Barker College

Trial 1999

Time 3 hrs.

Question 1 (Start a new page) **MARKS**

a) Find 3

i) $\int \frac{4}{x^2+4}\,dx$

ii) $\int \frac{4}{\sqrt{x^2+4}}\,dx$

iii) $\int \frac{4x}{\sqrt{x^2+4}}\,dx$

b) Evaluate 8

i) $\int_0^{\frac{\pi}{2}} e^x \cos x\,dx$

ii) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$

c)

i) Find polynomials $p(x), q(x)$ of degrees less than 2, such that $(x+2)p(x)+(x^2+4)q(x)=1$. 4

ii) Hence evaluate $\int_0^2 \frac{8dx}{(x+2)(x^2+4)}$.

Question 2 (Start a new page) **MARKS**

a) If $z = \dfrac{3+2i}{1-2i}$ then find **3**

i) $\bar{z}$

ii) $\arg \bar{z}$

b) **5**

i) Express $\sqrt{6i-8}$ in the form $a+ib$ where a,b are elements of the set of reals.

ii) Hence solve $2z^2-(3+i)z+2=0$ for z. Express your answer in the form $a+ib$.

c) Neatly sketch each of the following loci on separate Argand Diagrams. **4**

$\arg\left(\frac{z+1}{z-i}\right) = \frac{x+1+iy}{x+i(y-1)}$

i) $\arg\dfrac{z+1}{z-i} = \dfrac{2\pi}{3}$

ii) $z\bar{z} = z + \bar{z}$

$z\bar{z} - z - \bar{z} = 0$

d) **3**

i) Show on an Argand diagram the locus of z where $|z-4-3i| = 1$.

ii) What are the least values of $|z|$.

3

Question 3 (Start a new page) **Marks**

a)

i) Sketch $y = f(x)$, clearly labelling all essential features given that $f(x) = x^3 - 4x$. 10

On separate diagrams sketch showing labelling all essential features

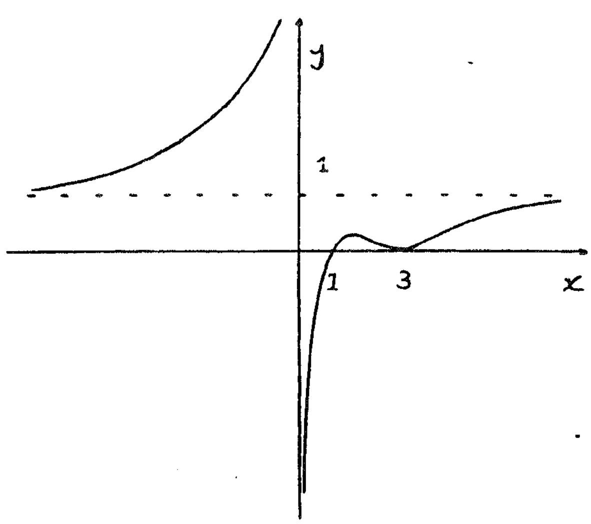
ii) $y^2 = f(x)$

iii) $y = f(\frac{1}{x})$

iv) $y = e^{f(x)}$

v) $|y| = |f(x)|$

b) 5



The diagram above is of the derivative of $y = f(x)$. i.e. The curve has equation $y = f'(x)$.

i) Sketch the function $y = f''(x)$.

ii) On a separate diagram sketch a possible graph of $y = f(x)$.

iii) Suggest a possible equation for $y = f'(x)$ in terms of x.

4

Question 4 (Start a new page) **MARKS**

a) **8**

Show that the normal to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ bisects the angle between the lines $x = 2ap$ and SP where S is the focus of the parabola.

b) **7**

i) Sketch the hyperbola with equation $\frac{x^2}{4} - \frac{y^2}{2} = 1$, carefully labelling all essential features.

ii) Show that the equation of the tangent to this hyperbola at $P(2\sec\theta, \sqrt{2}\tan\theta)$ is given by $\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{2}} = 1$.

iii) Hence prove that the area of the triangle bounded by this tangent and the asymptotes of the hyperbola is independent of the position of P.

Question 5 (Start a new page) **MARKS**

a) **7**

i) Prove that the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has a magnitude of πab.

ii) Find the volume of a mound with a circular base of equation $x^2+y^2=4$ which has semi-elliptical cross-sections parallel to the y axis, where the ratio of the major axis : minor axis = 2 : 1. The height of each cross-section is the length of the semi-minor axis.

b) **8**

i) Sketch the curve $y=x^2(x^2-1)$ shading the region bounded by the curve and the x-axis.

ii) Find the volume of the solid formed when this shaded area in part i) is rotated about the y-axis.

iii) What is the volume of the solid formed when the area encompassed by the relation $y^2=x^8-2x^6+x^4$ is rotated about the y-axis?

