Barker College

2001
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

AM WEDNESDAY 8 AUGUST

Total marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages
- Board-approved calculators may be used
- A table of standard integrals is provided on page 10.
- ALL necessary working should be shown in every question

Total marks (120)
Attempt Questions 1 – 8
ALL questions are of equal value

Answer each question on a SEPARATE sheet of paper

Marks

Question 1 [15 marks] [START A NEW PAGE]

(a) Find

$$\int \frac{dx}{x^2 - 16x + 80}$$ **2**

(b) Evaluate

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin\theta}\, d\theta$ **3**

(ii) $\displaystyle\int_0^2 \frac{8\, dx}{(x + 2)(x^2 + 4)}$ **4**

(iii) $\displaystyle\int_0^{\pi} e^x \cos x\, dx$ **3**

(c) Find $\displaystyle\int \frac{2x}{\sqrt{4x - x^2}}\, dx$

You may wish to use the substitution of $u = x - 2$. **3**

Marks

Question 2 [15 marks] [START A NEW PAGE]

(a) (i) Find all real numbers x and y such that $(x + iy)^2 = -3 + 4i$. **2**

(ii) Hence, solve the equation $z^2 - 3z + (3 - i) = 0$. **1**

(b) (i) Express $\sqrt{3} + i$ and $\sqrt{3} - i$ in modulus-argument form. **2**

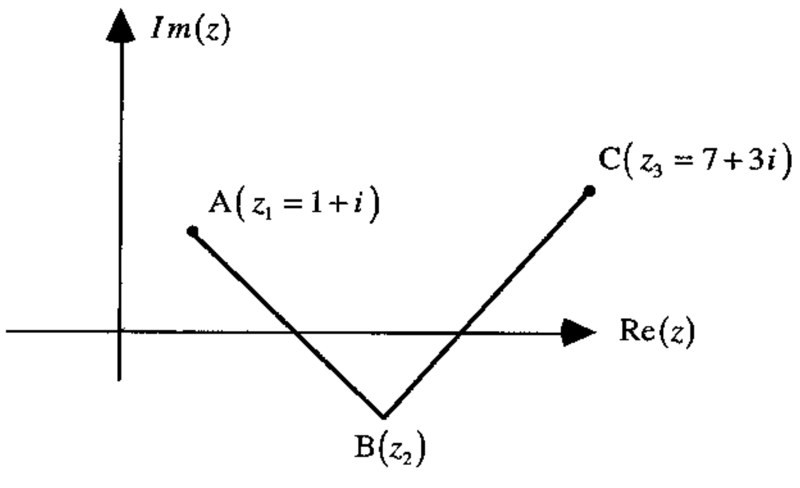
(ii) Hence, simplify $\left(\sqrt{3} + i\right)^{15} + \left(\sqrt{3} - i\right)^{15}$ **1**

(c) Sketch the locus specified by

(i) $|z| \leq |z - 2|$ and $-\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4}$ **3**

(ii) $\text{Re}\left(z - \frac{1}{z}\right) = 0$ (State the equation(s) of the locus). **4**

(d)



The points A and C represent the complex numbers

$z_1 = 1 + i$ and $z_3 = 7 + 3i$

Find the complex number z_2 represented by B such that ΔABC is isosceles and right angled at B. **2**

Marks

Question 3 [15 marks] [START A NEW PAGE]

(a) If $f(x) = (x - 1)(x - 3)$ then sketch

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = f(|x|)$ **2**

(iii) $|y| = f(x)$ **2**

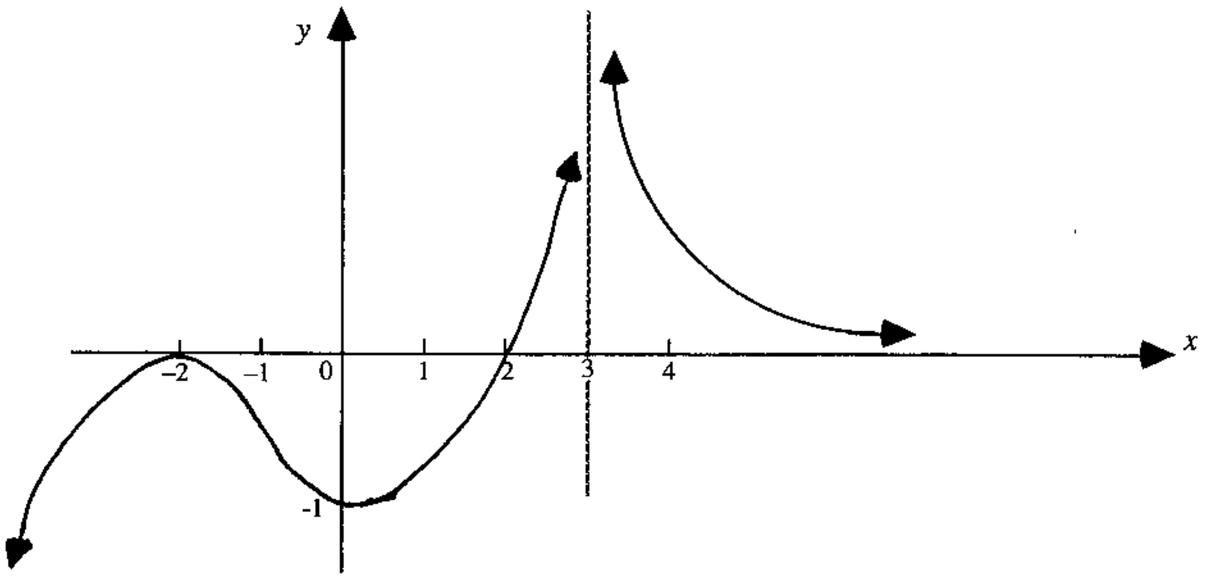
(b) (i) Find the stationary points and the asymptotes of the function

