Student Number_______________

ASCHAM SCHOOL

2016

YEAR 12

TRIAL

EXAMINATION

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black non-erasable pen.
- Board-approved calculators may be used.
- A BOSTES Reference sheet is provided.
- All necessary working should be shown in every question.

Total marks – 100

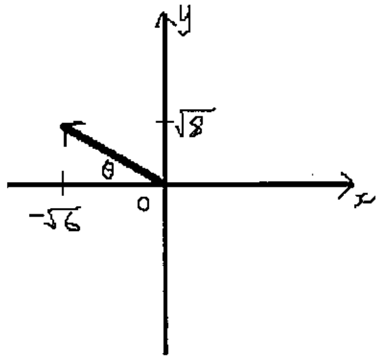
- Attempt Sections A and B.
- Section A is worth 10 marks.
- Answer Section A on the multiple choice answer sheet.
- Detach the multiple choice answer sheet from the back of the examination paper.
- Section B contains 6 questions worth 15 marks each.
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher.

SECTION A – 10 MULTIPLE CHOICE QUESTIONS 10 MARKS

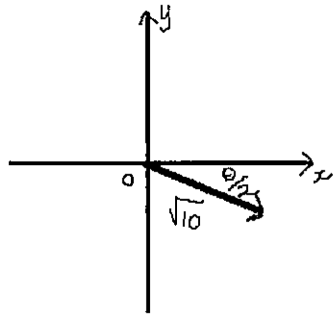
ANSWER ON THE ANSWER SHEET

1 If $6-8i = rcis\theta$, which of the following is a likely sketch of $\sqrt{6-8i}$?

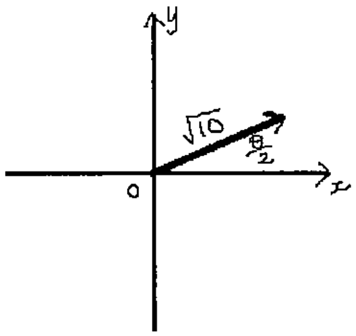
A



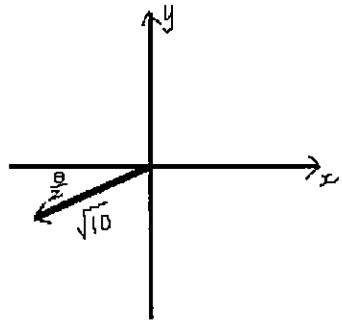
B



C



D



2 A polynomial with real coefficients has roots $1-i$ and $2+3i$. The degree of the polynomial could be:

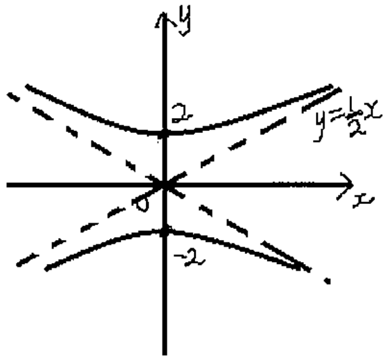
A 1

B 2

C 3

D 4

3 Which of the following describes the conic below?



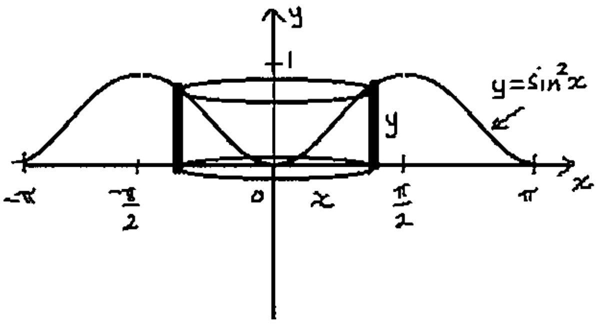
A $\dfrac{y^2}{4}-\dfrac{x^2}{16}=1$

B $\dfrac{y^2}{16}-\dfrac{x^2}{4}=1$

C $\dfrac{x^2}{16}-\dfrac{y^2}{4}=1$

D $\dfrac{x^2}{4}-\dfrac{y^2}{16}=1$

4 A correct expression for the volume generated by summing **shells** when the area under $y=\sin^2 x$ for $0 \le x \le \pi$ is rotated about the y-axis is given by:



A $\int_{-\pi}^{\pi} 2\pi yx \, dx$

B $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi x \sin^2 x \, dx$

C $\int_{0}^{\pi} \pi x(1-\cos 2x) \, dx$

D $\int_{0}^{\pi} x - x\cos 2x \, dx$

5 If a particle's motion is described by $\ddot{x} = v^3 + v$ where x is displacement and v is velocity at time t, then x as a function of v could be:

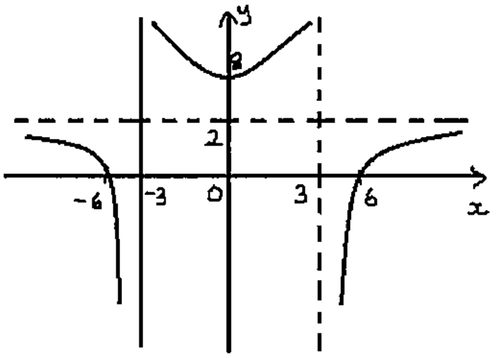
A $x = 3v^2 + 1$

B $x = \dfrac{v^4}{4} + \dfrac{v^2}{2}$

C $x = \displaystyle\int \frac{1}{v} + \frac{1}{v^2+1}\, dv$

D $x = \tan^{-1} v$

6 Which of the equations could describe the graph below?



A $y = \dfrac{2(x^2-9)}{x^2-36}$

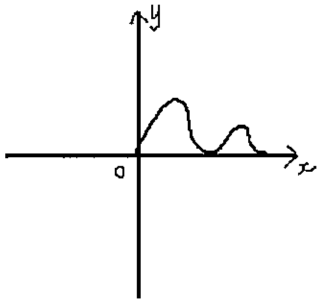
B $y = \dfrac{x^2-9}{2(x^2-36)}$

C $y = \dfrac{x^2-36}{2(x^2-9)}$

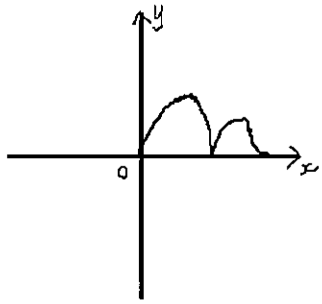
D $y = \dfrac{2(x^2-36)}{x^2-9}$.