Question 6 (Start a new page) **MARKS**

a) Show that $1+i$ is a root of the polynomial $P(x)=x^3+x^2-4x+6$ and hence completely factorize $P(x)$ over the field of complex numbers. **3**

b) **4**

i) If the polynomial $P(x)=x^4-4x^3+11x^2-14x+10$ has roots of the form $a+ib$ and $a-2ib$ where a, b are real, find the values of a and b.

ii) Find all the zeros of $P(x)$.

iii) Express $P(x)=x^4-4x^3+11x^2-14x+10$ as a product of two quadratic factors with rational coefficients.

c) **5**

i) Prove that if the polynomial $P(x)$ has a root α of multiplicity m then $P'(x)$ has a root α of multiplicity $m-1$.

ii) Given that the polynomial $P(x)=x^4+x^3-3x^2-5x-2$ has a root of multiplicity 3, find all the roots of $P(x)$.

d) If α,β,γ are the roots of the equation $x^3+qx+r=0$, prove that $(\beta-\gamma)^2+(\alpha-\beta)^2+(\alpha-\gamma)^2=-6q$. **3**

7

Question 7 (Start a new page) **MAR!**

a) If $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}}\,dx$ where $n = 0,1,2,3.......$ **7**

i) Show that $x^{n-1}\sqrt{x+1} = \frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}}$.

ii) Show that $(2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$ for $n = 1,2,3,......$

iii) Evaluate $\int_0^1 \frac{x^2}{\sqrt{x+1}}\,dx$.

b) **8**

i) Sketch on an argand diagram the roots of $z^5 - 1 = 0$.

ii) Show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

iii) Hence or otherwise find the exact values of $\cos\frac{2\pi}{5}$ and $\cos\frac{\pi}{5}$.

Question 8 (Start a new page) **MARKS**

a) Prove that if the opposite angles of a quadrilateral are supplementary then the quadrilateral must be cyclic. **4**

b) **5**

i) Show that $\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a+b}{1-ab}$.

ii) Simplify $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c$.

iii) If the equation $x^3 - 2x^2 + 3x + 4 = 0$ has roots α, β, γ show that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = \frac{\pi}{4}$.

c) **6**

i) Show that $f(x) = \frac{\sec x + \tan x}{2\sec x + 3\tan x}$ is a decreasing function in term of x for the domain $0 < x < \frac{\pi}{2}$.

ii) Deduce that $\frac{\pi}{28} > \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec x + \tan x}{2\sec x + 3\tan x} dx > \left(\sqrt{2} - 1\right)\frac{\pi}{12}$.

END OF PAPER

4 UNIT TRIAL 1999

1(a) (i) $\int \frac{4}{x^2+4}\,dx = 4 \cdot \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

$= 2\tan^{-1}\frac{x}{2} + c$

(ii) $\int \frac{4}{\sqrt{x^2+4}}\,dx = 4\log_e\left(x+\sqrt{x^2+4}\right)+c$

(iii) $\int \frac{4x}{\sqrt{x^2+4}} = \frac{2(x^2+4)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 4\sqrt{x^2+4} + c$

(b) (i) $\int_0^{\frac{\pi}{2}} e^x\cos x\,dx = \left[e^x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x\sin x\,dx$

$= e^{\frac{\pi}{2}} - \left\{\left[-e^x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x\cos x\,dx\right\}$

$2\int_0^{\frac{\pi}{2}} e^x\cos x\,dx = e^{\frac{\pi}{2}} - 1$

$\therefore \int_0^{\frac{\pi}{2}} e^x\cos x\,dx = \frac{e^{\frac{\pi}{2}}-1}{2}$

(ii) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{2}{2+\frac{1-u^2}{1+u^2}}\cdot\frac{du}{u^2+1}$

Let $u = \tan\frac{\theta}{2}$

$du = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$

$\therefore d\theta = \frac{2\,du}{u^2+1}$

$= \int_0^1 \frac{2}{2u^2+2+1-u^2}\,du$

$= \int_0^1 \frac{2}{u^2+3}\,du$

$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}\right]_0^1$

$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$

$= \frac{\pi}{3\sqrt{3}}$ or $\frac{\sqrt{3}\pi}{9}$

(c) $(x+2)\,p(x) + (x^2+4)\,q(x) = 1$

$\Rightarrow \frac{p(x)}{x^2+4} + \frac{q(x)}{x+2} = \frac{8}{(x+2)(x^2+4)}$

Let $p(x) = bx + c$ and $q(x) = a$

$(a+b)x^2 + (2b+c)x$

$a = 1,\ b = -1,$

$\int_0^2 \frac{8}{(x+2)(x^2+4)}\,dx = \int_0^2$

Q2

a) i) $Z = \frac{3+2i}{1-2i} \times \frac{1+2i}{1+2i}$

$= \frac{-1+8i}{5}$

$\therefore \bar{Z} = \frac{-1-8i}{5}$

ii) $\arg z = -\tan^{-1} 8 + \pi$

b) i) Let $(a+ib)^2 = 6i-8$

$a^2-b^2+2abi = 6i-8$

$a^2-b^2=-8$, $2ab=6 \Rightarrow ab=3$

$a^2 - \frac{9}{a^2} = -8$

or $a^4+8a^2-9=0$

$(a^2-1)(a^2+9)=0$, $a \in \mathbb{R}$

$\therefore a = \pm 1$. $b = \pm 3$

$\therefore \sqrt{6i-8} = \pm(1+3i)$

ii) $2z^2-(3+i)z+2=0$

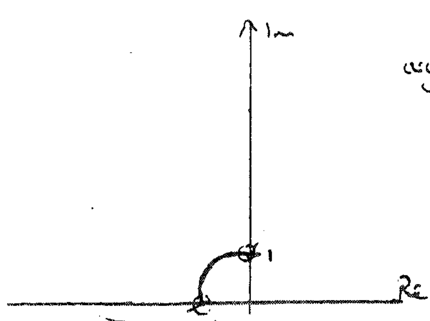
$z = \frac{3+i \pm \sqrt{(3+i)^2-16}}{4}$