$$y = \frac{(x + 1)^4}{x^4 + 1}$$ **2**

(ii) Sketch this function labelling all essential features. **1**

(iii) Use the graph to find the set of values of k for which $(x + 1)^4 = k(x^4 + 1)$ has two distinct real roots. **2**

(c) Given the graph of $y = f'(x)$ below, sketch the graph of $y = f(x)$. $y = f'(x)$ is the derivative of $y = f(x)$. **4**



Marks

Question 4 [15 marks] [START A NEW PAGE]

(a) An ellipse has the equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$

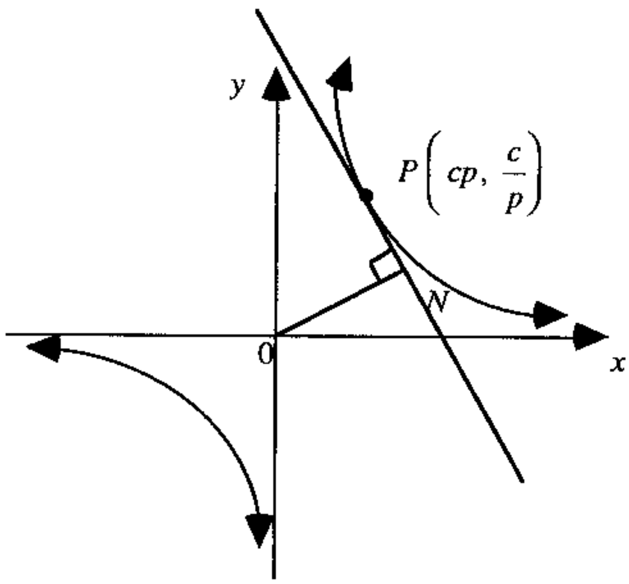
(i) Sketch the ellipse showing the foci S and S' and the directrices. **4**

(ii) Prove that the tangent at the point $P(4\cos\theta,\ 3\sin\theta)$ to the ellipse has the equation $\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$ **3**

(iii) The ellipse meets the y-axis at B and B'. The tangents at B and B' meet the tangent at P at the points Q and Q'. Prove that $BQ \,.\, B'Q' = 16$ **3**

(b) The line through 0 perpendicular to the tangent at $P\left(cp,\ \frac{c}{p}\right)$ on the rectangular hyperbola $xy = c^2$ meets the tangent at N.

Find the coordinates of N and show that as p varies, the locus of N is $\left(x^2 + y^2\right)^2 = 4c^2xy$. **5**



Marks

Question 5 [15 marks] [START A NEW PAGE]

(a) A solid has as its base the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$

If each section perpendicular to the major axis is an equilateral triangle, show that the volume of the solid is $128\sqrt{3}$ units^3. **4**

(b) The region $(x - 2R)^2 + y^2 \leq R^2$ is rotated about the y-axis forming a solid of revolution called a torus.
By summing volumes of cylindrical shells, show that the volume of the torus is $4\pi^2 R^3$ units^3. **6**

(c) The angles of elevation of the top of a tower P measured from three points A, B, C are α, β, γ respectively.
A, B, C are in a straight line such that $AB = BC = a$, but the line AC does not pass through S, the base of the tower.

