Barker College

2002
TRIAL
HIGHER SCHOOL
CERTIFICATE

Mathematics Extension 2

Staff Involved:

- DOK/GJR*
- BHC
- MRB

AM THURSDAY 8 AUGUST

40 copies

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used

Total marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value
- Write your Barker Student Number on ALL pages of your answer sheets
- A table of standard integrals is provided on page 10
- ALL necessary working should be shown in every question
- Start each question on a NEW page
- Write on one side of each answer page
- Marks may be deducted for careless or badly arranged work

Total marks (120)
Attempt Questions 1 – 8
ALL questions are of equal value

Answer each question on a SEPARATE sheet of paper

Marks

Question 1 (15 marks) **[BEGIN A NEW PAGE]**

(a) Find $\int_1^e \frac{dx}{x\left(1 + (\ln x)^2\right)}$ by substituting $u = \ln x$ **2**

(b) Find $\int \frac{x + 1}{x^2 + 4}\, dx$ **2**

(c) Find $\int \frac{x^2 + 4}{x + 1}\, dx$ **3**

(d) Evaluate $\int_0^{\frac{\pi}{4}} x^2 \sin x\, dx$ **4**

(e) Prove that $\int_0^{\frac{1}{4}} \sqrt{1 - 4x^2}\, dx = \frac{\pi}{24} + \frac{\sqrt{3}}{16}$ **4**

Marks

Question 2 (15 marks) **[BEGIN A NEW PAGE]**

(a) Given that $f(x) = e^{-x}$, sketch the following showing the main features.

(i) $y = -f(x)$ **1**

(ii) $y = 1 - f(x)$ **2**

(iii) $y = \dfrac{1}{1 - f(x)}$ **2**

(iv) $y = \left|\dfrac{1}{1 - f(x)}\right|$ **2**

(b) Next to each graph state whether it is odd, even or neither. **4**

(c) (i) For $x^2 + 2xy + y^5 = 4$, show that $\dfrac{dy}{dx} = \dfrac{-2x - 2y}{2x + 5y^4}$ **2**

(ii) A plane curve is defined implicitly by the equation

$$x^2 + 2xy + y^5 = 4.$$

This curve has a horizontal tangent at the point $P(x_1, y_1)$.

Show that x_1 is a root of the equation $x^5 + x^2 + 4 = 0$. **2**

Marks

Question 3 (15 marks) **[BEGIN A NEW PAGE]**

(a) If $z_1 = 1 + 2i$, $z_2 = 2 - i$ and $z_3 = -1 + i\sqrt{3}$, find $\left|\frac{z_1 z_2}{i z_3}\right|$ **3**

(b) Simplify $\dfrac{(2\cos\theta + 2i\sin\theta)^5(2\cos\theta + 2i\sin\theta)^{-3}}{(\cos 2\theta + i\sin 2\theta)^5}$ **3**

(c) Z is the point representing the complex number z on an Argand diagram.

(i) Describe in words the geometrical significance of the expressions

$$|z - 2| \quad \text{and} \quad \mathrm{Re}(z)$$ **2**

(ii) Hence, or otherwise, sketch the locus of Z given that

$$|z - 2| = \mathrm{Re}(z)$$

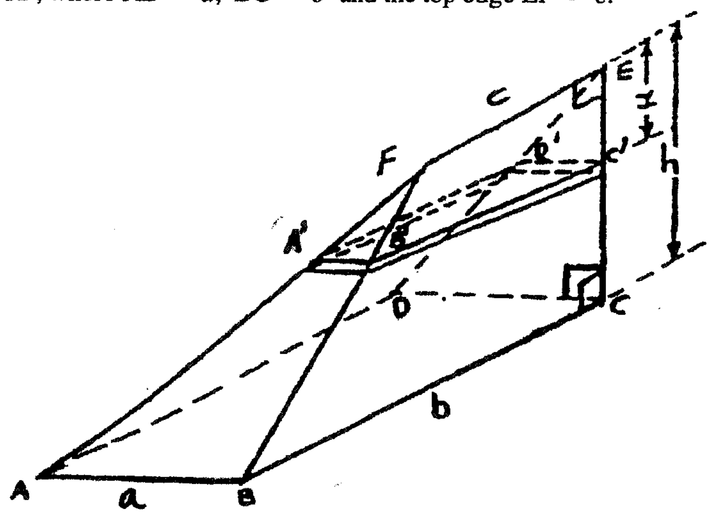
Show all important features of this locus. **3**

(d) Triangle OAB is an isosceles triangle with AO = OB and $\angle$OBA = $75°$. If O is the origin and A represents the complex number $-\sqrt{3} + i$, find **two** possible complex numbers represented by the point B, in the form $a + bi$. **4**

Marks

Question 4 (15 marks) **[BEGIN A NEW PAGE]**

(a) Consider solid ABCDEF whose height is h, and whose base is a rectangle ABCD, where AB $= a$, BC $= b$ and the top edge EF $= c$.