$= \frac{3+i \pm \sqrt{6i-8}}{4}$

$= \frac{3+i \pm (1+3i)}{4}$

$z = 1+i , \frac{1-i}{2}$

c) i)



$\arg(z+1) > \arg(z-i)$

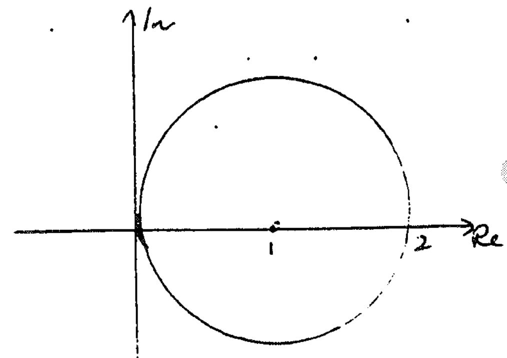
c (ii) $z = a+ib$

$\therefore (a+ib)(a-ib) = a+ib+a-ib$

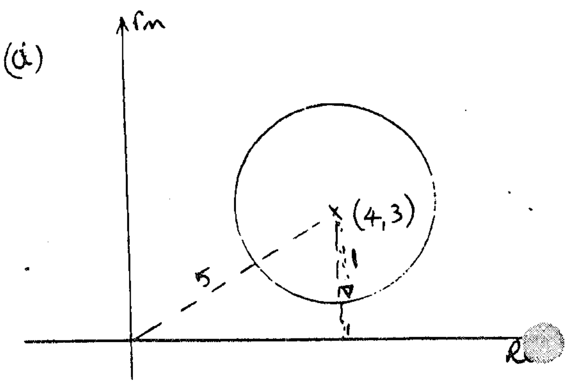
$a^2-b^2 = 2a$

$a^2-2a-b^2=0$

$(a-1)^2-b^2=1$



(d)



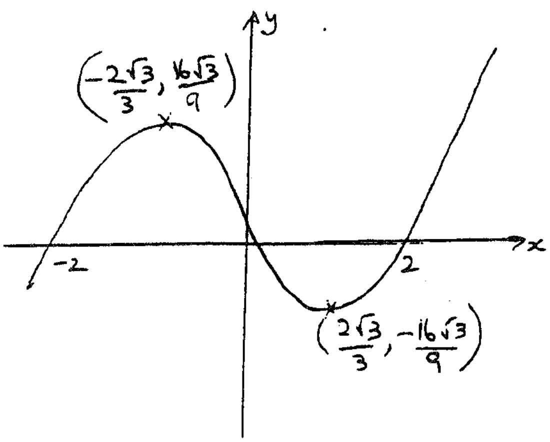
Least value of $|z| = 4$

3. (i) $f(x) = x^3 - 4x$

$f(x) = x(x-2)(x+2)$

$f'(x) = 3x^2 - 4$

$= (\sqrt{3}x - 2)(\sqrt{3}x + 2)$



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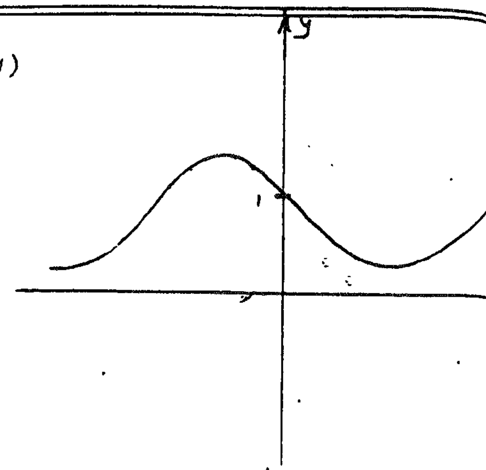

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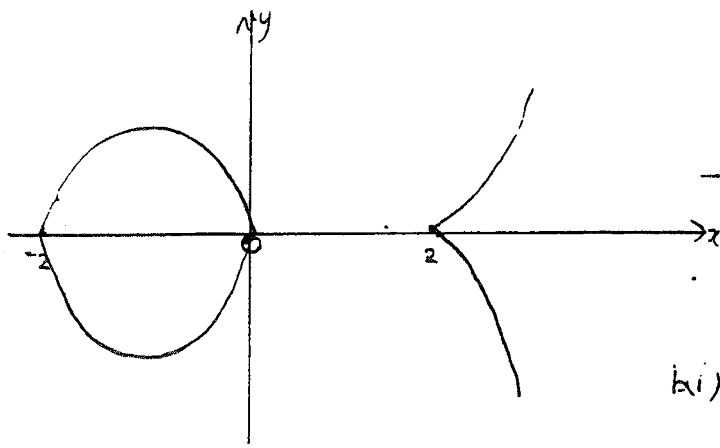
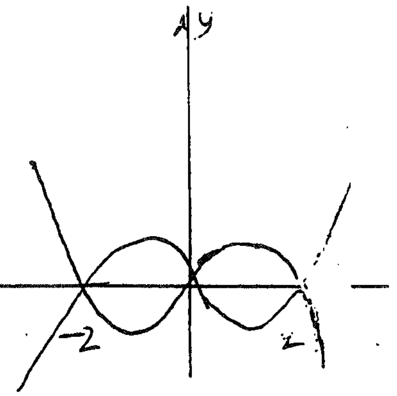
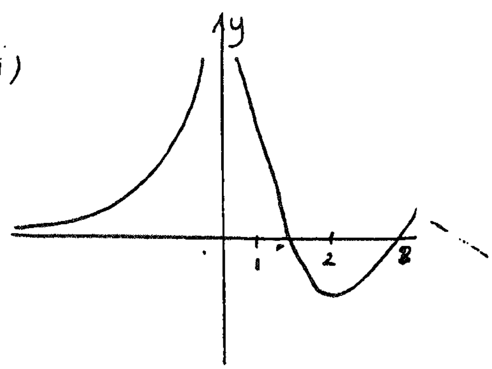