(i) If $\angle ABS = \theta$, show that

$$(CS)^2 = a^2 + h^2 \cot^2 \beta + 2ah \cot \beta \cos \theta$$ **2**

(ii) Prove that the height of the tower is

$$\frac{a\sqrt{2}}{\left\{\cot^2 \alpha + \cot^2 \gamma - 2\cot^2 \beta\right\}^{\frac{1}{2}}}$$ **3**

Marks

Question 6 [15 marks] [START A NEW PAGE]

(a) Given that a, b and c are the roots of the equation $x^3 + qx + r = 0$, find the cubic equation in y, in terms of q and r, whose roots are $(b + c - 2a)$, $(c + a - 2b)$ and $(a + b - 2c)$ **3**

(b) (i) Using $\tan 3\theta = \tan(2\theta + \theta)$, show that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$ **2**

(ii) Find the value of x for which $3\tan^{-1}x = \frac{\pi}{2} - \tan^{-1}(3x)$ where $\tan^{-1}x$ and $\tan^{-1}(3x)$ both lie between 0 and $\frac{\pi}{2}$ **4**

(c) (i) Using mathematical induction, prove that

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+1)(n+4)}{4(n+2)(n+3)}$$ **4**

(ii) Hence, find $\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)}$ **2**

Marks

Question 7 [15 marks] [START A NEW PAGE]

(a) (i) Given that $\int_0^{\pi} \sin^n\theta \, d\theta = I_n$, prove that

$nI_n = (n-1)I_{n-2}$ **3**

(ii) Hence, evaluate I_8 **1**

(iii) Use the result $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ to show that

$\int_0^{\pi} x\sin^n x \, dx = \left(\frac{\pi}{2}\right) I_n$ **2**

(b) (i) Use De Moivre's Theorem to prove that, if $2\cos\theta = x + \frac{1}{x}$,

then $2\cos n\theta = x^n + \frac{1}{x^n}$ **1**

(ii) Hence, or otherwise, solve the equation

$5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0$ **4**

(c) At the ends of three successive seconds, the distances of a point moving with Simple Harmonic Motion from its mean position, measured in the same direction, are 1, 5 and 5 metres.

Show that the period of the complete oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\frac{3}{5}\right)}$ seconds. **4**

Marks

Question 8 [15 marks] [START A NEW PAGE]

(a) A ball thrown from a point P with velocity V, at an inclination α to the horizontal reaches a point Q after t seconds.
Show that if PQ is inclined at θ to the horizontal, (where $\alpha > \theta$), then the direction of motion of the ball, when at Q, is inclined to the horizontal at an acute angle of $\tan^{-1}[2\tan\theta - \tan\alpha]$.
You may use the result without proof

$$x = V\cos\alpha \times t$$

$$y = V\sin\alpha \times t - \frac{1}{2}gt^2$$ **4**

(b) (i) A gun fires shells with muzzle velocity V. Ignoring air resistance, show that the range on a horizontal plane is $\dfrac{V^2\sin 2\theta}{g}$ where θ is the angle of elevation of the gun and g is the acceleration due to gravity. **2**

(ii) The gun and the target lie on the same horizontal plane. The gun fires, in the correct vertical plane, at the target using an angle of elevation α and the shell falls short by a distance p. When the angle of elevation is changed to β, the shell overshoots the target by a distance q.
Show that $\sin 2\theta = \dfrac{p\sin 2\beta + q\sin 2\alpha}{p + q}$ **4**

(c) Fred has three uniform tetrahedra (triangular pyramids). Each of these tetrahedra has one face black, one face white, one face red and one face green.
When tossed onto a table, three faces of each tetrahedron can be seen.
If the probability of any coloured face not being seen is equally likely, what is the probability that

(i) no black face can be seen? **1**

(ii) exactly 2 black faces can be seen? **1**

(iii) at least 2 red faces can be seen? **1**

(iv) 3 white faces and only 1 green face can be seen? **2**

END OF PAPER

BARKER COLLEGE 4U TRIAL 2001

Question 1

(a) $\int \frac{dx}{x^2-16x+80}$

$= \int \frac{dx}{(x-8)^2+16}$ ✓

$= \frac{1}{4}\tan^{-1}\left(\frac{x-8}{4}\right)+c$ ✓

(b) (i) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,d\theta$

$= \int_0^1 \frac{2}{(t+1)^2}\,dt$ ✓ let $\tan\frac{\theta}{2}=t$

$= -2\left[\frac{1}{t+1}\right]_0^1$ ✓ $\frac{dt}{d\theta}=\frac{1}{2}(1+t^2)$

$= 1$ ✓

(ii) $I=\int_0^2 \frac{8\,dx}{(x+2)(x^2+4)}$

$\frac{1}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$

$A=\frac{1}{8},\ B=-\frac{1}{8},\ C=\frac{1}{4}$ ✓

$I = 8\int_0^2 \frac{1}{8}\left(\frac{1}{x+2}\right) + \frac{-\frac{1}{8}x+\frac{1}{4}}{x^2+4}\,dx$

$=\left[\ln(x+2)\right]_0^2 - \frac{1}{2}\int\frac{2x}{x^2+4}\,dx$ ✓

$+ 2\int\frac{1}{x^2+4}\,dx$

$= \ln 2 - \frac{1}{2}\left[\ln(x^2+4)\right]_0^2 + \left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$ ✓