Consider a rectangular slice A'B'C'D' (parallel to the base ABCD) which is x units from the top edge with width Δx.
NOTE: B'C' $\|$ BC and A'B' $\|$ AB

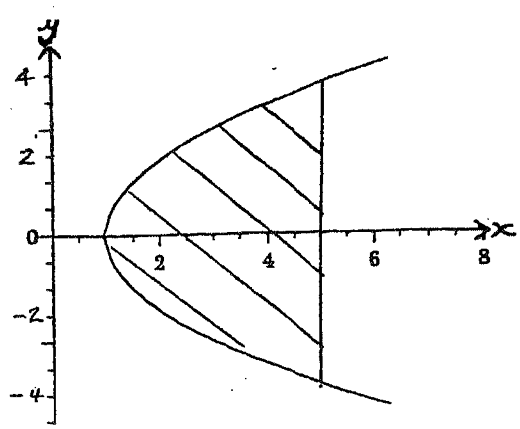
(i) Show that the volume Δv of the slice is given by **4**

$$\Delta v = \left(\frac{x}{h}a\right)\left(c + \frac{b-c}{h}x\right)\Delta x$$

(ii) Hence, show that the volume of the solid ABCDEF is **4**

$$\frac{ha}{6}(2b + c)$$

(b) The diagram shows the region bounded by the curve $y^2 = 4(x - 1)$ and the line $x = 5$. By using the method of cylindrical shells, or otherwise, find the volume of the solid formed by rotating the given region about the y-axis. **7**



Marks

Question 5 (15 marks) **[BEGIN A NEW PAGE]**

The normal at $P\,(ct,\ \frac{c}{t})$ to the hyperbola $xy = c^2$ meets the curve again at Q.

(a) Prove that the equation of the normal is $t^3x - ty = ct^4 - c$ **4**

(b) Find the coordinates of Q. **3**

(c) A line from P through the origin meets the hyperbola again at R.
Prove that PR is perpendicular to QR. **4**

(d) If M is the midpoint of PQ, find the equation of the locus M. **4**

Marks

Question 6 (15 marks) **[BEGIN A NEW PAGE]**

(a) α and β are the complex roots of $iz^2 + \sqrt{3}z - 1 = 0$.

(i) Find α and β in $a + ib$ form. **3**

(ii) Show that $\alpha^2\beta^2 + 1 = 0$. **1**

(b) Solve the equation $4x^3 - 12x^2 + 11x - 3 = 0$ given that the roots are in arithmetic sequence. **4**

(c) (i) Prove, by calculus if you wish, that the polynomial equation

$$\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x + c = 0$$

has no real roots if $c > 9\frac{1}{3}$ **5**

(ii) Find an approximation for the **largest** root of the polynomial equation in (i) above, if $c = -2$, using one application of Newton's Method. **2**

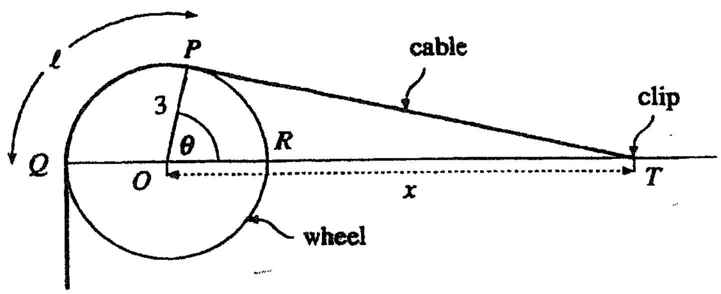
Question 7 (15 marks) **[BEGIN A NEW PAGE]** **Marks**

(a) Let n be a positive integer where $I_n = \int_1^2 (\log_e x)^n \, dx$

(i) Prove that $I_n = 2(\log_e 2)^n - nI_{n-1}$ **3**

(ii) Hence, evaluate $\int_1^2 (\log_e x)^4 \, dx$ **2**

(b)



A long cable is wrapped over a wheel of radius 3 metres and one end is attached to a clip at T. The centre of the wheel is at O and QR is a diameter. The point T lies on the line OR at a distance x metres from O.

The cable is tangential to the wheel at P and Q as shown.
Let $\angle POR = \theta$ (in radians).

The length of cable in contact with the wheel is ℓ metres; that is, the length of the arc between P and Q is ℓ metres.

(i) Explain why $\cos\theta = \dfrac{3}{x}$ **1**

(ii) Show that $\ell = 3\left[\pi - \cos^{-1}\left(\dfrac{3}{x}\right)\right]$ **2**

(iii) Show that $\dfrac{d\ell}{dx} = \dfrac{-9}{x\sqrt{x^2 - 9}}$ **2**

(iv) What is the significance of the fact that $\dfrac{d\ell}{dx}$ is negative? **1**

(v) Let $s = \ell + PT$
Given that $PT^2 = QT \times RT$, or otherwise, express s in terms of x **1**

(vi) The clip at T is moved away from O along the line OR at a constant speed of 2 metres per second.
Find the rate at which s changes when $x = 10$ **3**

Marks

Question 8 (15 marks) **[BEGIN A NEW PAGE]**

(a) It is given that the equation $ax^4 + 4bx + c = 0$ has a double root.
If α is the double root, show that $a\alpha^3 + b = 0$ and deduce that $ac^3 = 27b^4$ **4**

(b) $P(x)$ is divided by $(x - a)(x - b)$ so that a remainder $R(x)$ is obtained.
Show that the remainder is given by

$$R(x) = \left(\frac{P(a) - P(b)}{a - b}\right)x + \frac{aP(b) - bP(a)}{a - b}$$ **4**