b(i)



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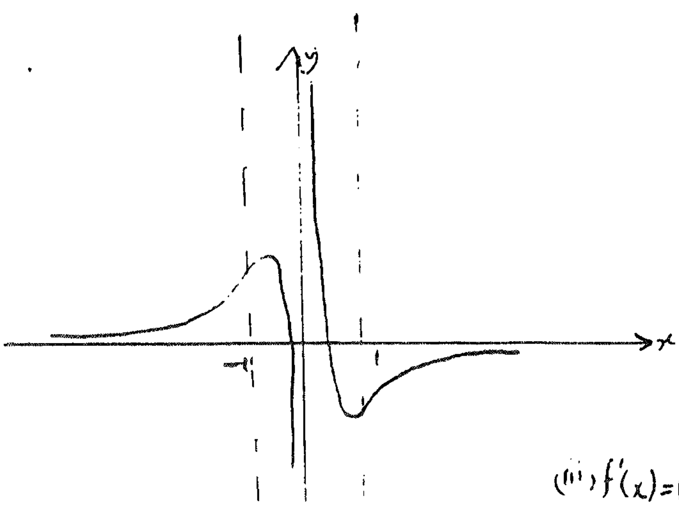

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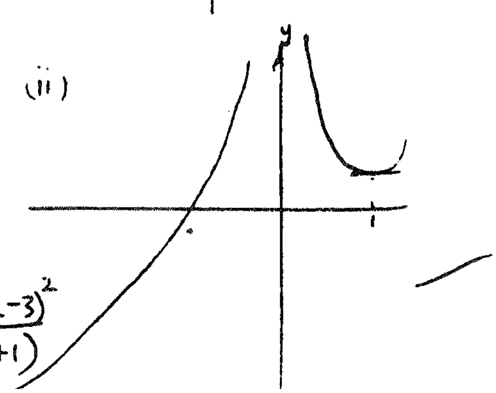
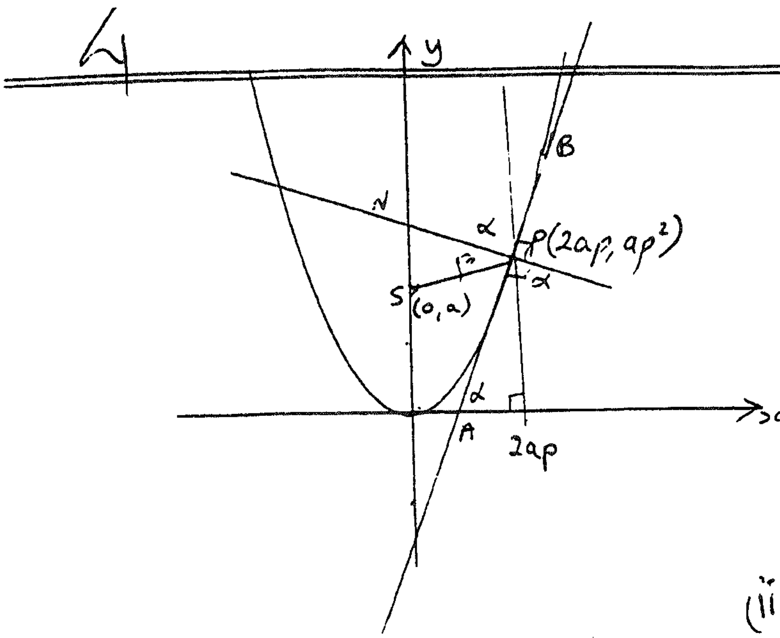


(iii) $f'(x) = \frac{(x-1)(x-3)^2}{x(x^2+1)}$

4



$M_{AB} = \tan\alpha = p$

$M_{SP} = \dfrac{ap^2 - a}{2ap}$

$= \dfrac{p^2-1}{2p}$

$M_{PN} = -\dfrac{1}{p}$

$\tan\beta = \left|\dfrac{\frac{p^2-1}{2p} + \frac{1}{p}}{1 - \frac{1}{p}\left(\frac{p^2-1}{2p}\right)}\right|$