7 The graph of $y=\sqrt{x(x-3)^2(4-x)^3}$ could be:

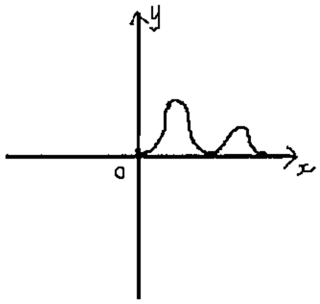
A



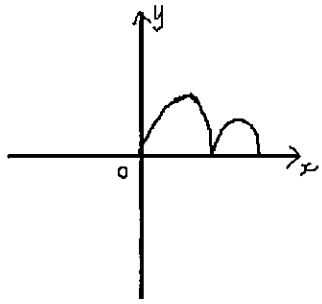
B



C



D



8 Which of the following is true?

A $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x\cos^4 x\, dx = 0$

B $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x + \cos^5 x\, dx = 0$

C $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\cos^4 x\, dx = 0$

D $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x + \cos^5 x\, dx = 0$

9 If $\sqrt{xy^2+5}=2$ then at the point $(-1,1)$, the value of $\frac{dy}{dx}$ is:

A -1

B $-\frac{1}{2}$

C $\frac{1}{2}$

D 1.

10 If $y=\cos^{-1}(\sin x)$ then the domain and range of the function are:

A $-\infty<x<\infty$
$-1<y<1$

B $-\infty<x<\infty$
$0\le y\le \pi$

C $-1\le x\le 1$
$-\frac{\pi}{2}\le y\le \frac{\pi}{2}$

D $-1\le x\le 1$
$0\le y\le \pi$

SECTION 2 – 6 QUESTIONS EACH WORTH 15 MARKS

Question 11 – Begin a new writing booklet

a **i** Simplify $(3-2i)(4+i)$. **1**

ii Simplify $\dfrac{3-2i}{4+i}$. **2**

iii If $\arg(3-2i)=\alpha$ and $\arg(4+i)=\beta$, find **2**

$\arg\left[(3-2i)(4+i)^2\right]-\arg\left[\dfrac{3+2i}{4+i}\right]$ in terms of α and β.

Give your answer in simplest form.

b Sketch the following conics, showing distinguishing features:

i $\dfrac{x^2}{4}-\dfrac{y^2}{9}=1$ **2**

ii $\left(3t,\dfrac{3}{t}\right)$ **2**

iii $(6\cos\theta,5\sin\theta)$ **2**

c Consider $P(x)=4x^3+12x^2-15x+4$

i Show that $P(x)$ has a double root at $x=\dfrac{1}{2}$. **2**

ii Hence factorise $P(x)$ over the real plane. **2**

Question 12 – Begin a new writing booklet

a If z is a complex number, sketch $|z+3|+|z-3|=8$ showing features. **2**

b Use the fact that $x^2+y^2 \geq 2xy$, or otherwise, to prove that:

i $\frac{a}{b}+\frac{b}{a} \geq 2$ [You may assume $a,b>0$.] **1**

ii $p^2+q^2+r^2 \geq pq+qr+rp$ **2**

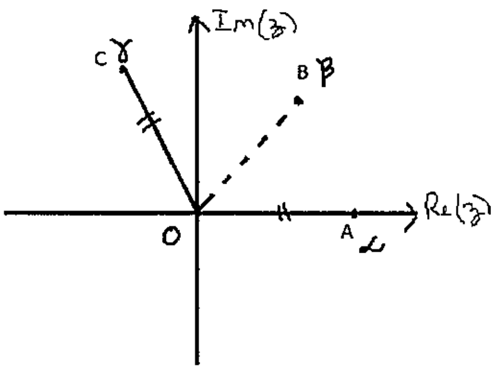
iii $p^3+q^3 \geq pq(p+q)$ (assume that $p,q \geq 0$) **3**

and hence

iv $2(p^3+q^3+r^3) \geq pq(p+q)+qr(q+r)+rp(r+p)$ **2**

c In an Argand diagram, the quadrilateral $OABC$ is a rhombus where $\angle AOC = 120°$. The point A represents the complex number $(\sqrt{2}+0i)$. Let A, B and C represent the complex numbers α, β and γ respectively.

Diagram not to scale.



Copy the diagram.

i Find the polar form of C (mod-arg form). **2**

ii Using part (i) or otherwise, find B in $a+ib$ form. **2**

iii Explain why $\arg\left(\frac{-\alpha}{\beta-\alpha}\right)=60°$. **1**

Question 13 – Begin a new writing booklet

a Consider the rectangular hyperbola $xy = 4$ with points $P\left(2p, \dfrac{2}{p}\right)$ where $p > 0$ and $Q\left(2q, \dfrac{2}{q}\right)$ where $q < 0$.

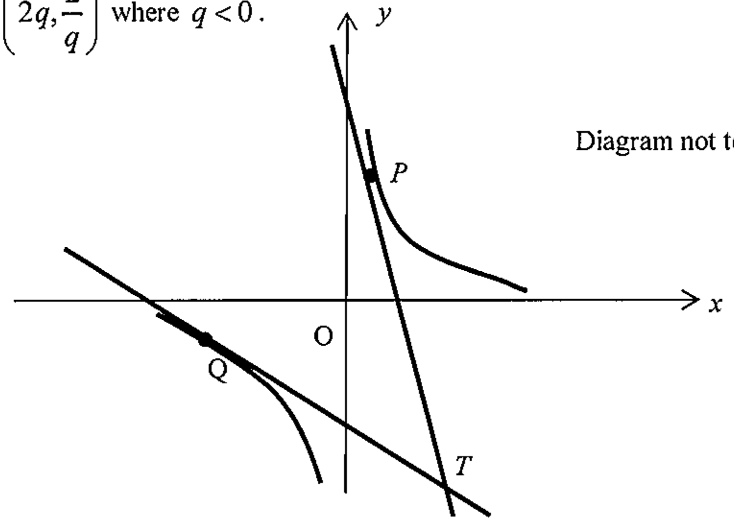


Diagram not to scale.

i Show that the equation of the tangent at point P is $x + p^2 y = 4p$. **2**

ii The tangents at P and Q meet at T. Show that the coordinates of T are $\left(\dfrac{4pq}{p+q}, \dfrac{4}{p+q}\right)$. **2**

iii Find the relationship between p and q if

(α) T lies on the line $y = -x$ **1**

(β) the two tangents never meet. **1**

iv Explain why the tangents can never be perpendicular. **1**

b The graph of $y = f(x)$, where $f(x) = (5-x)(x-1)$, is sketched below. At $x = A$ and $x = B$, $y = 1$. Sketch the graphs of the following:

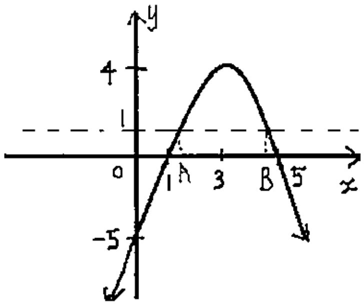


Diagram not to scale.