$= \frac{\pi}{4} + \frac{1}{2}\ln 2$ ✓

(iii) $I=\int_0^{\pi} e^x\cos x\,dx$

$= e^x\cos x + \int e^x\sin x\,dx$ ✓

$= e^x\cos x + \left[\sin x\, e^x - \int e^x\cos x\,dx\right]$

$= e^x\cos x + \sin x\, e^x - I$ ✓

$\therefore 2I = \left[e^x(\cos x+\sin x)\right]_0^{\pi}$

$I = \frac{1}{2}\left[-e^{\pi}-1\right]$ ✓

(c) $\int \frac{2x}{\sqrt{4x-x^2}}\,dx$ let $u=x-2$, $du=dx$

$=\int \frac{2(u+2)}{\sqrt{4(u+2)-(u+2)^2}}\,du$ ✓

$= 2\int \frac{u+2}{\sqrt{4-u^2}}\,du$

$= 2\int \frac{u}{\sqrt{4-u^2}}\,du + 4\int\frac{1}{\sqrt{4-u^2}}\,du$

$= -2\sqrt{4-u^2} + 4\sin^{-1}\left(\frac{u}{2}\right)$ ✓

$= -2\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)+C$ ✓

Question 2

(a) $(x+iy)^2=-3+4i$

$x^2-y^2+2xyi = -3+4i$

$xy=2$

$x^2-y^2=-3$ ✓

Solving simult...

$x=2,\ y=1$

$x=-1,\ y=-2$ ✓

(b) $z^2-3z+(3-i)=0$

$z = \frac{3\pm\sqrt{9-4(3-i)}}{2}$

$= \frac{3\pm\sqrt{-3+4i}}{2}$

$= 2+i$ or $1-i$ ✓

Question 2

(b) (i) $\sqrt{3}+i = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ ✓

$\sqrt{3}-i = 2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)$ ✓

(ii) $(\sqrt{3}+i)^{15}+(\sqrt{3}-i)^{15}$

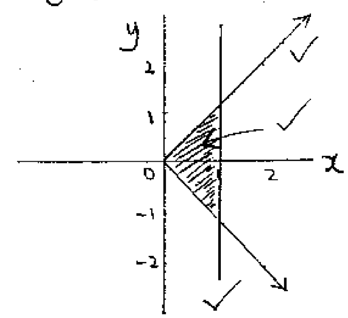
$= 2^{15}\left(\cos\frac{15\pi}{6}+i\sin\frac{15\pi}{6}\right)$

$+ 2^{15}\left(\cos\frac{15\pi}{6}-i\sin\frac{15\pi}{6}\right)$

$= 2\times 2^{15}\cos\frac{15\pi}{6}$

$= 0$ ✓

(c) (i)



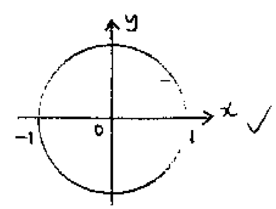
(ii) $Re\left(z-\frac{1}{z}\right)=0$

$Re\left(x+iy-\frac{x-iy}{x^2+y^2}\right)=0$ ✓

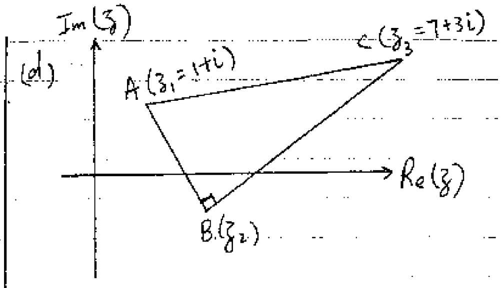
$x-\frac{x}{x^2+y^2}=0$

$\therefore x^2+y^2=1$ ✓

or $x=0$ ✓



(d)



$z_2 = x+iy$

$i\vec{BC} = \vec{BA}$

$i[(7-x)+(3-y)i] = (1-x)+i(1-y)$

$-(3-y)+i(7-x) = (1-x)+i(1-y)$

Equating real part,

$-3+y = 1-x$

$x+y=4$ —(1)

Equating imag part,

$7-x=1-y$

$-x+y=-6$ —(2) ✓

solving (1) + (2) simult.