$= |p|$

sts $\alpha, \beta < \frac{\pi}{2}$

and $\tan\alpha = \tan\beta$ $\quad \underline{\underline{\alpha = \beta}}$

$\underline{\underline{\text{angle is bisected}}}$

(ii) $\dfrac{2x}{4} - \dfrac{2y}{2} \cdot \dfrac{dy}{dx} = 0$

$\therefore \dfrac{dy}{dx} = \dfrac{x}{2y}$

when $x = 2\sec\theta \quad y = \sqrt{2}\tan\theta$

$\therefore \dfrac{dy}{dx} = \dfrac{\sec\theta}{\sqrt{2}\tan\theta}$

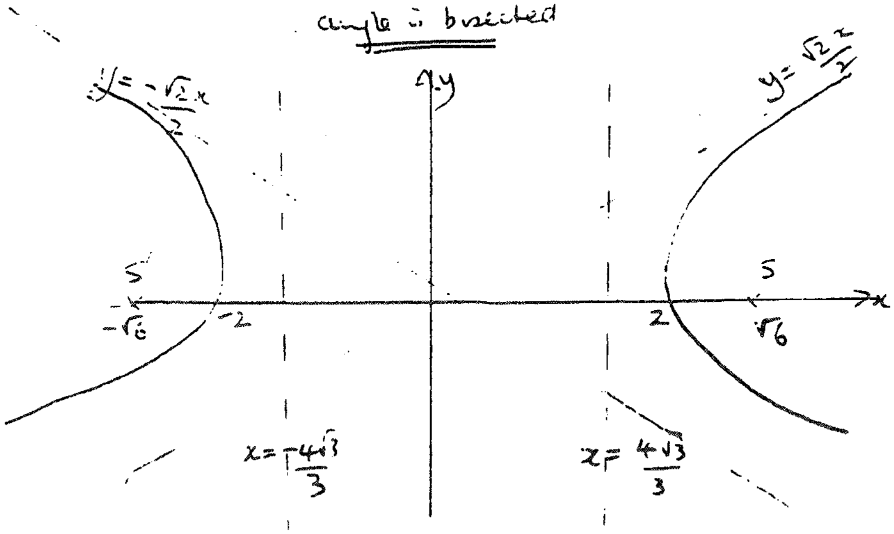
$\therefore y - \sqrt{2}\tan\theta = \dfrac{\sec\theta}{\sqrt{2}\tan\theta}$

$x\sec\theta - \sqrt{2}\tan\theta \cdot y = 2\sec^2\theta - 2\tan$

$= 2$

$\therefore \dfrac{x\sec\theta}{2} - \dfrac{y\tan\theta}{\sqrt{2}} = 1$

is equation of tangent



4(iii) Find the points of intersection at asymptotes and tangent

$\therefore y = \frac{\sqrt{2}}{2}x$

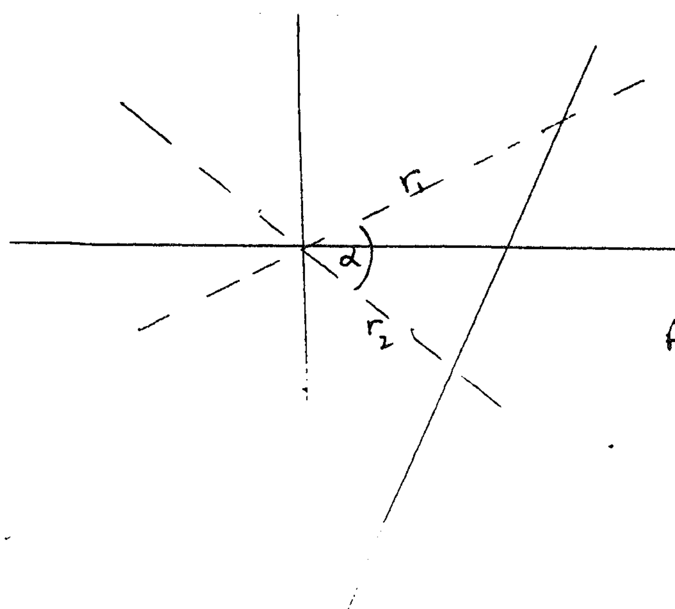
$\therefore \frac{x\sec\theta}{2} - \frac{x\tan\theta}{2} = 1$

$\therefore x = \frac{2}{\sec\theta - \tan\theta}, \quad y = \frac{\sqrt{2}}{\sec\theta - \tan\theta}$

for $y = -\frac{\sqrt{2}}{2}x$

$\frac{x\sec\theta}{2} + \frac{x\tan\theta}{2} = 1$

$\therefore x = \frac{2}{\sec\theta + \tan\theta}, \quad y = \sec\theta - \tan\theta$



Area $\Delta = \frac{1}{2}r_1 r_2 \sin\alpha$
where α is constant

Now $r_1^2 = \left(\frac{2}{\sec\theta - \tan\theta}\right)^2 + \left(\frac{\sqrt{2}}{\sec\theta - \tan\theta}\right)^2$

$\therefore r_1 = \frac{\sqrt{6}}{\sec\theta - \tan\theta}$ and $r_2 = \frac{\sqrt{6}}{\sec\theta + \tan\theta}$

Area $= \frac{1}{2}\frac{\sqrt{6}}{\sec\theta - \tan\theta} \cdot \frac{\sqrt{6}}{\sec\theta + \tan\theta} \cdot \sin\alpha$

$= \frac{3}{2}\ \frac{1}{\sec^2\theta - \tan^2\theta}\sin\alpha$

$= \frac{3}{2}\sin\alpha$ which is constant

$\therefore$ Area is independent of the position of P

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(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