i $y = \dfrac{1}{f(x)}$ **2**

ii $y = \dfrac{x}{f(x)}$ **2**

iii $y = e^{f(x)}$ **2**

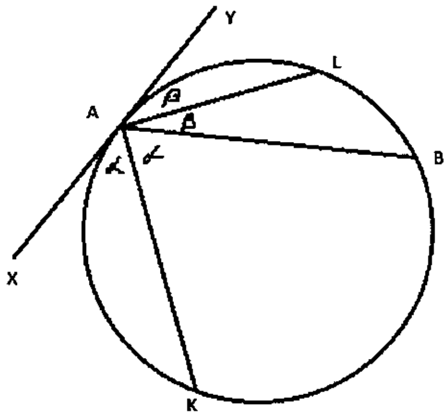
iv $y = \ln\left(f(x)\right)$ **2**

Question 14 – Begin a new writing booklet

a Given that $u_1 = 2,\ u_2 = 22$ and $u_n = 6u_{n-1} - 5u_{n-2}$ for $n \geq 3, n \in \mathbb{N}$, then prove by Mathematical Induction that $u_n = 5^n - 3$ for $n \geq 1, n \in \mathbb{N}$. **3**

b In the diagram, A, L, B, K lie on a circle. XY is a tangent at A. AL bisects $\angle YAB$ and AK bisects $\angle XAB$.

Diagram not to scale.



Copy or trace the diagram into your writing booklet.

Prove:

i $LA = LB$ **2**

ii KL is diameter of the circle **1**

iii $KL \perp AB$. **1**

c If p is a complex 6^{th} root of 1,

i prove that $1 + p + p^2 + p^3 + p^4 + p^5 = 0$ **1**

ii prove that $1 + p + p^8 + p^9 + p^{16} + p^{17} = 0$ **2**

iii factorise $z^6 - 1$ in two ways **2**

iv hence show that $p^4 + p^2 + 1 = (p^2 + p + 1)(p^2 - p + 1)$ **1**

v hence or otherwise find all complex roots of $z^4 + z^2 + 1 = 0$. **2**

Question 15 – Begin a new writing booklet

a Sketch $y = \cos\left(\sin^{-1} x\right)$. **2**

b Find or state the area of the semi-ellipse shown using any method. **1**

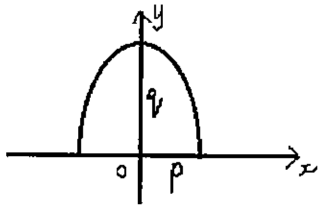


Diagram not to scale.

c A *Smiggle* pencil sharpener is similar to a solid described by the following. It has a front view of a semicircle with equation $y = \sqrt{b^2 - z^2}$ where z is a variable. Cross sections taken parallel to the side view are similar to the semi-ellipse above where the lengths p and q vary. The base of the solid can also be described as an ellipse with equation $\frac{z^2}{b^2} + \frac{x^2}{a^2} = 1$ .The diagram below gives an image.

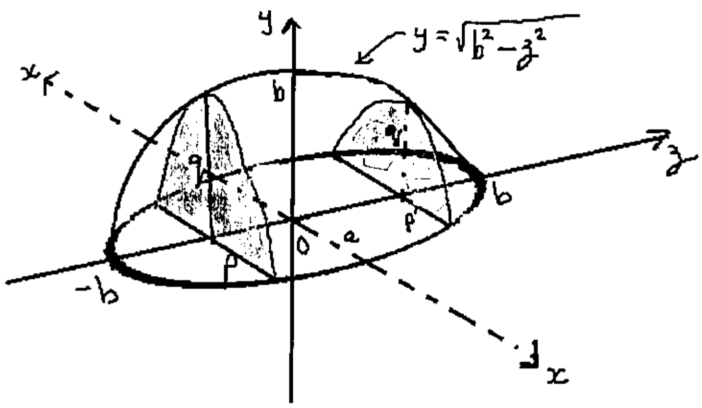


Diagram not to scale.

(i) Show that the volume of one slice taken perpendicular to the z- axis is given by $V \approx \frac{\pi a}{2b}\left(b^2 - z^2\right)\Delta z$. **2**

(ii) Hence, by summing slices, show that the volume of the resulting solid is $\frac{2\pi ab^2}{3}$ cubic units. **2**

Question 15 continues on the next page...

Question 15 continued

d Two particles P and Q each of mass 1 kg are travelling through a medium with resistance to the direction of motion of βv and βu respectively where v and u in m/s are the respective velocities of P and Q at time t. Both P and Q are moving under gravity g where $g = 10\,\text{m/s}^2$.

P is launched vertically upwards from O with an initial speed V m/s and simultaneously Q is allowed to fall from rest from a height H metres in the same vertical line as P. They meet at time T seconds.

i For P derive the equation relating the velocity v and the time t. Show that $t = \dfrac{1}{\beta}\ln\left|\dfrac{\beta V + g}{\beta v + g}\right|$. [Assume the launch point of P is the origin.] **3**

ii For Q derive the equation relating the velocity u and the time t. [Take the initial position of Q as the origin.] Show that $t = \dfrac{1}{\beta}\ln\left|\dfrac{g}{g - \beta u}\right|$. **2**

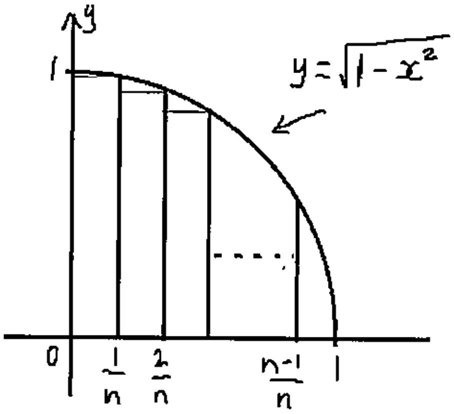
iii Noting that P and Q meet at time T seconds find a relationship at that instant between v and u and hence show that $u = \dfrac{g(V - v)}{\beta V + g}$. **3**

Question 16 – Begin a new writing booklet

a Show that $\int x^n e^x \, dx = x^n e^x - n\int x^{n-1}e^x \, dx$. **2**

Hence find $\int x^3 e^x \, dx$.

b Consider the areas of the n rectangles each of width $\frac{1}{n}$ under the curve $y = \sqrt{1-x^2}$ for $0 \le x \le 1$ as shown below.



i Show that the sum A_n of the areas of the rectangles is given by **2**

$$A_n = \frac{1}{n^2}\left(\sqrt{n^2-1} + \sqrt{n^2-2^2} + \sqrt{n^2-3^2} + \ldots + \sqrt{n^2-(n-1)^2}\right)$$

ii Hence show that **2**

$$\lim_{n \to \infty} \frac{1}{n^2}\left(\sqrt{n^2-1} + \sqrt{n^2-2^2} + \sqrt{n^2-3^2} + \ldots\right) \le \frac{\pi}{4}.$$

Question 16 continues on the next page...