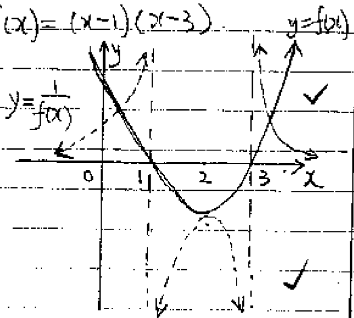
$x=5,\ y=-1$

$z_2 = 5-i$ ✓

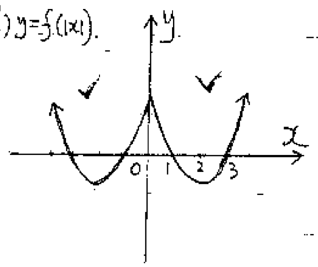
Question 3

(a) $f(x) = (x-1)(x-3)$

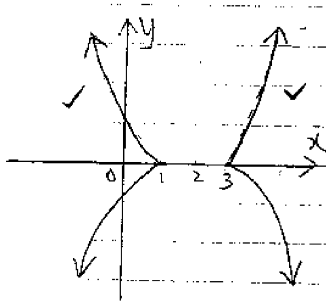
(i) $y = \frac{1}{f(x)}$ ✓ ✓



(ii) $y = f(|x|)$ ✓ ✓



(iii) $|y| = f(x)$ ✓ ✓



(b)(i) $y = \frac{(x+1)^4}{x^4+1}$

$$\frac{dy}{dx} = \frac{(x^4+1)4(x+1)^3 - (x+1)^4 4x^3}{(x^4+1)^2}$$

$$= \frac{4(x+1)^3(1-x^3)}{(x^4+1)^2}$$

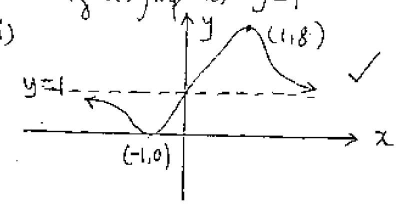
$y' = 0 \Rightarrow x = -1, 1$

Stationary points are $(-1, 0)$, $(1, 8)$ ✓

As $x \to \infty$, $y \to 1$ ✓

Horiz. asymp is $y = 1$

(ii) ✓

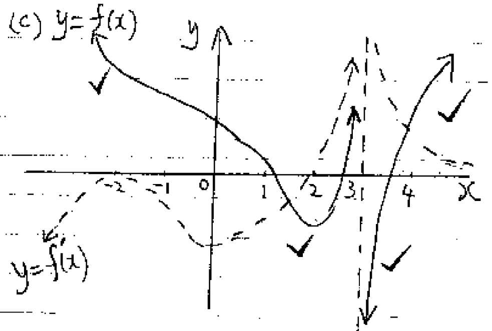


(iii) $(x+1)^4 = k(x^4+1)$ has two distinct real roots

$\therefore$ the line $y = k$ must meet the curve at two points

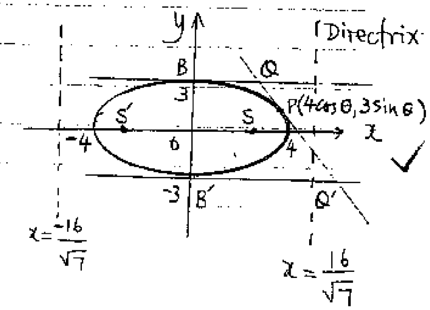
$0 < k < 1$ ✓ and $1 < k < 8$ ✓

(c) $y = f(x)$ ✓ ✓ ✓ ✓



Question 4

(a) (i) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ ✓



$b^2 = a^2(1-e^2)$

$9 = 16(1-e^2)$

$e = \frac{\sqrt{7}}{4}$ ✓

$S(ae, 0) = (\sqrt{7}, 0)$ ✓

$S'(-\sqrt{7}, 0)$

Directrix $x = \pm\frac{a}{e}$

$= \pm 16/\sqrt{7}$ ✓

(ii) $\frac{dy}{dx} = \frac{-9x}{16y}$ ✓

$= \frac{-3\cot\theta}{4}$

Equation of the tangent at P

$y - 3\sin\theta = \frac{-3\cot\theta}{4}(x - 4\cos\theta)$ ✓

$\frac{y\sin\theta}{3} + \frac{x\cos\theta}{4} = 1$ ✓

(iii) At Q, $y = 3$

$\therefore \frac{3\sin\theta}{3} + \frac{x\cos\theta}{4} = 1$

$\therefore x_Q = \frac{4(1-\sin\theta)}{\cos\theta}$ ✓

At Q', $y = -3$

$\therefore \frac{-3\sin\theta}{3} + \frac{x\cos\theta}{4} = 1$

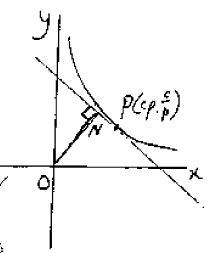
$x_{Q'} = \frac{4(1+\sin\theta)}{\cos\theta}$ ✓

$BQ \times B'Q'$

$= \frac{4(1-\sin\theta)}{\cos\theta} \times \frac{4(1+\sin\theta)}{\cos\theta} = 16$ ✓

