\frac{4\pi}{5}\right)=\cos\left(\frac{4\pi}{5}\right), \cos\frac{2\pi}{5}=\cos\left(-\frac{2\pi}{5}\right)$

$\therefore$ Sum of roots $=\frac{-b}{a}=\frac{0}{16}=0$

$\therefore \cos 0+\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\left(\frac{-2\pi}{5}\right)+\cos\left(-\frac{4\pi}{5}\right)=0$

$\therefore 1+2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=0$

$\therefore \cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}.$

and product $=\frac{-f}{a}=\frac{1}{16}$ (2)

$\therefore 1\times\cos\frac{2\pi}{5}\times\cos\frac{4\pi}{5}\times\cos\frac{-2\pi}{5}\times\cos\left(\frac{-4\pi}{5}\right)=\frac{+1}{16}$

$\therefore \cos^2\frac{2\pi}{5}\cos^2\frac{4\pi}{5}=\frac{1}{16}$

iii) let $\cos\frac{2\pi}{5}=a$, $\cos\frac{4\pi}{5}=b$

$\therefore a+b=-\frac{1}{2}$ $\quad a^2b^2=\frac{1}{16}$ (2)

$ab=\pm\frac{1}{4}$

but $\cos\frac{4\pi}{5}<0$ so $ab=-\frac{1}{4}$ only

$\therefore a=\frac{-1}{4b}$ $\therefore \frac{-1}{4b}+b=\frac{-1}{2}$

$\therefore 1-4b^2=2b \Rightarrow 4b^2+2b-1=0$

$\Rightarrow b=\frac{-2\pm\sqrt{20}}{8}=\frac{-1\pm\sqrt{5}}{4}$ but $b<0$ $\therefore b=\frac{-1-\sqrt{5}}{4}$

Solutions to
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Q15 cont'd:

b iii) cont'd $\therefore a = \frac{-1}{2} - \left(\frac{-1-\sqrt5}{4}\right) = \frac{-1+\sqrt5}{4}$

$\therefore \cos\frac{2\pi}{5} = \frac{-1+\sqrt5}{4}$, $\cos\frac{4\pi}{5} = \frac{-1-\sqrt5}{4}$

c) i) RTP: $\frac{\sin(n+\frac12)x}{2\sin\frac{x}{2}} + \cos(n+1)x = \frac{\sin(n+\frac32)x}{2\sin\frac{x}{2}}$

Proof: $\frac{\sin(n+\frac12)x}{2\sin\frac{x}{2}} + \cos(n+1)x$

$= \frac{\sin(n+\frac12)x}{2\sin\frac{x}{2}} + \cos((n+\frac12)x + \frac12 x)$ (A, B) (2)

$= \frac{\sin(n+\frac12)x}{2\sin\frac{x}{2}} + \cos(n+\frac12)x\cos\frac12 x - \sin(n+\frac12)x\sin\frac12 x$

$= \frac{\sin(n+\frac12)x + 2\sin\frac{x}{2}\cos(n+\frac12)x\cos\frac12 x - 2\sin(n+\frac12)x\sin^2\frac12 x}{2\sin\frac{x}{2}}$

$= \frac{\sin(n+\frac12)x + \cos(n+\frac12)x\sin x - 2\sin(n+\frac12)x\sin^2\frac12 x}{2\sin\frac{x}{2}}$

$= \frac{\sin(n+\frac12)x\left[1-2\sin^2\frac12 x\right] + \cos(n+\frac12)x\sin x}{2\sin\frac{x}{2}}$

$= \frac{\sin(n+\frac12)x\cos x + \cos(n+\frac12)x\sin x}{2\sin\frac{x}{2}}$

$= \frac{\sin(n+\frac12+1)x}{2\sin\frac{x}{2}} = \frac{\sin(n+\frac32)x}{2\sin\frac{x}{2}}$. QED

ii) RTP: let $P(n)$ be the proposition that $\frac12 + \cos x + \cos 2x + \cos 3x + \dots + \cos nx = \frac{\sin(n+\frac12)x}{2\sin\frac{x}{2}}$.

Proof: Let $P(1)$ be $\frac12 + \cos x = \frac{\sin(\frac32 x)}{2\sin\frac{x}{2}}$.

LHS $= \frac12 + \cos x$ [or use (i) result where $n=0$.]

$= \frac12 + \cos(\frac12 x + \frac12 x)$

$= \frac12 + \cos^2\frac12 x - \sin^2\frac12 x$

$= \frac{1+2\cos^2\frac12 x - 2\sin^2\frac12 x}{2} \times \frac{\sin\frac{x}{2}}{\sin\frac{x}{2}}$

$= \frac{\sin\frac{x}{2} + 2\cos^2\frac12 x\sin\frac{x}{2} - 2\sin^2\frac12 x\sin\frac{x}{2}}{2\sin\frac{x}{2}}$

$= \frac{\sin\frac{x}{2}\left(1-2\sin^2\frac{x}{2}\right) + 2\sin\frac{x}{2}\cos\frac{x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}}$

$= \frac{\sin\frac{x}{2}\cos x + \sin x\cos\frac{x}{2}}{2\sin\frac{x}{2}} = \frac{\sin(\frac{x}{2}+x)}{2\sin\frac{x}{2}}$ = RHS

$\therefore P(1)$ true.

Assume $P(k)$ true i.e. $k \in \mathbb{N}, k \geq 1$

$\frac12 + \cos x + \cos 2x + \dots + \cos kx = \frac{\sin(k+\frac12)x}{2\sin\frac{x}{2}}$

RTP: $P(k+1)$ true i.e.