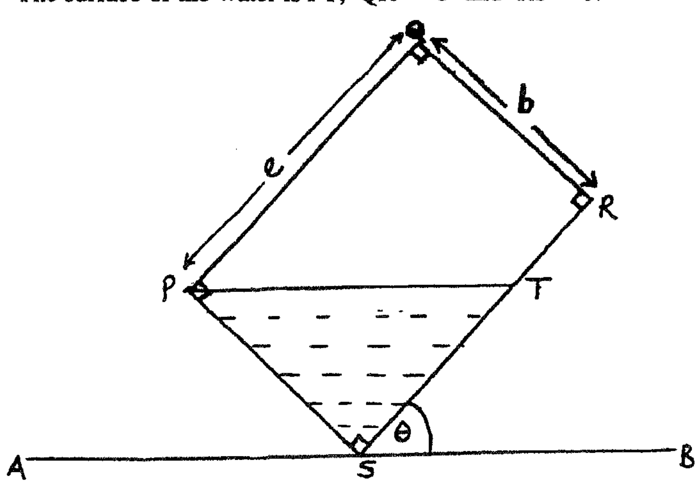
(c) Using the fact that $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$, or otherwise,

(i) find a general solution of the equation $\sin 3x = -\cos 2x$ **3**

(ii) find the smallest positive solution of the equation

$$\sin 3x = -\cos 2x$$ **1**

(d) A rectangular fish tank PQRS is tilted at an angle of θ to the horizontal surface AB.
The surface of the water is PT, QR $= b$ and RS $= e$.



If the fish tank is lowered so that SR lies on AB, prove that the height, h, of the water in the tank is given by

$$h = \frac{b^2 \cot\theta}{2e}$$ **3**

End of Paper

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4U TRIAL 2002

Question One

(a) $\int_1^e \frac{dx}{x(1+(\ln x)^2)} = \int_0^1 \frac{du}{1+u^2}$ (1)

Let $u = \ln x$, $du = \frac{1}{x}\,dx$

$= \tan^{-1} u \Big|_0^1$

$= \tan^{-1} 1 - \tan^{-1} 0$

$= \pi/4$ (1)

2

(b) $\int \frac{x+1}{x^2+4}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+4} + \int \frac{1}{x^2+4}\,dx$ (1)

$= \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$ (1)

2

(c) $\int \frac{x^2+4}{x+1}\,dx = \int (x-1) + \frac{5}{x+1}\,dx$ (1)

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \quad +4 \\ & x^2+x \\ \hline & -x+4 \\ & -x-1 \\ \hline & 5 \end{array}$$ (1)

$= \frac{x^2}{2} - x + 5\ln(x+1) + c$ (1)

3

(d) $\int_0^{\pi/4} x^2 \sin x\,dx = -x^2\cos x\Big|_0^{\pi/4} + 2\int_0^{\pi/4} x\cos x\,dx$ (1)

by parts with $u = x^2$, $dv = \sin x\,dx$, $du = 2x\,dx$, $v = -\cos x$

$= \frac{-\pi^2}{16\sqrt{2}} + 2\int_0^{\pi/4} x\cos x\,dx$

$= \frac{-\pi^2}{16\sqrt{2}} + 2x\sin x\Big|_0^{\pi/4} - 2\int_0^{\pi/4}\sin x\,dx$ (1)

by parts with $u = x$, $dv = \cos x\,dx$, $du = 1\,dx$, $v = \sin x$

Q1 cont

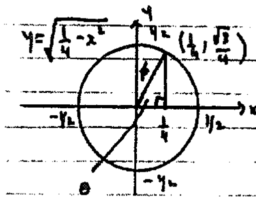
$= \frac{-\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}} + 2\Big[\cos x\Big]_0^{\pi/4}$ (1)

$= \frac{-\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}} + \sqrt{2} - 2$ (1)

$= \sqrt{2} - 2 + \frac{8\pi - \pi^2}{16\sqrt{2}}$

4

(e) $\int_0^{1/4} \sqrt{1-4x^2}\,dx = 2\int_0^{1/4}\sqrt{\frac{1}{4} - x^2}\,dx$ (1)



$= 2 \times \text{Area}_\triangle + 2 \times \text{Area}$ (1)

$= 2\times\left(\frac{1}{2}\times\frac{1}{4}\times\frac{\sqrt{3}}{4}\right) + 2\times\left(\frac{1}{12}\times\pi\times\left(\frac{1}{2}\right)^2\right)$ (1)

$= \frac{\sqrt{3}}{16} + \frac{\pi}{24}$ # (1)

4

$\theta = \tan^{-1}\left(\frac{\sqrt{3}/4}{1/4}\right) = \tan^{-1}\sqrt{3} = 60^\circ$

$\therefore \phi = 30^\circ \equiv \frac{1}{12}$ of circle area

ALT: use trig substitution

$x = \frac{1}{2}\sin\theta$

$x^2 = \frac{1}{4}\sin^2\theta$

$\sqrt{1-4x^2} = \cos\theta$

$dx = \frac{1}{2}\cos\theta\,d\theta$

$x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$x = \frac{1}{4} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$\int_0^{1/4}\sqrt{1-4x^2}\,dx = \int_0^{\pi/6}\frac{1}{2}\cos^2\theta\,d\theta$ (1) includes subn

$= \frac{1}{4}\int_0^{\pi/6}(1+\cos 2\theta)\,d\theta$ (1)

$= \frac{1}{4}\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\pi/6}$ (1)