(b) $y = \frac{c^2}{x}$

$y' = \frac{-c^2}{x^2}$



Eqn of tangent at P,

$y - \frac{c}{p} = \frac{-1}{p^2}(x - cp)$

$x + p^2 y = 2cp \quad -(1)$ ✓

Equation of ON is

$y = p^2 x \quad -(2)$ ✓

Solving (1), (2) simult to find coord N

$x_N = \frac{2cp}{1+p^4}$ ✓

$y_N = \frac{2cp^3}{1+p^4}$ ✓

using $x(1+p^4) = 2cp$ and $\frac{y}{x} = p^2$,

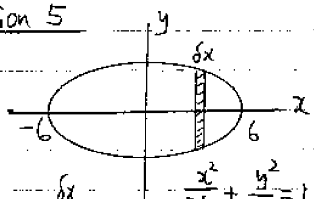
$x^2(1+p^4)^2 = (2cp)^2$

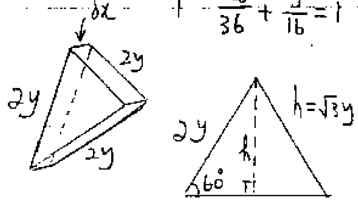
$x^2\left(1 + \frac{y^2}{x^2}\right)^2 = 4c^2\left(\frac{y}{x}\right)$

$(x^2+y^2)^2 = 4c^2xy$ ✓

Question 5

(a)





cross-sectional area of $\Delta = \sqrt{3}y^2$

$\delta V = A\delta x$

$= \sqrt{3}y^2\delta x$ ✓

Vol of the solid

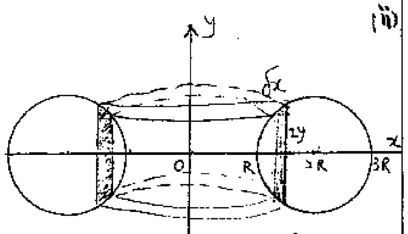
$= \int \sqrt{3}\, y^2 dx$

$= \sqrt{3}\int_{-6}^{6} 16\left(1-\frac{x^2}{36}\right)dx$ ✓

$= 16\sqrt{3}\left[\left(x-\frac{x^3}{108}\right)\right]_{-6}^{6}$ ✓

$= 128\sqrt{3}$ units^3 ✓

(b)



$\delta V = 2\pi x \cdot 2y\,\delta x$

$= 4\pi x y\,\delta x$ ✓

Volume $= \int_R^{3R} 4\pi x\sqrt{R^2-(x-2R)^2}\,dx$ ✓

Let $x-2R = R\sin\theta$

$dx = R\cos\theta\, d\theta$

volume of the torus

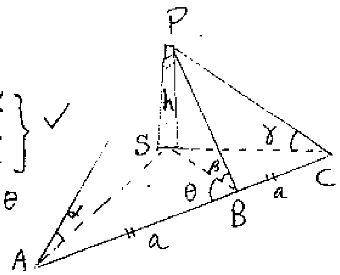
$= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{(R^2-R^2\sin^2\theta)}.\ (2R+R\sin\theta)\ R\cos\theta\, d\theta$ ✓

$= 4\pi R^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2\cos^2\theta+\cos^2\theta\sin\theta)\,d\theta$

$= 4\pi R^3\left[(\theta+\sin 2\theta)-\frac{1}{3}\cos^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$ ✓ ✓

$= 4\pi^2 R^3$ ✓

(c)



(i) AS = h cot α
BS = h cot β
CS = h cot γ ✓

∠SBC = 180 − θ

∠ABS = θ

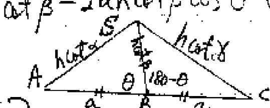
In ΔBSC,

$(SC)^2 = a^2+h^2\cot^2\beta-2ah\cot\beta\cos(180-\theta)$,

(ii) $h^2\cot^2\gamma = a^2+h^2\cot^2\beta+2ah\cot\beta\cos\theta$ — ① ✓

In ΔABS,

$h^2\cot^2\alpha = a^2+h^2\cot^2\beta-2ah\cot\beta\cos\theta$ — ② ✓



Adding, ① + ②

$h^2(\cot^2\alpha+\cot^2\gamma) = 2a^2+2h^2\cot^2\beta$ ✓

$h^2(\cot^2\alpha+\cot^2\gamma-2\cot^2\beta) = 2a^2$

$h = \frac{a\sqrt{2}}{[\cot^2\alpha+\cot^2\gamma-2\cot^2\beta]^{\frac{1}{2}}}$ ✓

Question 6

(a) a, b, c are roots of $x^3+qx+r=0$

Let $y = b+c-2a$

$= (b+c+a)-3a$

$= 0-3a$

$\therefore a = -\frac{y}{3}$ ✓

$\left(-\frac{y}{3}\right)^3+q\left(-\frac{y}{3}\right)+r=0$ ✓

$y^3+9qy-27r=0$ ✓

(b) (i) $\tan 3\theta = \tan(2\theta+\theta)$

RHS $= \frac{\tan 2\theta+\tan\theta}{1-\tan 2\theta\tan\theta}$ ✓

$= \frac{\frac{2\tan\theta}{1-\tan^2\theta}+\tan\theta}{1-\frac{2\tan\theta}{1-\tan^2\theta}\tan\theta}$