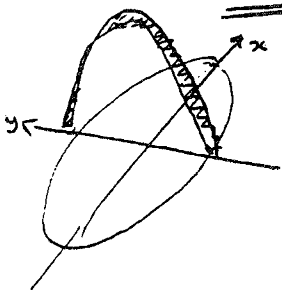
$\therefore y=\frac{b}{a}\sqrt{a^2-x^2}$

$\therefore \text{Area}=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$=4\cdot\frac{b}{a}\cdot\frac{\pi a^2}{4}$

$=\underline{\underline{\pi ab}}$

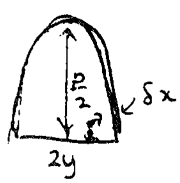
(ii)



$V=\frac{1}{2}\pi a\cdot\frac{b}{2}\,\delta x$

$=\frac{\pi a^2}{8}\,\delta x$

$=\frac{\pi y^2}{2}\,\delta x$



$\therefore V=2\int_0^2\frac{\pi}{2}y^2\,dx$

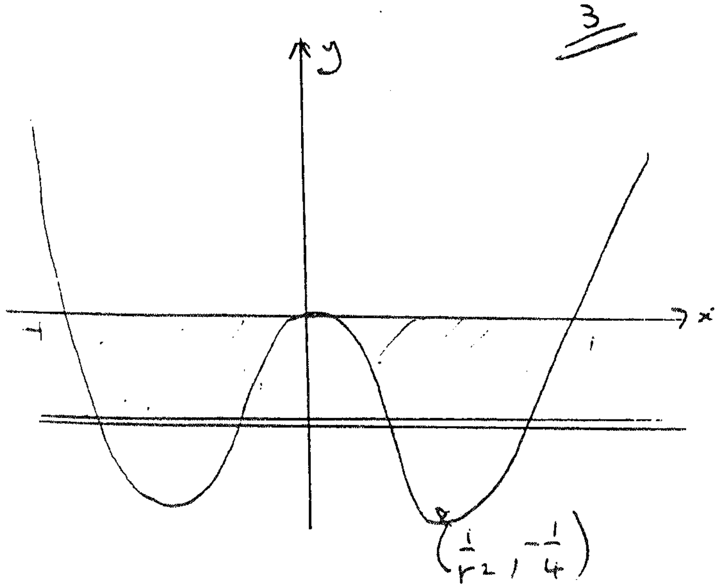
$=\pi\int_0^2 4-x^2\,dx$

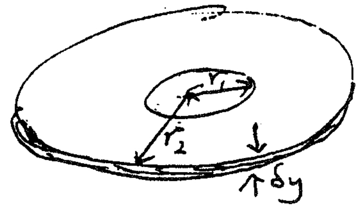
$=\pi\left[4x-\frac{x^3}{3}\right]_0^2$

$=\pi\left[\frac{16}{3}\right]$

$=\underline{\underline{\frac{16\pi}{3}}}$

(b)





Volume of slice $\pi\left(r_2^2-r_1^2\right)\delta y$

As $x^4-x^2-y=0$

$\therefore x^2=\frac{1\pm\sqrt{1+4y}}{2}$

$\therefore r_2^2=\frac{1+\sqrt{1+4y}}{2}$

and $r_1^2=\frac{1-\sqrt{1+4y}}{2}$

$\therefore r_2^2-r_1^2=\sqrt{1+4y}$

$\therefore V=\pi\int_{-\frac{1}{4}}^0\sqrt{1+4y}\,dy$

$=\pi\left[\frac{2}{3\times 4}(1+4y)^{\frac{3}{2}}\right]_{-\frac{1}{4}}^0$

$=\underline{\underline{\frac{\pi}{6}}}$

As $y=\pm x^2(x^2-1)$

Volume is $\frac{2\times\pi}{6}$

$=\underline{\underline{\frac{\pi}{3}\text{ units}^3}}$

(a) $1+i$

$(1+i)^2 = 1+i^2+2i$

$= 2i$

$(1+i)^3 = (1+i)2i$

$= 2i-2$

$P(1+i) = 2i-2+2i-4(1+i)+6$

$=0$

$\therefore 1+i$ is a root.

$\therefore 1-i$ is a root

x^2-2x+2 is a factor

$\therefore P(x) = (x-(1+i))(x-(1+i))(x+3)$

$= (x^2-2x+3)(x+3)$

(b)(i) Sum of roots $= 4 = 4a$

$\therefore a=1$

Product of roots $= 10$ $\therefore (a^2-b^2)(a^2-4b^2) = 10$

$a=1$, $(1-b^2)(1-4b^2) = 0$

$\therefore (4b^2-9)(b^2+1) = 0$

$\therefore b^2 = \frac{9}{4}$

$\therefore b = \pm\frac{3}{2}$

(ii) $\therefore$ Roots are $1+\frac{3}{2}i$, $1-\frac{3}{2}i$, $1-3i$, $1+3i$

(iii) $\therefore$ $P(x) = (x^2-2x-\frac{5}{4})(x^2-2x-8)$

(c)(i) Let $P(x) = (x-\alpha)^m Q(x)$ where $x-\alpha \nmid Q(x)$

$\therefore P'(x) = m(x-\alpha)^{m-1}Q(x) + (x-\alpha)^m Q'(x)$

$= (x-\alpha)^{m-1}[mQ(x) + (x-\alpha)Q'(x)]$

now $x-\alpha \mid (x-\alpha)Q'(x)$ but not $mQ(x)$

$\therefore x-\alpha \nmid [mQ(x) + (x-\alpha)Q'(x)]$

$\therefore$ root α has multiplicity m in $P(x)$

multiplicity $m-1$ in $P'(x)$

(ii)