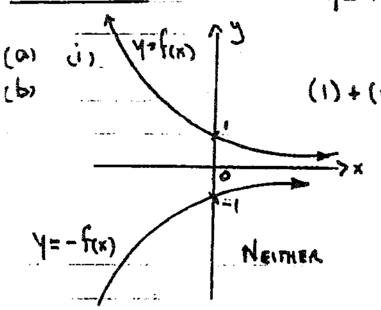
$= \frac{1}{4}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$

$= \sqrt{3} + \pi$ (1)

Question Two $y = f(x) = e^{-x}$

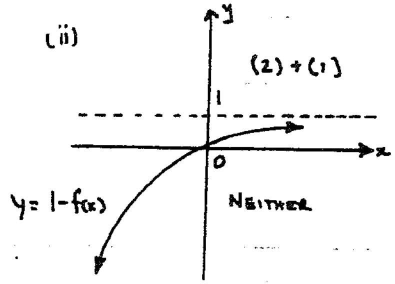
(a) (i)
(b)

(1) + (1)



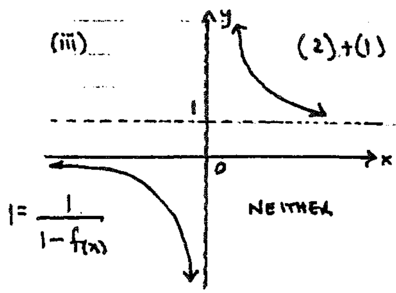
(ii)

(2) + (1)



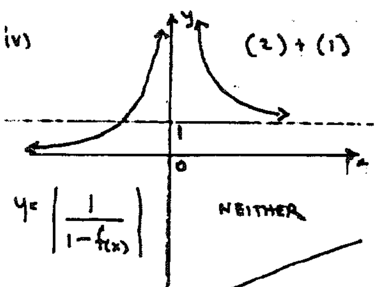
(iii)

(2) + (1)



(iv)

(2) + (1)



7 + 4

(c) (i) $x^2 + 2xy + y^5 = 4$

$\Rightarrow 2x + (2xy' + 2y) + 5y^4 y' = 0$ (1)

$\Rightarrow y'(2x + 5y^4) = -2x - 2y$

$\Rightarrow y' = \dfrac{-2x - 2y}{2x + 5y^4}$ (1)

2

Q2 cont (c) (ii) $\frac{dy}{dx} = 0 \Rightarrow -2x - 2y = 0$

$\Rightarrow y = -x$

so $P(x_1, y_1) = P(x_1, -x_1)$ (1)

put P in $x^2 + 2xy + y^5 = 4$

$\Rightarrow x_1^2 + 2x_1(-x_1) + (-x_1)^5 = 4$

$\Rightarrow -x_1^2 - x_1^5 = 4$

$\Rightarrow x_1^5 + x_1^2 + 4 = 0$ } (1)

hence x_1 is a root of $x^5 + x^2 + 4 = 0$

2

Question Three

(a) $Z_1 = 1+2i \Rightarrow |Z_1| = \sqrt{5}$

$Z_2 = 2-i \Rightarrow |Z_2| = \sqrt{5}$ (1)

$Z_3 = -1 + i\sqrt{3} \Rightarrow |Z_3| = 2$

$\left|\dfrac{Z_1 Z_2}{iZ_3}\right| = \dfrac{|Z_1||Z_2|}{|i||Z_3|}$ (1)

$= \dfrac{\sqrt{5} \times \sqrt{5}}{1 \times 2}$

$= \dfrac{5}{2}$ } (1)

3

(b) $\dfrac{(2\cos\theta + 2i\sin\theta)^5 (2\cos\theta + 2i\sin\theta)^{-3}}{(\cos 2\theta + i\sin 2\theta)^5} = \dfrac{(2\cos\theta + 2i\sin\theta)^2}{(\cos\theta + i\sin\theta)^{10}}$ (1) (1)

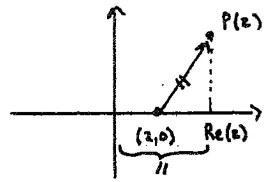
$= \dfrac{4}{(\cos\theta + i\sin\theta)^8}$ or $4(\cos\theta + i\sin\theta)^{-8}$ or $4(\cos 8\theta - i\sin 8\theta)$ (1

3

(c) (i) $|Z-2|$ is the vector from (2,0) to P(z) (1)

Re(z) is the x-axis position of P(z) (vector from (0,0) to) (1)

Q3 (c) (ii)



$|z-2| = Re(z)$ let $z = x+iy$

$\Rightarrow |(x+iy)-2| = Re(x+iy)$ (1)

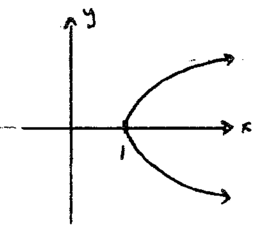
$\Rightarrow \sqrt{(x-2)^2+y^2} = x$

$(x-2)^2+y^2 = x^2$ (1)

$x^2-4x+4+y^2 = x^2$

$\Rightarrow y^2 = 4x-4 = 4(x-1)$

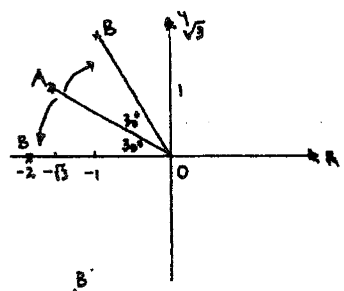
Hence Locus is:

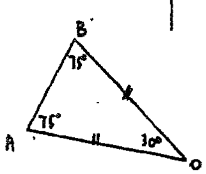


(1)