Question 16 continued

c It is claimed that Isaac Newton was responsible for the cat flap to enable cats to exit and enter houses using a two-way flap that swings. The movement of the flap, after a cat has set it in oscillation back and forth over time t seconds, can be modelled by the equation $x = 15e^{-t}\cos t$ where $t \geq 0$ and x cm is the sideways movement of the flap.

i Between which two graphs is $x = 15e^{-t}\cos t$ contained? **1**

ii Write down the first four solutions to the equation $e^{-t}\cos t = 0$ for $t \geq 0$. **2**

iii Sketch the graph of $x = 15e^{-t}\cos t$ showing features. **2**

iv The flap squeaks while the sideways movement back and forth exceeds 0.04 cm. Use one approximation of Newton's method to determine how long the flap swings in either direction until it stops squeaking. [Hint: Work out a sensible first approximation by examining your graph.] **4**

The end! ☺

detach

Student Number ....

ASCHAM SCHOOL

YEAR 12 Trial Mathematics Extension 2 Exam

MULTIPLE-CHOICE ANSWER SHEET

1.	A ○	B ●	C ○	D ○
2.	A ○	B ○	C ○	D ●
3.	A ●	B ○	C ○	D ○
4.	A ○	B ○	C ●	D ○
5.	A ○	B ○	C ○	D ●
6.	A ○	B ○	C ○	D ●
7.	A ○	B ●	C ○	D ○
8.	A ●	B ○	C ○	D ○
9.	A ○	B ○	C ●	D ○
10.	A ○	B ●	C ○	D ○

Multiple Choice:

1. B 2. D 3. A 4. C 5. D
6. D 7. B 8. A 9. C 10. B

Question 11:

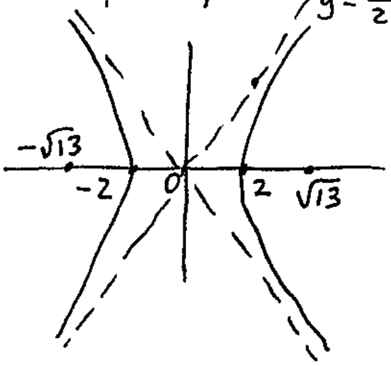
a) i) $(3-2i)(4+i) = 12+3i-8i+2$
$= 14-5i$ (1)

ii) $\frac{3-2i}{4+i} \times \frac{4-i}{4-i} = \frac{12-3i-8i-2}{16+1}$
$= \frac{10-11i}{17}$ (2)

iii) $\arg\left[(3-2i)(4+i)^2\right] - \arg\left[\frac{3+2i}{4+i}\right]$
$= \alpha + 2\beta - (-\alpha - \beta)$
$= \alpha + 2\beta + \alpha + \beta$ (2)
$= 2\alpha + 3\beta$.

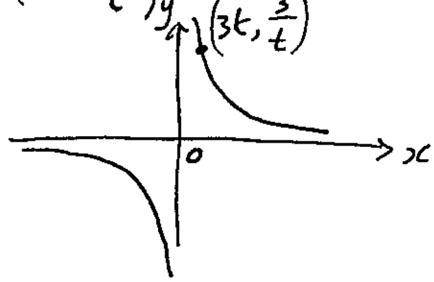
b) i) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ $a=2$ $b=3$

$e^2 = 1 + \frac{b^2}{a^2}$
$= 1 + \frac{9}{4}$
$e = \frac{\sqrt{13}}{2}$



(2)

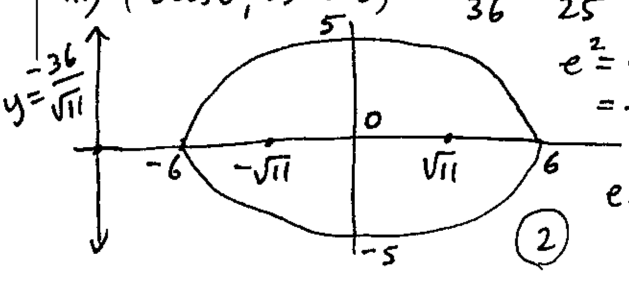
ii) $\left(3t, \frac{3}{t}\right)$ $\therefore xy = 3^2 = 9$ $e = \sqrt{2}$



(2)

iii) $(6\cos\theta, 5\sin\theta)$ $\therefore \frac{x^2}{36} + \frac{y^2}{25} = 1$

$e^2 = 1 - \frac{25}{36}$
$= \frac{11}{36}$
$e = \frac{\sqrt{11}}{6}$



(2)

Q11 cont'd:

c) i) $P(x) = 4x^3 + 12x^2 - 15x + 4$
$P'(x) = 12x^2 + 24x - 15$
$P\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 + 12\left(\frac{1}{2}\right)^2 - 15\left(\frac{1}{2}\right) + 4$
$= \frac{4}{8} + 12 \times \frac{1}{4} - \frac{15}{2} + 4$
$= 0$
$P'\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right)^2 + 24\left(\frac{1}{2}\right) - 15$
$= 12 \times \frac{1}{4} + 12 - 15$ (2)
$= 0$

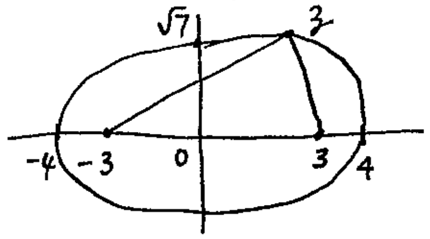
Since $P\left(\frac{1}{2}\right) = P'\left(\frac{1}{2}\right) = 0$ there is a double root at $x = \frac{1}{2}$.

$\therefore x - \frac{1}{2}$ is a factor or $(2x-1)$
$\therefore P(x) = (2x-1)^2(x-\beta)$ (2)
$\therefore 4 = (-1)^2 \times -\beta \Rightarrow \beta = -4$
$\therefore P(x) = (2x-1)^2(x+4)$

Solutions to
Title: 2016 Y12 Ascham Trial Math Ext 2 p2

Q12.

a) $|z+3|+|z-3|=8$



$ae=3$

$a=4$

$\therefore e=\frac{3}{4}$

$e^2=\frac{9}{16}$

$=1-\frac{b^2}{a^2}$

$\therefore b=\sqrt{7}$

Form is $PS+PS'=2a$

$\therefore \frac{x^2}{16}+\frac{y^2}{7}=1$ (2)

b) $x^2+y^2 \geq 2xy$

i) RTP: $\frac{a}{b}+\frac{b}{a} \geq 2$ $[a,b>0]$

Proof: Using $a^2+b^2 \geq 2ab$

Dividing by ab: $\frac{a^2}{ab}+\frac{b^2}{ab} \geq \frac{2ab}{ab}$

(1) $\therefore \frac{a}{b}+\frac{b}{a} \geq 2$ QED

ii) From above: $p^2+q^2 \geq 2pq$

$q^2+r^2 \geq 2qr$

$r^2+p^2 \geq 2rp$

Adding: $2p^2+2q^2+2r^2 \geq 2(pq+qr+rp)$

(2) $\therefore p^2+q^2+r^2 \geq pq+qr+rp$ QED

iii) RTP: $p^3+q^3 \geq pq(p+q)$ $[p,q \geq 0]$

Proof: Consider the difference:

$p^3+q^3-pq(p+q)$

$=(p+q)(p^2-pq+q^2)-pq(p+q)$

$=(p+q)(p^2-pq+q^2-pq)$ (3)

$=(p+q)(p^2-2pq+q^2)$

$=(p+q)(p-q)^2$

Since $p,q \geq 0$, $p+q \geq 0$ and

$(p-q)^2 \geq 0$

$\therefore (p+q)(p-q)^2 \geq 0$

$\therefore p^3+q^3 \geq pq(p+q)$. QED

Q12 Cont'd

b) iv) RTP: $2(p^3+q^3+r^3)$

$\geq pq(p+q)+qr(q+r)+rp(r+p)$

Proof: From (iii)

$p^3+q^3 \geq pq(p+q)$

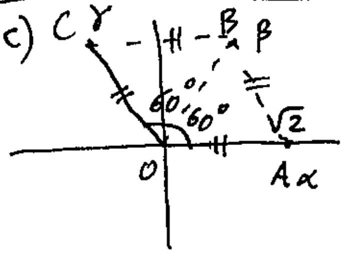
$q^3+r^3 \geq qr(q+r)$ (2)

$r^3+p^3 \geq rp(r+p)$ Adding:

$2(p^3+q^3+r^3) \geq pq(p+q)+qr(q+r)+rp(r+p)$

QED.

c)



$|\alpha|=\sqrt{2}$

$\arg \alpha = 0°$

i) $\gamma = \alpha \times \text{cis}\, 120°$

$=\sqrt{2} \times (\cos 120° + i\sin 120°)$

$=\sqrt{2} \times \left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$ (2)

$=\frac{-1}{\sqrt{2}}+\frac{i\sqrt{3}}{\sqrt{2}}$

or in polar form $\gamma = r\,\text{cis}\,\theta$

$=\sqrt{2}\,\text{cis}\,120°$

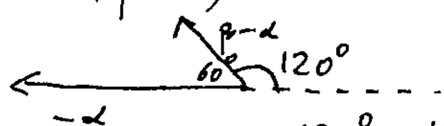
ii) $\vec{OB}=\vec{OA}+\vec{OC}$ or $\beta=\alpha+\gamma$

$=\sqrt{2}+0i+\frac{-1}{\sqrt{2}}+\frac{i\sqrt{3}}{\sqrt{2}}$

$=\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)+i\frac{\sqrt{3}}{\sqrt{2}}$. (2)

$=\frac{1}{\sqrt{2}}+\frac{i\sqrt{3}}{\sqrt{2}}$

iii) $\arg\left(\frac{-\alpha}{\beta-\alpha}\right)=\arg(-\alpha)-\arg(\beta-\alpha)$



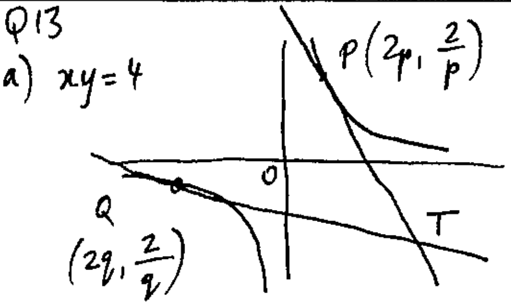
$=180°-120°$

$=60°$ (1)

Solutions to
Title: 2016 Y12 Ascham Trial Math Ext 2 p3

Q13

a) $xy = 4$



i) $y = \frac{4}{x} = 4x^{-1}$

$y' = -4x^{-2} = \frac{-4}{x^2}$ If $x = 2p$ $y' = \frac{-4}{4p^2}$

$\therefore m = \frac{-1}{p^2}$

$\therefore y - y_1 = m(x - x_1)$

$y - \frac{2}{p} = \frac{-1}{p^2}(x - 2p)$

$p^2 y - 2p = -x + 2p$ (2)

$x + p^2 y = 4p$ QED.

ii) Tangent at Q is: $x + q^2 y = 4q$ · ①

Solve simultaneously: $x + p^2 y = 4p$ ②

② − ① $(p^2 - q^2)y = 4(p - q)$

$p \neq q$ $(p+q)(p-q)y = 4(p-q)$

$\therefore y = \frac{4}{p+q}$

$\Rightarrow x + p^2\left(\frac{4}{p+q}\right) = 4p$

$x = 4p - p^2\left(\frac{4}{p+q}\right)$

$= \frac{4p(p+q) - 4p^2}{p+q}$

$= \frac{4p^2 + 4pq - 4p^2}{p+q}$ (2)

$= \frac{4pq}{p+q}$

$\therefore T$ is $\left(\frac{4pq}{p+q}, \frac{4}{p+q}\right)$.

Q13 cont'd:

a) iii) (α) For T: $x = \frac{4pq}{p+q}$, $y = \frac{4}{p+q}$

$\therefore ypq = x$. (1)

If $y = -x$ then $pq = -1$ $\therefore q = \frac{-1}{p}$.

(β) If never meet then Parallel (1)

$\therefore \frac{-1}{p^2} = -\frac{1}{q^2}$ $\therefore p^2 = q^2$

$\therefore p = \pm q$ but

$p \neq q$ as in different quadrants

$\therefore p = -q$. OR T is undefined

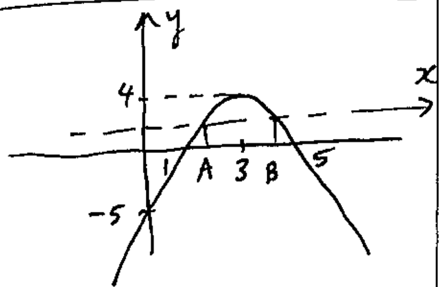
$\therefore p + q = 0$ $\therefore p = -q$.

iv) If perpendicular then $m_p \times m_Q = -1$

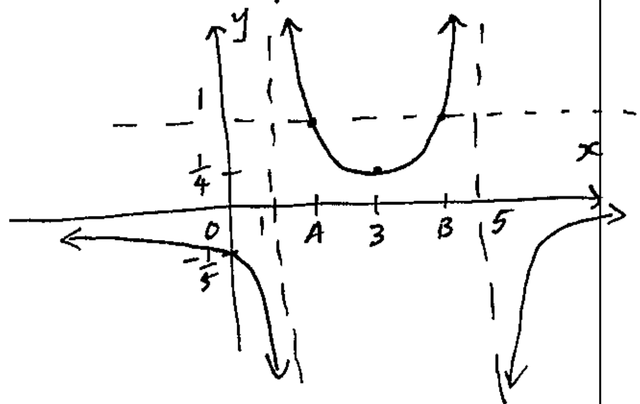
$\therefore \frac{-1}{p^2} \times \frac{-1}{q^2} = -1$ $\therefore p^2 q^2 = -1$.