$= \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$ ✓

(ii) Let $\theta = \tan^{-1}x$

$\therefore \tan(3\tan^{-1}x) = \frac{3x-x^3}{1-3x^2}$

using $3\tan^{-1}x = \frac{\pi}{2}-\tan^{-1}(3x)$

$\tan(3\tan^{-1}x) = \tan\left(\frac{\pi}{2}-\tan^{-1}(3x)\right)$ ✓

$= \cot(\tan^{-1}3x)$

$= \frac{1}{\tan[\tan^{-1}(3x)]}$ ✓

$\frac{3x-x^3}{1-3x^2} = \frac{1}{3x}$

$3x^4-12x^2+1=0$ ✓

$x^2 = \frac{12\pm\sqrt{144-12}}{6}$

$\therefore x = 0.292$ only ✓

$x = 1.979$ is not the solut.

(c) (i) $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots$

$+\frac{1}{(n+1)(n+2)(n+3)}$

$= \frac{(n+1)(n+4)}{4(n+2)(n+3)}$

When n = 0, LHS $= \frac{1}{6}$, RHS $= \frac{1}{6}$ ✓

Assume that it is true for n = K

$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\cdots+\frac{1}{(K+1)(K+2)(K+3)}$

$= \frac{(K+1)(K+4)}{4(K+2)(K+3)}$ ✓

RTP: $S_{K+1} = S_K+T_{K+1}$

$= \frac{(K+2)(K+5)}{4(K+3)(K+4)}$

$S_{K+1} = \frac{1}{(K+2)(K+3)(K+4)}+\frac{(K+1)(K+4)}{4(K+2)(K+3)}$

$= \frac{4+(K+1)(K+4)(K+4)}{4(K+3)(K+4)(K+2)}$

$= \frac{(K+1)(K^2+8K+16)+4}{4(K+2)(K+3)(K+4)}$

$= \frac{(K+2)^2(K+5)}{4(K+2)(K+3)(K+4)}$

$= \frac{(K+2)(K+5)}{4(K+3)(K+4)}$ ✓

Since it is true for n = 0,
it is proven true for n = k+1.
∴ it is true for n = 0+1, = 1
true for n = 1+1 = 2
∴ it is true for all n ≥ 0 ✓

(ii) $\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{(r+1)(r+2)(r+3)}$

$= \frac{(n+1)(n+4)}{4(n+2)(n+3)}$

$= \frac{(\frac{n}{n}+\frac{1}{n})(\frac{n}{n}+\frac{4}{n})}{4(\frac{n}{n}+\frac{2}{n})(\frac{n}{n}+\frac{3}{n})}$ ✓

$= \frac{1\times1}{4\times1\times1} = \frac{1}{4}$ ✓

Question 7

(a) (i) $I_n = \int_0^{\pi} \sin^n\theta \, d\theta$

$= \int \sin\theta . \sin^{n-1}\theta \, d\theta$

$= \left[-\cos\theta . \sin^{n-1}\theta\right]_0^{\pi} - \int -\cos\theta . (n-1)\sin^{n-2}\theta . \cos\theta \, d\theta$ ✓

$= 0 + (n-1)\int \cos^2\theta \sin^{n-2}\theta \, d\theta$

$= (n-1)\int_0^{\pi} (1-\sin^2\theta)\sin^{n-2}\theta \, d\theta$

$= (n-1)\int_0^{\pi} \sin^{n-2}\theta \, d\theta - (n-1)\int_0^{\pi} \sin^n\theta \, d\theta$ ✓

$= (n-1) I_{n-2} - (n-1) I_n$

$n I_n = (n-1) I_{n-2}$ ✓

(ii) $I_8 = \frac{7}{8} I_6$

$= \frac{7}{8} \cdot \frac{5}{6} \cdot I_4$

$= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot I_2$

$= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0$

$= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \pi$

$= \frac{35\pi}{128}$ ✓

(iii) $\int_0^{\pi} x \sin^n x \, dx = \int_0^{\pi} (\pi - x)\sin^n(\pi - x) \, dx$