3

(d)





$|OA| = 2$ ✓

$\arg(OA) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$

$\therefore \angle AOR = 30°$ ✓

If ΔOAB isos and $\angle OBA = 75° \Rightarrow \angle AOB = 30°$

Want two vectors length 2, 30° either side of $\overrightarrow{OA}$

From diagram $\overrightarrow{OB} = \underline{-2}$ (1)

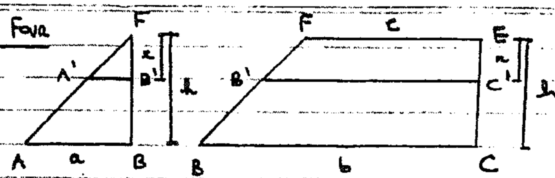
or $\overrightarrow{OB} = 2\,\text{cis}\,\frac{2\pi}{3}$

$= \underline{-1 + i\sqrt{3}}$ (1)

III

Question Four

(a)



(1) For setup of Problem

i) at $x=0$ $z=0$; $x=h$ $z=a$

at $x=0$ $y=c$; $x=h$ $y=b$

$z = mx+b$

$\underline{0=b}$

$\underline{a = mh} \Rightarrow m = \frac{a}{h}$

$\therefore \underline{z = \left(\frac{a}{h}\right)x}$ (1)

$\underline{y = mx+d}$

$\underline{c = d}$

$\underline{b = mh+d}$

$mh = b-c$

$\Rightarrow m = \frac{b-c}{h}$

$\therefore \underline{y = \left(\frac{b-c}{h}\right)x + c}$ (1)

at height x $\quad A(x) = zy$

$= \left(\frac{xa}{h}\right)\left(c + \left(\frac{b-c}{h}\right)x\right)$

for a slice of Δx chosen sufficiently small

$\Rightarrow \Delta V = A(x).\Delta x$ (1)

$= \underline{\left(\frac{xa}{h}\right)\left(c + \left(\frac{b-c}{h}\right)x\right)\Delta x}$

4

Q4 (a)(ii)

from (i) $\Rightarrow$ $V = \int_0^h \frac{\Delta V}{\Delta x}\,dx$

$$= \int_0^h \left(\frac{xa}{h}\right)\left(c + \frac{b-c}{h}x\right)dx \quad (1)$$

$$= \frac{a}{h}\int_0^h \left(xc + \frac{b-c}{h}x^2\right)dx$$

$$= \frac{a}{h}\left[\frac{x^2c}{2} + \frac{1}{3}\left(\frac{b-c}{h}\right)x^3\right]_0^h \quad (1)$$

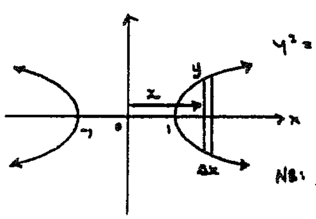
$$= \frac{a}{h}\left[\frac{ch^2}{2} + \frac{1}{3}\left(\frac{b-c}{h}\right)h^3\right] \Big\} \quad (1)$$

$$= \frac{ach}{2} + \frac{ah(b-c)}{3}$$

$$= \frac{ah}{6}\{2b + c\} \; * \quad (1)$$

4

(b)



$y^2 = 4(x-1)$

cylinder at x has volume $2\pi x^2 y$

cylinder at $(x+\Delta x)$ has volume $2\pi(x+\Delta x)^2 y$

NB: cylinder heights are $2y$. (1)

$\therefore$ Volume of shell is $2\pi(x+\Delta x)^2 y - 2\pi x^2 y$ (1)

$$= 2\pi y(x^2 + 2x\Delta x + \Delta x^2 - x^2)$$

$$= 2\pi y(2x\Delta x + \Delta x^2) \quad (1)$$

as Δx is sufficiently small ignore Δx^2 $\Rightarrow$ $\Delta V = 4\pi xy.\Delta x$ (1)

Q4 (b) cont

$\therefore$ $V = \int_1^5 \frac{\Delta V}{\Delta x}\,dx$

$$= \int_1^5 4\pi xy\,dx$$

(**) $= 4\pi \int_1^5 x \cdot 2\sqrt{x-1}\,dx$ } as $y = \pm 2\sqrt{x-1}$ (1)

$$= 8\pi \int_1^5 x\sqrt{x-1}\,dx$$

(*) let $u = x-1$, $x = u+1$, $du = dx$ } $= 8\pi\int_0^4 (u+1)\sqrt{u}\,du$

$$= 8\pi\left[\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right]_0^4 \quad (1)$$

$$= 8\pi\left[\frac{64}{5} + \frac{16}{3}\right]$$

$$= \frac{2176\pi}{15} \text{ units}^3 \quad (1)$$

7

Alts at (*) use

$u^2 = x-1$

$x = u^2+1$

$dx = 2u.du$

end up with $8\pi\int_0^2 (2u^4 + 2u^2)\,du$

at (**) substitute to y instead of x

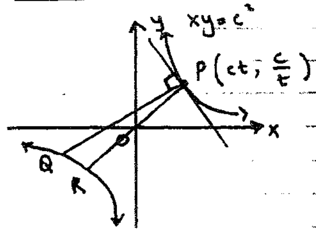
$y^2 = 4(x-1) \Rightarrow x = \frac{y^2+4}{4}$

$dx = \frac{y}{2}\,dy$

end up with $4\pi\int_0^4 \left(\frac{y^2+4}{4}\right).y.\frac{y}{2}\,dy$

$$= \frac{\pi}{2}\int_0^4 (y^4 + 4y^2)\,dy$$

Question Five



(i) $xy = c^2$

$\Rightarrow xy' + y = 0 \Rightarrow y' = \frac{-y}{x}$ (1)

at P $\Rightarrow y' = \frac{-c/t}{ct} = \frac{-c}{ct^2} = -\frac{1}{t^2}$

$\therefore$ grad of normal is t^2 (1)

eqn of normal: $y - \frac{c}{t} = t^2(x - ct)$ (1)

$\Rightarrow y - \frac{c}{t} = t^2x - ct^3$

$\Rightarrow ty - c = t^3x - ct^4$ (1)

or $t^3x - ty = ct^4 - c$ as required.