$P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

$P'(x) = 4x^3 + 3x^2 - 6x - 5$

$P''(x) = 12x^2 + 6x - 6$

$= 6(2x^2 + x - 1)$

$= 6(2x-1)(x+1)$

$\therefore P''(-1) = 0 \quad P''(\tfrac{1}{2}) = 0 \quad P'(-1) = 0$

As $P(-1) = 1 - 1 - 3 + 5 - 2$

$= 0$

$\therefore$ $(x+1)$ is a root of $P(x)$

$\therefore P(x) = (x+1)^3(x-2)$

gives roots $x = -1, 2$

(d)

$2(\alpha^2+\beta^2+\gamma^2) - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 2\left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \gamma\alpha + \beta\gamma)\right] - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 2(\alpha+\beta+\gamma)^2 - 6(\alpha\beta + \alpha\beta + \beta\gamma)$

$= 2 \cdot 0^2 - 6q$

$= -6q$

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(a) i) $\frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}} = \frac{x^n\sqrt{x+1} + x^{n-1}\sqrt{x+1}}{x+1}$

$= \sqrt{x+1}\left[\frac{x^{n-1}(x+1)}{x+1}\right]$

$= \sqrt{x+1}\,.\,x^{n-1}$

(ii) $I_n = \int_0^1 \frac{x^n}{\sqrt{x+2}}\,dx$

$= \left[x^n.\,2\sqrt{x+1}\right]_0^1 - \int_0^1 n x^{n-1}\, 2\sqrt{x+1}\,dx$

$= 2\sqrt{2} - n\int_0^1 x^{n-1}.\,2\sqrt{x+1}\,dx$

$= 2\sqrt{2} - 2n\int_0^1 \frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}}\,dx$

$\therefore I_n(2n+1) = 2\sqrt{2} - 2nI_{n-1}$

(iii) $I_0 = \int_0^1 \frac{1}{\sqrt{x+1}}\,dx$

$= \left[2\sqrt{x+1}\right]_0^1$

$= 2(\sqrt{2}-1)$

$I_1 \cdot (3) = 2\sqrt{2} - 2I_0$

$= 2\sqrt{2} - 2(2\sqrt{2}-2)$

$= 4 - 2\sqrt{2}$

$I_1 = \frac{4}{3} - \frac{2}{3}\sqrt{3}$

$5I_2 = 2\sqrt{2} - 4I_1$ and $3I_1 = 2\sqrt{2} - 2I_0$

$\therefore I_1 = \frac{2}{3}\sqrt{2} - \frac{4}{3}(\sqrt{2}-1)$

$= \frac{4}{3} - \frac{2}{3}\sqrt{2}$

$I_2 = \frac{2}{5}\sqrt{2} - \frac{4}{5}\left(\frac{4}{3} - \frac{2}{3}\sqrt{2}\right)$

$= \frac{14}{15}\sqrt{2} - \frac{16}{15} \qquad = \frac{2}{15}(7\sqrt{2}-5)$

b(iii) $\cos\frac{4\pi}{5} = \cos 2.\frac{2\pi}{5}$

$= 2\cos^2\frac{2\pi}{5} - 1$

Let $\omega = \cos\frac{2\pi}{5}$

$\therefore \omega + 2\omega^2 - 1 = -\frac{1}{2}$

$4\omega^2 + 2\omega - 1 = 0$

$\omega = \frac{-2\pm\sqrt{20}}{8}$

$= \frac{-1\pm\sqrt{5}}{4}$

But $\cos\frac{2\pi}{5} > 0$

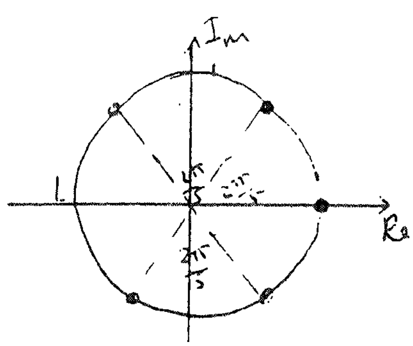
$\therefore \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$

$\cos\frac{4\pi}{5} = 2.\left(\frac{\sqrt{5}-1}{4}\right)^2 - 1$

$= \frac{-1-\sqrt{5}}{4}$

$\therefore \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$

(i)



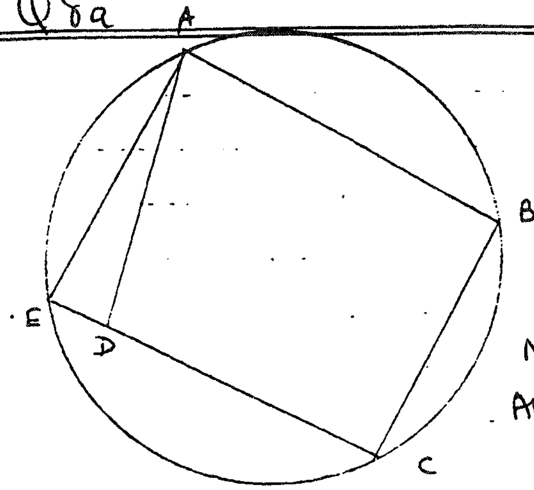
(ii) $\Sigma\,\text{roots} = 1 + \text{cis}\,\frac{2\pi}{5} + \text{cis}\,\frac{4\pi}{5} + \text{cis}\,\frac{-2\pi}{5} + \text{cis}\,\frac{-4\pi}{5}$