(1) No solution. $\therefore$ can't be perp.

b) $y = f(x)$



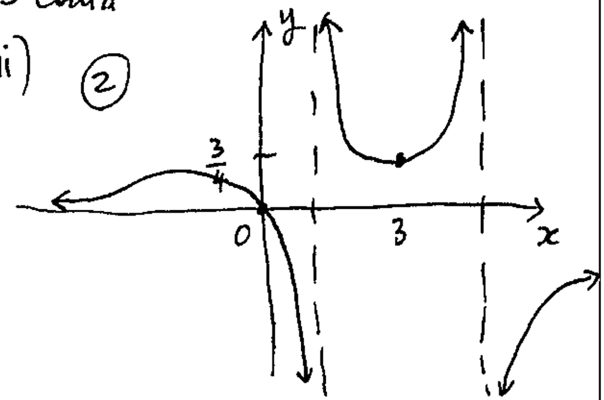
i)



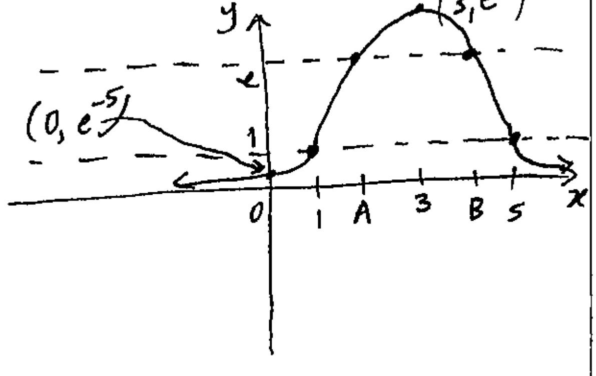
(2)

Q 13 cont'd

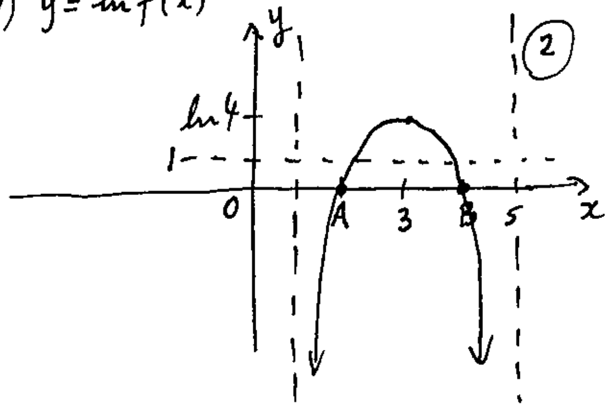
b) ii) (2)



iii) $y = e^{f(x)}$ (2)



iv) $y = \ln f(x)$ (2)



Q14

a) $u_1 = 2$, $u_2 = 22$ $u_n = 6u_{n-1} - 5u_{n-2}$, $n \geq 3$

RTP: $u_n = 5^n - 3$ for $n \geq 1$, $n \in \mathbb{N}$.

Proof: Let $P(n)$ be the proposition that if $u_1 = 2$, $u_2 = 22$ and $u_n = 6u_{n-1} - 5u_{n-2}$ for $n \geq 3$ then $u_n = 5^n - 3$, $n \geq 1$.

Test for $P(1)$: $u_1 = 2$, and $u_1 = 5^1 - 3 = 2$ $\therefore$ $P(1)$ works.

Assume $P(k)$ true for some $k \geq 1$, i.e. $u_1 = 2$, $u_2 = 22$, $u_k = 6u_{k-1} - 5u_{k-2}$ and then $u_k = 5^k - 3$ (& $u_{k-1} = 5^{k-1} - 3$).

RTP: if $u_1 = 2$, $u_2 = 22$ and $u_{k+1} = 6u_k - 5u_{k-1}$ then $u_{k+1} = 5^{k+1} - 3$.

Proof: Consider $u_{k+1} = 6u_k - 5u_{k-1}$

$= 6(5^k - 3) - 5(5^{k-1} - 3)$ using $P(k)$

$= 6.5^k - 18 - 5.5^{k-1} + 15$

$= 6.5^k - 5^k - 3$

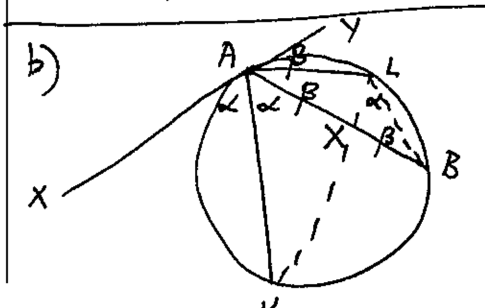
$= 5.5^k - 3$

$= 5^{k+1} - 3$ (3)

$=$ RHS of $P(k+1)$

$\therefore$ $P(n)$ true by Math Induction for all $n \geq 1$, $n \in \mathbb{N}$.

b)



Construct LB, LK

Let X be intersection of LK & AB.

i) $\angle LBA = \beta^\circ$ ($\angle$ in alternate segment $=$ $\angle$ between tangent & chord)

$\therefore LA = LB$ (base $\angle$s equal, ΔALB is isosceles, opposite sides equal.) (2)

ii) $2\alpha + 2\beta = 180^\circ$ ($\angle$s on straight line $= 180^\circ$)

$\therefore \alpha + \beta = 90$

$\therefore \angle LAK = 90^\circ$ (adjacent $\angle$s) (1)

$\therefore$ LK is diameter ($\angle$ in semicircle $= 90^\circ$)

iii) $\angle KLB = \alpha^\circ$ ($\angle$s standing on arc KB are equal at circumference) (1)

$\therefore \angle LXB = 90^\circ$ (supplementary with $\alpha + \beta$ in ΔLXB)

$\therefore LK \perp AB$.

c) $p^6 = 1$

$\therefore p^6 - 1 = 0$

(i) $\therefore (p-1)(p^5 + p^4 + p^3 + p^2 + p + 1) = 0$

$\therefore p = 1$ but 1 is real or (1)

$p^5 + p^4 + p^3 + p^2 + p + 1 = 0$.