4

(ii) Solve $t^3x - ty = ct^4 - c$ — ①

& $xy = c^2$ — ②

put ② in ① ($y = c^2/x$) $\Rightarrow t^3x - \frac{c^2t}{x} = ct^4 - c$ (1)

$\therefore t^3x^2 - c(t^4-1)x - c^2t = 0$

$\therefore x = \frac{c(t^4-1) \pm \sqrt{c^2(t^4-1)^2 + 4t^4c^2}}{2t^3}$

$= \frac{c(t^4-1) \pm \sqrt{c^2(t^4+1)^2}}{2t^3}$ (1)

$= \frac{2ct^4}{2t^3}$ or $\frac{-2c}{2t^3}$

$= ct$ or $-\frac{c}{t^3}$ (1) 3

Hence $y = \frac{c^2}{x} = \frac{c}{t}$ or $-ct^3$ $\therefore Q = \left(\frac{-c}{t^3}, -ct^3\right)$

(c) by symmetry $R = \left(-ct, \frac{-c}{t}\right)$ (1)

$G_{PR} = \frac{\frac{c}{t} + \frac{c}{t}}{ct + ct} = \frac{\frac{2c}{t}}{2ct} = \frac{1}{t^2}$ (1)

$G_{QR} = \frac{-ct^3 + \frac{c}{t}}{-\frac{c}{t^3} + ct} = \frac{-ct^6 + ct^2}{-c + ct^4} = \frac{\cancel{ct^2(1-t^4)}}{\cancel{-c(1-t^4)}}$ (1)

$= -t^2$

$G_{PR} \cdot G_{QR} = \frac{1}{t^2} \times -t^2 = -1$ Hence $PR \perp QR$. (1) 4

(d) $M = \left(\frac{ct - c/t^3}{2}, \frac{c/t - ct^3}{2}\right)$

Locus of M:

Let $x = \frac{ct - c/t^3}{2}$ and $y = \frac{c/t - ct^3}{2}$

$= \frac{c(t^4-1)}{2t^3}$ $\quad = \frac{c(1-t^4)}{2t}$

Eliminate t:

$\frac{2x}{c} = \frac{t^4-1}{t^3}$ — ① $\qquad -\frac{2y}{c} = \frac{t^4-1}{t}$ — ②

Now ② ÷ ① $\Rightarrow \frac{-y}{x} = t^2$ — ③

② cubed $\Rightarrow \left(\frac{-2y}{c}\right)^3 = \left(\frac{t^4-1}{t}\right)^3 = \frac{(t^4-1)^3}{t^3}$

$= (t^4-1)^2 \cdot \left(\frac{t^4-1}{t^3}\right)$

with ① $= (t^4-1)^2 \cdot \left(\frac{2x}{c}\right)$ (1)

$$\therefore \left(\frac{c}{2x}\right) \times \frac{-8y^3}{c^3} = (t^4-1)^2$$

$$\text{so} \quad t^4 - 1 = \sqrt{\frac{-4y^3}{c^4 x}}$$

$$\therefore t^4 = 1 + \frac{2y}{c}\sqrt{\frac{-y}{x}} \quad \text{— (4)}$$

(1)

(3)^2 & subs (4)

$$\Rightarrow t^4 = \frac{y^2}{x^2} = 1 + \frac{2y}{c}\sqrt{\frac{-y}{x}}$$

$$\Rightarrow \frac{y^2}{x^2} - 1 = \frac{2y}{c}\sqrt{\frac{-y}{x}} \qquad (1)$$

$$\Rightarrow \left(\frac{cy}{2x^2} - \frac{c}{2y}\right)^2 = \frac{-y}{x}$$

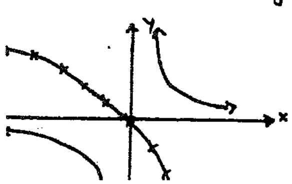
$$\Rightarrow \frac{c^2y^2}{4x^4} + \frac{c^2}{4y^2} - \frac{\cancel{2}c^2\cancel{y}}{\cancel{2}\cdot 2x^2\cancel{y}} = \frac{-y}{x}$$

$$\Rightarrow c^2\left(\frac{y^2}{x^4} + \frac{1}{y^2} - \frac{2}{x^2}\right) = \frac{-4y}{x}$$

$$\times x^4y^2 \Rightarrow c^2\left(y^4 + x^4 - 2x^2y^2\right) = -4x^3y^3$$

$$\Rightarrow c^2(x^2-y^2)^2 = -4x^3y^3$$

$$\therefore c^2(y^2-x^2)^2 + 4x^3y^3 = 0 \quad \text{is the locus} \quad (1)$$



Geometrically it behaves like half a hyperbola but is in fact *not* hyperbolic.