$= 1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5}$

But $\Sigma\,\text{roots} = 0$

$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$

Q8a



Draw a circle through ABC and assume it will not pass through D. (ie ABCD is not cyclic!)
Produce CD to E a point on the circle

Now $A\hat{D}C + A\hat{B}C = 180°$ (opp angle quad)
Also $A\hat{E}C + A\hat{B}C = 180°$ (opp angle of cyclic quad)

$\therefore A\hat{E}C = A\hat{D}C$

But $A\hat{E}C$, $A\hat{D}C$ are corresponding angles
$\therefore AE \parallel DA$

But this is not possible as A is common to both
$\therefore$ Assumption is incorrect $\therefore$ ABCD is cyclic

b(i)

$$\tan^{-1}(\tan(\tan^{-1}a - \tan^{-1}b) = \tan^{-1}\left[\frac{\tan(\tan^{-1}a) + \tan(\tan^{-1}b)}{1 - \tan(\tan^{-1}a)\tan(\tan^{-1}b)}\right]$$

$$= \tan^{-1}\left(\frac{a+b}{1-ab}\right)$$

(ii)

$$\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \tan^{-1}\left[\frac{\frac{a+b}{1-ab} + c}{1 - \frac{a+b}{1-ab}\cdot c}\right]$$

$$= \tan^{-1}\left[\frac{a+b+c-abc}{1-(ab+ac+bc)}\right]$$

(iii)

$$\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = \tan^{-1}\left(\frac{2-4}{1-3}\right) = \tan^{-1}\left(\frac{-2}{-2}\right)$$

$$= \frac{\pi}{4}$$

8 (c) (i) $f(x) = \dfrac{\sec x + \tan x}{2\sec x + 3\tan x}$ $\qquad (2\sec x \tan x + 2\sec^2 x)$

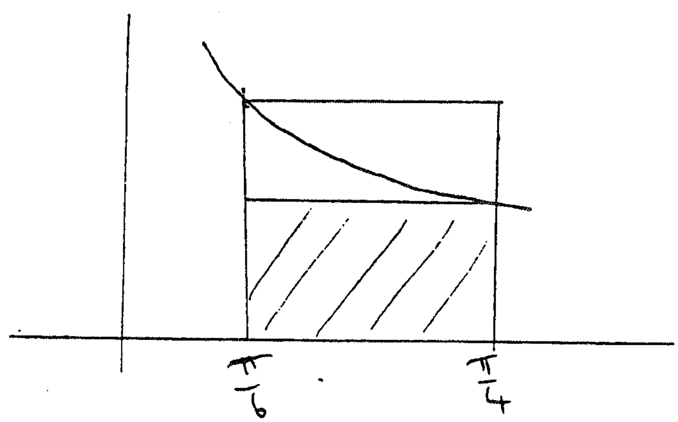
$$f'(x) = \frac{(2\sec x + 3\tan x)(\sec x \tan x + \sec^2 x) - (\sec x + \tan x)(2\sec \ldots + \ldots)}{(2\sec x + 3\tan x)^2}$$

$$= \frac{\sec x\,(\tan^2 x - \sec^2 x)}{(2\sec x + 3\tan x)^2}$$

$$= \frac{-\sec x}{(2\sec x + 3\tan x)^2}$$ now $\sec x > 0$ for $0 \le x < \frac{\pi}{2}$

$\therefore$ $f'(x) < 0$ for $0 \le x < \frac{\pi}{2}$

(ii)



$$A_{upper} = \left(\frac{\pi}{4} - \frac{\pi}{6}\right) \cdot \frac{\sec\frac{\pi}{6} + \tan\frac{\pi}{6}}{2\sec\frac{\pi}{6} + 3\tan\frac{\pi}{6}} = \frac{\pi}{12}\left[\frac{\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{2 \cdot \frac{2}{\sqrt{3}} + 3 \cdot \frac{1}{\sqrt{3}}}\right]$$

$$= \frac{\pi}{28}$$

$$A_{lower} = \left(\frac{\pi}{4} - \frac{\pi}{6}\right) \cdot \frac{\sec\frac{\pi}{4} + \tan\frac{\pi}{4}}{2\sec\frac{\pi}{4} + 3\tan\frac{\pi}{4}} = \frac{\pi}{12}\,\frac{\sqrt{2}+1}{2\sqrt{2}+3} \times \frac{2\sqrt{2}-3}{2\sqrt{2}-3}$$

$$= \frac{\pi}{12} \cdot \frac{4 - 3 + \sqrt{2}}{8 - 9}$$

$$= \frac{\pi}{12}(\sqrt{2} - 1)$$

$$\therefore \frac{\pi}{28} > \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec x + \tan x}{2\sec x + 3\tan x}\,dx > (\sqrt{2} - 1)\,\frac{\pi}{12}$$