$\therefore p^5 + p^4 + p^3 + p^2 + p + 1 = 0$

(ii) $\therefore p^8 = p^2$, $p^9 = p^3$, $p^{16} = p^4$, $p^{17} = p^5$

$\therefore 1 + p + p^8 + p^9 + p^{16} + p^{17}$

$= 1 + p + p^2 + p^3 + p^4 + p^5$ (2)

$= 0$

iii) $z^6 - 1 = (z^2 - 1)(z^4 + z^2 + 1)$ (2)

$= (z-1)(z+1)(z^4 + z^2 + 1)$

or $= (z^3 - 1)(z^3 + 1)$

$= (z-1)(z^2 + z + 1)(z+1)(z^2 - z + 1)$

iv) Equating in p: $\therefore p^4 + p^2 + 1 = (p^2 + p + 1) \times (p^2 - p + 1)$ (1)

v) $\therefore$ factors of $z^2 \pm z + 1$ or roots:

(2) $z = \dfrac{\mp 1 \pm \sqrt{(\pm 1)^2 - 4(1)(1)}}{2} = \dfrac{\mp 1 \pm i\sqrt{3}}{2}$

Solutions to
Title: 2016 Y12 Ascham Math Ext 2 p6 Trial

Q15

a) $y = \cos(\sin^{-1}x)$ Let $\alpha = \sin^{-1}x$

$\therefore -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

$-1 \le x \le 1$

$\therefore \sin\alpha = x$

$\therefore y = \cos\alpha$

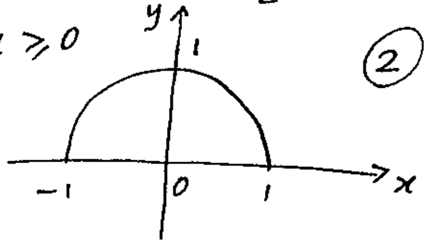
$\therefore \cos^2\alpha + \sin^2\alpha = 1$

$\therefore x^2 + y^2 = 1$

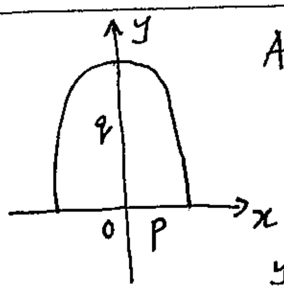
but $\cos\alpha \ge 0$ for $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

$\therefore y \ge 0$ (2)

$\therefore$



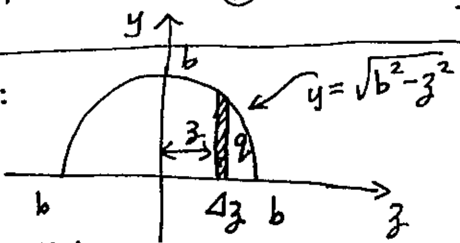
b)



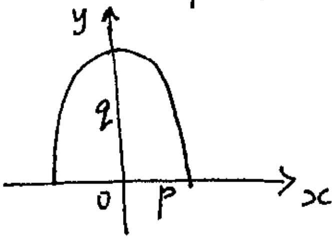
Area of an ellipse $= \pi ab$

$\therefore$ Area $= \frac{1}{2} pq\pi$. (1)

c) Front:



i) Slice:

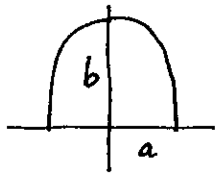


$\text{Area}_{1\,slice} = \frac{\pi pq}{2}$

$\text{Vol}_{1\,slice} = \frac{\pi pq}{2}\Delta z$

In terms of z: $q = \sqrt{b^2 - z^2}$

Similar slice to middle:



$\therefore \frac{p}{a} = \frac{q}{b}$

$\therefore p = \frac{aq}{b}$

$\therefore \text{Vol}_{1\,slice} = \frac{\pi}{2}\frac{aq}{b}\sqrt{b^2 - z^2}\,\Delta z$

$= \frac{\pi a}{2b}\left(\sqrt{b^2 - z^2}\right)^2 \Delta z$

c) Cont'd (i)

$\therefore V_{1\,slice} = \frac{\pi a}{2b}(b^2 - z^2)\Delta z$. (2)

(ii) $\therefore \text{Vol} = \lim_{\Delta z \to 0} 2\sum_{0}^{b} \frac{\pi a}{2b}(b^2 - z^2)\Delta z$

$= 2\int_0^b \frac{\pi a}{2b}(b^2 - z^2)\,dz$

$= \frac{\pi a}{b}\left[b^2 z - \frac{z^3}{3}\right]_0^b$

$= \frac{\pi a}{b}\left[b^3 - \frac{b^3}{3} - (0 - 0)\right]$

$= \frac{\pi a}{b} \times \frac{2b^3}{3}$

$= \frac{2\pi ab^2}{3}\ u^3$ (2)

d) $F = ma$ but $m = 1$

P → 0 (upward): $\ddot{x} = -\beta v - g$

0 → Q (downward): $\ddot{x} = -\beta u + g$

i) $\ddot{x} = -(\beta v + g)$

$\frac{dv}{dt} = -(\beta v + g)$

$\frac{dv}{-(\beta v + g)} = dt$ (3)

$-\int_V^v \frac{dv}{\beta v + g} = \int_0^t dt$

$\left[-\frac{1}{\beta}\ln|\beta v + g|\right]_V^v = t - 0$

$-\frac{1}{\beta}\left[\ln|\beta v + g| - \ln|\beta V + g|\right] = t$

$\therefore t = \frac{1}{\beta}\ln\left|\frac{\beta V + g}{\beta v + g}\right|$. QED.

P.T.O.

Q15 cont'd:

ii) $\ddot{x} = g - \beta u$

$\frac{du}{dt} = g - \beta u$

$\frac{du}{g - \beta u} = dt$

$\int_0^u \frac{du}{g-\beta u} = \int_0^t dt$

$-\frac{1}{\beta}\left[\ln|g-\beta u|\right]_0^u = t - 0$

$-\frac{1}{\beta}\left[\ln|g-\beta u| - \ln|g-0|\right] = t$

$\therefore t = \ln\left|\frac{g}{g-\beta u}\right|$ (2)

iii) P and Q meet at time T:

$\therefore \frac{1}{\beta}\ln\left|\frac{\beta V+g}{\beta v+g}\right| = \frac{1}{\beta}\ln\left|\frac{g}{g-\beta u}\right|$

$\therefore \frac{\beta V+g}{\beta v+g} = \frac{g}{g-\beta u}$ Noting both >1 so $\ln|\,|>0$

$\therefore g - \beta u = \frac{g(\beta v+g)}{\beta V+g}$ (3)

$\beta u = g - \frac{g(\beta v+g)}{\beta V+g}$

$= \frac{g(\beta V+g) - g(\beta v+g)}{\beta V+g}$

$= \frac{g\beta V + g^2 - g\beta v - g^2}{\beta V+g}$

$\therefore u = \frac{g\beta(V-v)}{\beta(\beta V+g)}$ QED.