$\frac12 + \cos x + \cos 2x + \dots + \cos kx + \cos(k+1)x = \frac{\sin(k+1+\frac12)x}{2\sin\frac{x}{2}}$. (3)

Proof: Consider the LHS of $P(k+1)$:

$\frac12 + \cos x + \cos 2x + \dots + \cos kx + \cos(k+1)x$

$= \frac{\sin(k+\frac12)x}{2\sin\frac{x}{2}} + \cos(k+1)x$ using $P(k)$

$= \frac{\sin(k+\frac32)x}{2\sin\frac{x}{2}}$ from (i)

= RHS of $P(k+1)$ $\therefore P(k+1)$ true.

$\therefore P(n)$ is true for all $n \in \mathbb{N}, n \geq 1$ by Math Induction.

Q16

a) $\cos 5x = \sin 9x$ (2)

$\therefore \cos 5x = \cos(\frac{\pi}{2} - 9x)$

$\therefore 5x = 2k\pi \pm (\frac{\pi}{2} - 9x)$ $k \in \mathbb{Z}$

$\therefore 5x = 2k\pi + \frac{\pi}{2} - 9x$ OR $5x = 2k\pi - (\frac{\pi}{2} - 9x)$

$\therefore 14x = 2k\pi + \frac{\pi}{2}$ OR $4x = -(2k\pi - \frac{\pi}{2})$

$\therefore x = \frac{2k\pi + \frac{\pi}{2}}{14}$ OR $x = -\frac{(4k\pi - \pi)}{4}$

$\therefore x = \frac{\pi(4k+1)}{28}$ OR $x = -\frac{\pi(4k-1)}{4}$

b) $M = 4t + 30 + 8\sin\frac{\pi t}{2}$, $0 \leq t \leq 70$

i) $-1 \leq \sin\left(\frac{\pi t}{2}\right) \leq 1$

$\therefore -8 \leq 8\sin\frac{\pi t}{2} \leq 8$

$\therefore 4t + 30 - 8 \leq 4t + 30 + 8\sin\frac{\pi t}{2} \leq 4t + 30 + 8$

$\therefore 4t + 22 \leq y \leq 4t + 38$ (2)

$\therefore$ Lines are $y = 4t + 22$ and $y = 4t + 38$

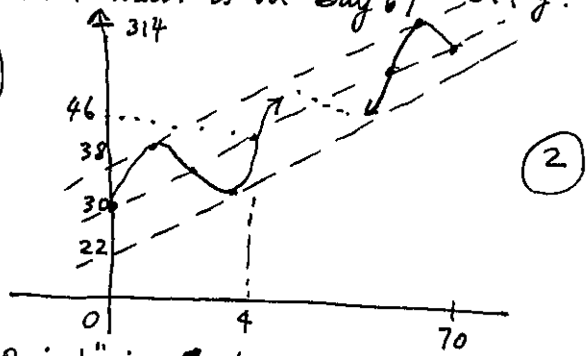
Solutions to
Title: Ascham 2014 Y12 Trial Math Ext 2 Exam

16 b) ii) If $t = 70$ $\quad y = 4(70) + 30 + 8\sin(35\pi)$
$= 310 + 0 = 310$

But if $t = 69$ $\quad y = 4(69) + 30 + 8\sin\left(\frac{68\pi + \pi}{2}\right)$
$= 306 + 8 \times 1$
$= 314$

$\therefore$ Max mass is on Day 69 314 g.

iii)



(2)

"Period" is 4 days so chick probably gets fed about every 4 days.

c) $G_n = \int_0^\infty e^{-t} t^{n-1}\,dt \quad n = 1, 2, 3, \ldots$

i) $G_{n+1} = \int_0^\infty e^{-t} t^n\,dt$ (1)

ii) $G_n = \int_0^\infty e^{-t} t^{n-1}\,dt$

$= uv - \int v\,du$ where $u = e^{-t}$, $du = -e^{-t}dt$, $dv = t^{n-1}dt$, $v = \frac{1}{n}t^n$

$= \left[e^{-t}\frac{1}{n}t^n\right]_0^\infty - \int_0^\infty \frac{1}{n}t^n \cdot -e^{-t}\,dt$

$= 0 \cdot \frac{1}{n}t^\infty - e^0 \cdot \frac{1}{n}0^n + \frac{1}{n}\int_0^\infty t^n \cdot e^{-t}\,dt$

$= 0 + \frac{1}{n}\int_0^\infty e^{-t} t^n\,dt$

$\therefore G_n = \frac{1}{n} G_{n+1}$ (2)

$\therefore G_{n+1} = n G_n$ QED

iii) $G_1 = \int_0^\infty e^{-t} t^{1-1}\,dt$

$= \int_0^\infty e^{-t} \cdot 1\,dt$

$= \left[-e^{-t}\right]_0^\infty$ (2)

$= -0 - -e^{-0}$

$= e^0 = 1$ QED

16 cont'd

c) iii) RTP: $G_n = (n-1)!$

Now $G_1 = 1$ and $G_{n+1} = nG_n$.

$\therefore G_2 = 1 \cdot G_1 = 1 \times 1 = 1$

$G_3 = 2 \cdot G_2 = 2 \cdot 1$

$G_4 = 3 \cdot G_3 = 3 \cdot 2 \cdot 1 = 3!$

$\therefore G_5 = 4 \cdot G_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$

$\vdots$

$G_n = (n-1)G_{n-1}$

$= (n-1) \cdot (n-2)(n-3) \ldots \times 2 \times 1$

$\therefore G_n = (n-1)! \quad n = 1, 2, 3, \ldots$ QED

The End! ☺