4

Question Six

(a)(i) $iz^2 + \sqrt{3}z - 1 = 0$

$a = i$, $b = \sqrt{3}$, $c = -1$ $\Rightarrow$ $z = \dfrac{-\sqrt{3} \pm \sqrt{3+4i}}{2i}$ (1)

let $\sqrt{3+4i} = x+iy$

$\Rightarrow 3+4i = (x^2-y^2) + 2ixy$

$\Rightarrow x^2 - y^2 = 3$, $xy = 2$ $\}$ $|x| > |y|$

$\therefore x = \pm 2,\ y = \pm 1$ (1)

$$= \frac{-\sqrt{3} \pm (2+i)}{2i}$$

$\therefore \alpha = \dfrac{(2-\sqrt{3})+i}{2i}$ sim $\beta = \dfrac{-1+(2+\sqrt{3})i}{2}$

$$= \frac{1-(2-\sqrt{3})i}{2} \qquad (1)$$

3

(ii) $\alpha^2\beta^2 + 1 = (\alpha\beta)^2 + 1$ now $\alpha\beta = \prod$ of roots $= \dfrac{-1}{i} = i$ (1)

$= (i)^2 + 1$

$= 0.$

1

b) $4x^3 - 12x^2 + 11x - 3 = 0$ — (*)

Possible 3 roots in A.P.

thus, let roots be $a-d$, a, $a+d$ (1)

$\sum \alpha = 3a$

$\sum \alpha\beta = a(a-d) + a(a+d) + (a-d)(a+d)$

$= 3a^2 - d^2$

$\sum \alpha\beta\gamma = a(a-d)(a+d) = a^3 - ad^3$ (1)

also from (*) $\sum \alpha = 3$, $\sum \alpha\beta = 11/4$, $\sum \alpha\beta\gamma = \frac{3}{4}$ — equating with above:

$3a = 3 \Rightarrow a = 1$ (1)

$\Rightarrow 3a^2 - d^2 = 11/4 \Rightarrow 3 - d^2 = 11/4 \Rightarrow d = 1/2$

check: $a^3 - ad^3 = 3/4 \Rightarrow 1 - d^2 = 3/4$ ---

Q6 (c) cont

(i) As it is a quartic W shape will only have no real roots if minimum values > 0 (1)

Let $y = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^3 + 4x + c = 0$

$y' = x^3 - x^2 - 4x + 4$

$y'' = 3x^2 - 2x - 4$

$y' = 0 \Rightarrow x^3 - x^2 - 4x + 4 = 0$ (1)

$(x-1)(x-2)(x+2) = 0$

$\therefore x = -2, 1, 2$ are turning points

easily seen that $x = -2$ min (1)

1 max

2 min

at $x = -2 \Rightarrow y = 4 + \frac{8}{3} - 8 - 8 + c = -9\frac{1}{3} + c$

for no roots $y > 0 \Rightarrow c > 9\frac{1}{3}$

at $x = 2 \Rightarrow y = \frac{4}{3} + c$ (1)

for no roots $y > 0 \Rightarrow c > -\frac{4}{3}$

as this is smaller than $9\frac{1}{3}$

Need $c > 9\frac{1}{3}$ to ensure both minimums above x-axis. # (1)

5

Q6(c)(ii) cont

(ii) Let $y = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x - 2$

$\Rightarrow$ largest root is when $x > 2$ (i.e. $x = 2$ a min)

$y' = x^3 - x^2 - 4x$ (1)

Let $x_0 = 3 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_n)}$ Newton's Method

$= 3 - \frac{13/4}{10}$

$= \frac{107}{40}$ (1)

$\doteqdot 2.675$

2

QUESTION SEVEN

(a) (i) $I_n = \int_1^2 (\log x)^n \, dx = \int_1^2 (\log x)^n . 1 \, dx$ (1)

Let $u = (\log x)^n$, $dv = 1\,dx$

$\Rightarrow du = \frac{n(\log x)^{n-1}}{x}dx$, $v = x$

$= x(\log x)^n \Big|_1^2 - n\int_1^2 (\log x)^{n-1}$ (1)

$= 2(\log 2)^n - n I_{n-1}$ (1)

3

(ii) Now $I_4 = \int_1^2 (\log x)^4 \, dx$ (1)

by (i) $\Rightarrow$ $I_4 = 2(\log 2)^4 - 4I_3$ $\Rightarrow I_1 = 2\log 2 - 1$

$I_3 = 2(\log 2)^3 - 3I_2$ $\therefore I_2 = 2(\log 2)^2 - 4\log 2 + 2$

$I_2 = 2(\log 2)^2 - 2I_1$ $\therefore I_3 = 2(\log 2)^3 - 6(\log 2)^2 + 12\log 2 - 6$

$I_1 = 2\log 2 - I_0$ $\therefore I_4 = 2(\log 2)^4 - 8(\log 2)^3$

$I_0 = \int_1^2 1\,dx = x\Big|_1^2 = 1$ $+ 24(\log 2)^2 - 48\log 2 + ⋯$

(1)

(b) (i) $\cos\theta = \frac{3}{x}$ as $PT \perp PO$ as cable is tangent. (1) 1

(ii) $l = 3(\pi - \theta)$ arc length, as $\cos\theta = \frac{3}{x}$ (1)