$= \frac{g(V-v)}{\beta V+g}$.

Q16

a) $\int x^n e^x dx = uv - \int v\,du$ Let $u = x^n$, $du = nx^{n-1}dx$, $dv = e^x dx$, $v = e^x$

$= x^n e^x - \int e^x . nx^{n-1} dx$

$= x^n e^x - n\int x^{n-1} e^x dx$

QED.

$\therefore \int x^3 e^x dx = x^3 e^x - 3\int x^2 e^x dx$

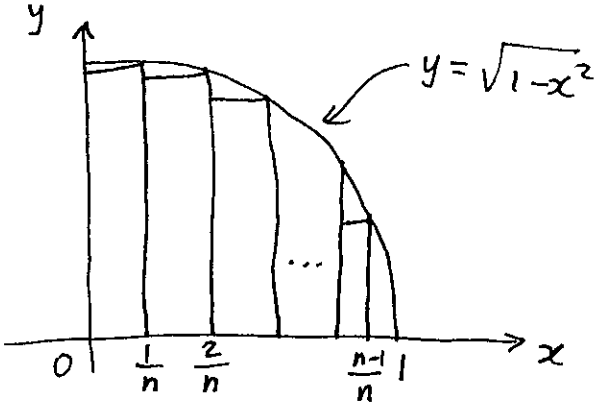
$= x^3 e^x - 3\left[x^2 e^x - 2\int x e^x dx\right]$ (2)

$= x^3 e^x - 3x^2 e^x + 6\int x e^x dx$

$= x^3 e^x - 3x^2 e^x + 6\left[x e^x - 1\int x^0 e^x dx\right]$

$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$

b)



i) Area(1) $= lb$

$= \sqrt{1-\left(\frac{1}{n}\right)^2} \times \frac{1}{n}$

$= \sqrt{1-\frac{1}{n^2}} \times \frac{1}{n}$

$= \sqrt{\frac{n^2-1}{n^2}} \times \frac{1}{n}$

$= \frac{1}{n^2}\sqrt{n^2-1}$ (2)

Area(2) $= lb$

$= \sqrt{1-\left(\frac{2}{n}\right)^2} \times \frac{1}{n}$

$= \frac{1}{n^2}\sqrt{n^2-4}$

⋮

Area(n-1) $= \sqrt{1-\left(\frac{n-1}{n}\right)^2} \times \frac{1}{n}$

$= \frac{1}{n^2}\left(\sqrt{n^2-(n-1)^2}\right)$

$A(n) = 0$.

$\therefore A_n = A(1) + A(2) + A(3) + \ldots + A(n)$

$= \frac{1}{n^2}\sqrt{n^2-1} + \frac{1}{n^2}\sqrt{n^2-2^2} + \ldots + \frac{1}{n^2}\sqrt{n^2-(n-1)^2}$

$= \frac{1}{n^2}\left[\sqrt{n^2-1} + \sqrt{n^2-2^2} + \ldots + \sqrt{n^2-(n-1)^2}\right]$

ii) As $n \to \infty$ the area $\to \frac{1}{4}$ circle from underneath

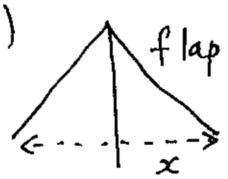
$\therefore 0 \le A_n$ as $n \to \infty \le \frac{1}{4}\pi r^2$

$\therefore \lim_{n\to\infty} \frac{1}{n^2}\left(\sqrt{n^2-1} + \sqrt{n^2-2^2} + \ldots\right) \le \frac{1}{4}\pi \times 1^2$

$\le \frac{\pi}{4}$

QED (2)

c)



$x = 15e^{-t}\cos t$, $t \ge 0$

$15e^{-t} > 0$

i) $-1 \le \cos t \le 1$ (1)

$\therefore -15e^{-t} \le 15e^{-t}\cos t \le 15e^{-t}$

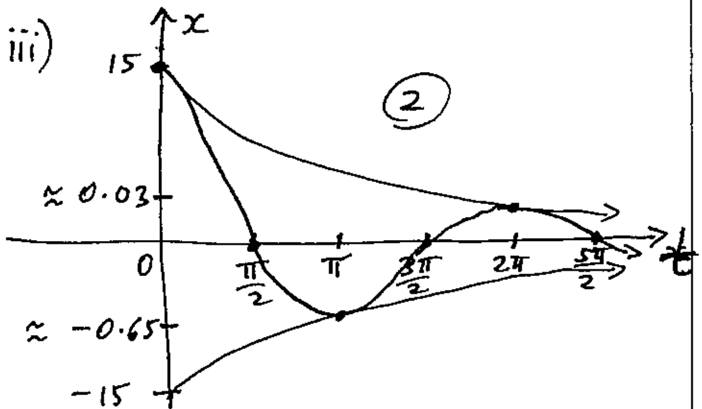
$\therefore$ Graphs $x = -15e^{-t}$ and $x = 15e^{-t}$.

ii) $e^{-t}\cos t = 0$ for $t \ge 0$ (2)

$\therefore \cos t = 0$

$t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ sec.

iii) (2)



When $t = \pi$ $x = 15e^{-\pi}\cos\pi$

$= -15e^{-\pi} \doteqdot -0.648$

$t = 2\pi$ $x = 15e^{-2\pi}\cos 2\pi$

$= 15e^{-2\pi} \doteqdot 0.028\ldots$

Solutions to
Title: 2016 Y12 Ascham Math Ext 2 Trial p9

c) iv) Q16 cont'd

Choose t such that $0.04 \leq x \leq 0.04$.

By graph $x \approx \pm 0.04$ before $t = \frac{3\pi}{2}$.

If $t = \frac{5\pi}{4}$ $x = 15e^{-\frac{5\pi}{4}} \cos\frac{5\pi}{4}$

$= -0.2089\ldots$

Too big. Try closer to $\frac{3\pi}{2}$ $(4.712\ldots)$

$t = 4.6$ $x =$ too small

$t = 4.3$ $x = -0.08\ldots$

$t = 4.4$ $x = -0.0565\ldots$ OK.

Let $f(t) = 15e^{-t}\cos t - 0.04$

$f'(t) = u'v + v'u$

$= -15e^{-t}\cos t + -\sin t . 15e^{-t}$

$= -15e^{-t}(\cos t + \sin t)$

Using Newton's method with $t_1 = 4.4$

$$t_2 = t_1 - \frac{f(t_1)}{f'(t_1)}$$

$$= 4.4 - \frac{15e^{-4.4}\cos 4.4 - 0.04}{-15e^{-4.4}(\cos 4.4 + \sin 4.4)}$$

(4)

$$= 4.4 - \frac{-0.09658\ldots}{0.2318455\ldots}$$

$= 4.8166\ldots$ over

Try $t = 4.5$?