$= \int_0^{\pi} (\pi - x)\sin^n x \, dx$

$= \pi\int_0^{\pi} \sin^n x \, dx - \int_0^{\pi} x\sin^n x \, dx$ ✓

$2\int_0^{\pi} x\sin^n x \, dx = \pi\int_0^{\pi} \sin^n x \, dx$

$\therefore \int_0^{\pi} x \sin^n x \, dx = I_n\left(\frac{\pi}{2}\right)$ ✓

(b) $x = \cos\theta + i\sin\theta$

(i) $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$= 2\cos\theta$

$x^n + \frac{1}{x^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$ ✓

(ii) $5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0$

$5\left(x^2 + \frac{1}{x^2}\right) - 11\left(x + \frac{1}{x}\right) + 16 = 0$ ✓

$5 . 2\cos 2\theta - 11 . 2\cos\theta + 16 = 0$

$10\cos^2\theta - 11\cos\theta + 3 = 0$ ✓

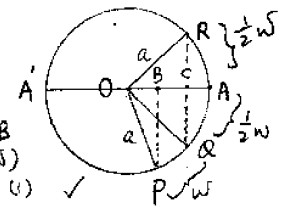
$(5\cos\theta - 3)(2\cos\theta - 1) = 0$

$\cos\theta = \frac{3}{5}$ or $\cos\theta = \frac{1}{2}$

$x = \frac{3}{5} + \frac{4}{5}i$, $x = \frac{3}{5} - \frac{4}{5}i$ ✓

$x = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $x = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ ✓

(c)



OB = 1 unit,
OC = 5 units

$OB = a\cos\angle POB$

$1 = a\cos(w + \frac{1}{2}w)$

$= a\cos\frac{3w}{2}$ — (1) ✓

$OC = a\cos\angle QOA$

$5 = a\cos\frac{w}{2}$ — (2) ✓

$\frac{(2)}{(1)}$ $5 = \frac{\cos\frac{1}{2}w}{\cos\frac{3w}{2}}$

$5\cos 3\left(\frac{w}{2}\right) = \cos\frac{1}{2}w$

$5\left[4\cos^3\left(\frac{1}{2}w\right) - 3\cos\frac{1}{2}w\right] = \cos\frac{1}{2}w$

$\cos^2\left(\frac{1}{2}w\right) = \frac{16}{20} \Rightarrow \cos^2\left(\frac{w}{2}\right) = \frac{4}{5}$

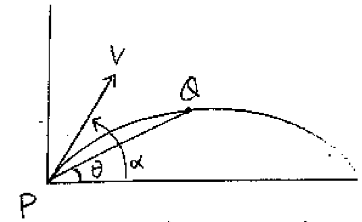
$\cos w = 2\cos^2\frac{1}{2}w - 1$

$= \frac{3}{5}$ ✓

Period $T = \frac{2\pi}{w} = \frac{2\pi}{\cos^{-1}\left(\frac{3}{5}\right)}$ ✓

Question 8

(a)



Let $\dot{x}$ and $\dot{y}$ be the horizon. and vertical components of velocity when the ball is at Q

$\dot{x} = V\cos\alpha$, $\dot{y} = V\sin\alpha - gt$ ✓

$\tan\theta = \frac{y}{x}$

$= \frac{V\sin\alpha \, t - \frac{1}{2}gt^2}{V\cos\alpha \, t}$ ✓

$\therefore \tan\theta = \tan\alpha - \frac{gt}{2V\cos\alpha}$

The direction of motion at Q is inclined to the horizon. at

$\tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) = \tan^{-1}\left[\tan\alpha - \frac{gt}{V\cos\alpha}\right]$ ✓

But $\frac{gt}{V\cos\alpha} = 2\tan\alpha - 2\tan\theta$

$\therefore \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) = \tan^{-1}\left[\tan\alpha - 2\tan\alpha + 2\tan\theta\right]$

$= \tan^{-1}\left[2\tan\theta - \tan\alpha\right]$ ✓

(b) (i) $x = V\cos\theta \, t$

$y = V\sin\theta . t - \frac{1}{2}gt^2$

Time of flight $\Rightarrow y = 0$

$\Rightarrow t = \frac{2V\sin\theta}{g}$ ✓

$\therefore$ Range $= V\cos\theta \times \frac{2V\sin\theta}{g}$

$R = \frac{V^2\sin 2\theta}{g}$ ✓

(ii) $R - p = \frac{V^2\sin 2\alpha}{g}$ — (1) ✓

$R + q = \frac{V^2\sin 2\beta}{g}$ — (2) ✓

$\frac{V^2\sin 2\theta}{g} - p = \frac{V^2\sin 2\alpha}{g}$

$\frac{V^2\sin 2\theta}{g} + q = \frac{V^2\sin 2\beta}{g}$

$\frac{p}{q} = \frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin\theta}$ ✓

$(p+q)\sin 2\theta = p\sin 2\beta + q\sin 2\alpha$

$\sin 2\theta = \frac{p\sin 2\beta + q\sin 2\alpha}{p + q}$ ✓

(c) (i) $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$ ✓

(ii) ${}^3C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$

$= \frac{27}{64}$ ✓

(iii) ${}^3C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) + {}^3C_3\left(\frac{3}{4}\right)$

$= \frac{27}{32}$ ✓

(iv) ${}^3C_2\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{2}{4}\right)$ ✓

$= \frac{3}{32}$ ✓

— END OF PAPER —