$\Rightarrow l = 3\left[\pi - \cos^{-1}(3/x)\right]$ (1) 2

(iii) $\frac{dl}{dx} = -3 \cdot \frac{-1}{\sqrt{1-\left(\frac{3}{x}\right)^2}} \cdot \frac{-3}{x^2}$ (1)

$= \frac{-9}{\sqrt{x^4 - 9x^2}}$

$= \frac{-9}{x\sqrt{x^2-9}}$ as $x > 3$ as $T > R$

$\Rightarrow \frac{dl}{dx} < 0$ (1) 2

(iv) $\frac{dl}{dx} < 0 \Rightarrow$ rate of change of l against x is decreasing
ie. l decreases as x increases.
NB: as $x \to \infty \Rightarrow \theta \to 90° \Rightarrow l \to \frac{3\pi}{2}$ (ie l is finite $\neq 0$). (1) 1

(v) $s = l + PT$ $PT = \sqrt{x^2-9}$ by pythagoras.

$s = 3\pi - 3\cos^{-1}(3/x) + \sqrt{x^2-9}$ (1) 1

(vi) $\frac{ds}{dx} = \frac{-9}{x\sqrt{x^2-9}} + \frac{1}{2}\left(x^2-9\right)^{-1/2} \cdot 2x$

$= \frac{-9}{x\sqrt{x^2-9}} + \frac{x}{\sqrt{x^2-9}}$

$= \frac{x^2-9}{x\sqrt{x^2-9}} = \frac{\sqrt{x^2-9}}{x}$ (1)

so $\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \frac{\sqrt{x^2-9}}{x} \cdot 2$ (1)

Question Eight

(a) $ax^4 + 4bx + c = 0$ has a double root at $x = \alpha$

$\Rightarrow 4ax^3 + 4b = 0$ has a single root at $x = \alpha$

$\Rightarrow 4a\alpha^3 + 4b = 0$

$\Rightarrow a\alpha^3 + b = 0 \Rightarrow \alpha^3 = \frac{-b}{a}$ (1)

now $a\alpha^4 + 4b\alpha + c = 0$ (1) (put $x = \alpha$ in eqn)

$\alpha(a\alpha^3 + 4b) + c = 0$

$\alpha(-b + 4b) + c = 0$

$3b\alpha + c = 0$ (1)

$\alpha = \frac{-c}{3b} \Rightarrow \alpha^3 = \frac{-c^3}{27b^3}$ 4

$\therefore -\frac{b}{a} = \frac{-c^3}{27b^3}$ (1)

$\Rightarrow -27b^4 = -ac^3$

ie. $ac^3 = 27b^4$ as required.

b) $P(x) = (x-a)(x-b)Q(x) + R(x)$

as divisor is quadratic $\Rightarrow R(x) = mx + n$ (1)

$\therefore P(x) = (x-a)(x-b)Q(x) + (mx+n)$

now $P(a) = ma + n$ — ①
$P(b) = mb + n$ — ②

① − ② $\Rightarrow m = \frac{P(a) - P(b)}{a-b}$ (1)

$\therefore n = P(a) - ma$ (1)

$= P(a) - \frac{P(a)-P(b)}{a-b} \cdot a$

Q8 (b) cont

So $n = \dfrac{aP(b) - bP(a)}{a-b}$ (1)

$\therefore R(x) = \left[\dfrac{P(a)-P(b)}{a-b}\right]x + \dfrac{aP(b)-bP(a)}{a-b}$ 4

(c) (i) $\sin 3x = -\cos 2x$

$\sin 3x = -\sin\left(\frac{\pi}{2} - 2x\right)$ by hint

$= \sin\left(2x - \pi/2\right)$ (1)

$\therefore 3x = (2x - \pi/2) + 2n\pi$ or $3x = \pi - (2x - \pi/2) + 2n\pi$

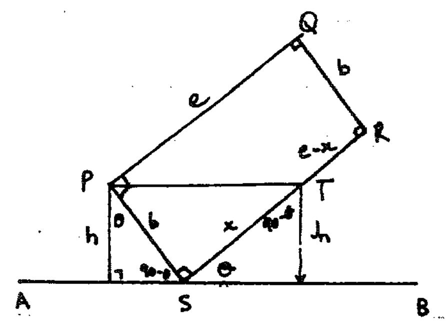
$\Rightarrow x = 2n\pi - \frac{\pi}{2}$ — (1) $\quad 5x = 2n\pi + \frac{3\pi}{2}$

(1) $\quad x = \frac{2n\pi}{5} + \frac{3\pi}{10}$ — (2)

(1) 3

(ii) smallest soln when $n=0$ in (2) $\Rightarrow x = \frac{3\pi}{10}$. (1) 1

(d)



Q8 (d) cont $\sin(90-\theta) = \frac{h}{b} \Rightarrow \cos\theta = \frac{h}{b}$

$\therefore h = b\cos\theta$ (not needed)

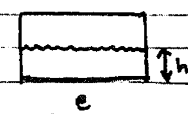
also $\cot\theta = \frac{x}{b}$

$x = b\cot\theta$ (1)

Area $\Delta SPT = \frac{bx}{2} = \frac{b \cdot b\cot\theta}{2} = \frac{b^2}{2}\cot\theta$

(1)

lie tank flat



Area of Water must be the same so

$he = \dfrac{b^2\cot\theta}{2}$ (1)

$\therefore h = \dfrac{b^2\cot\theta}{2e}$ as required.

\# 3