Barker College

2003
TRIAL
HIGHER SCHOOL
CERTIFICATE

Mathematics Extension 2

Staff Involved:

- MRB*
- BHC*
- DOK
- BTP

AM WEDNESDAY 6 AUGUST

35 copies

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- Write your Barker Student Number on ALL pages of your answer sheets
- A table of standard integrals is provided on page 10

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value
- ALL necessary working should be shown in every question
- Start each question on a NEW page
- Write on one side of each answer page
- Marks may be deducted for careless or badly arranged work

Total marks – 120
Attempt Questions 1 – 8
ALL questions are of equal value

Answer each question on a SEPARATE sheet of paper

Marks

Question 1 (15 marks) **[BEGIN A NEW PAGE]**

(a) By using the substitution $u = \sec x$, or otherwise,

find $\displaystyle\int \frac{\tan x}{\sec^2 x}\, dx$ **2**

(b) By completing the square, find

$$\int \frac{1}{\sqrt{1 + 2x - x^2}}\, dx$$ **2**

(c) Find $\displaystyle\int x^2 e^{-x}\, dx$ **2**

(d) Evaluate

(i) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1 - \tan x}{1 + \tan x}\, dx$ **2**

(ii) $\displaystyle\int_0^2 \frac{x}{(x + 1)(x + 2)}\, dx$ **3**

(e) Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\, dx$ where n is an integer and $n \geq 3$

(i) Show that $I_n + I_{n-2} = \dfrac{1}{n - 1}$ **3**

(ii) Hence evaluate I_7 **1**

Marks

Question 2 (15 marks) **[BEGIN A NEW PAGE]**

(a) If $w = \dfrac{1 + 2i}{3 + 4i}$ then

(i) express w in the form $a + ib$ **2**

(ii) find the value of $|w|$ **2**

(b) (i) Express $1 - \sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$ **1**

(ii) If $z = \dfrac{1 - \sqrt{3}i}{2}$, then show that $z^2 = \dfrac{-1 - \sqrt{3}i}{2}$ **1**

(iii) Write down an equation with real coefficients that has both z and z^2 as roots. **1**

(c) Describe in geometric terms the locus in the Argand plane represented by:

(i) $|z - 2i| = 1$ where $z = x + iy$ **1**

(ii) $2|z| = z + \bar{z} + 4$ where $z = x + iy$ **3**

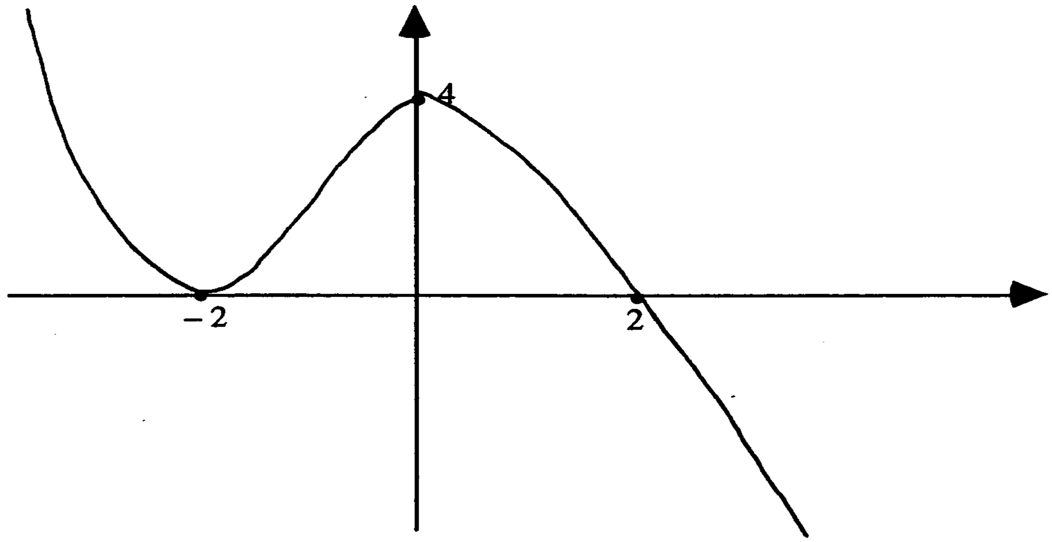
(d) If $3 - i$ is a root of $P(z) = z^3 + rz^2 + sz + 20$, where r and s are real numbers

(i) State why $3 + i$ is also a root **1**

(ii) Factorize $P(z)$ over the complex field **3**

Question 3 (15 marks) **[BEGIN A NEW PAGE]**

(a)



The diagram shows $y = f(x)$.

Draw separate one-third page sketches of the graphs of the following:

(i) $y = f\left(\frac{1}{x}\right)$ **2**

(ii) $|y| = f(|x|)$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(b) (i) Prove that the curve $y = \frac{4x}{1 + x^2}$ has a local minimum at $A(-1, -2)$, a local maximum at $B(1, 2)$ and a point of inflexion at $O(0, 0)$. **2**

(ii) Sketch the curve. **2**

(iii) If $g'(x) = \frac{4x}{1 + x^2}$ and $g(0) = 0$, then sketch $y = g(x)$ clearly labelling all essential features. **3**

Marks

Question 4 (15 marks) **[BEGIN A NEW PAGE]**

(a) The ellipse $\mathcal{E}$ has cartesian equation $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

(i) Write down its eccentricity, the coordinates of its foci S and S' and the equation of each directrix, where S lies on the positive side of the x–axis. **3**

(ii) Sketch $\mathcal{E}$ clearly labelling all essential features. **2**

(iii) If P lies on $\mathcal{E}$, then prove that the sum of the distances PS and PS' is independent of P. **2**

(iv) The tangent at P intersects the directrix at T where T is the point (a, b) and $a > 0$.
Show that PT subtends a right angle at S. **2**

(b) $P\left(p, \dfrac{1}{p}\right)$ and $Q\left(q, \dfrac{1}{q}\right)$ are two variable points on the rectangular hyperbola $xy = 1$

(i) If M is the midpoint of the chord PQ and if OM is perpendicular to PQ, express q in terms of p. **2**

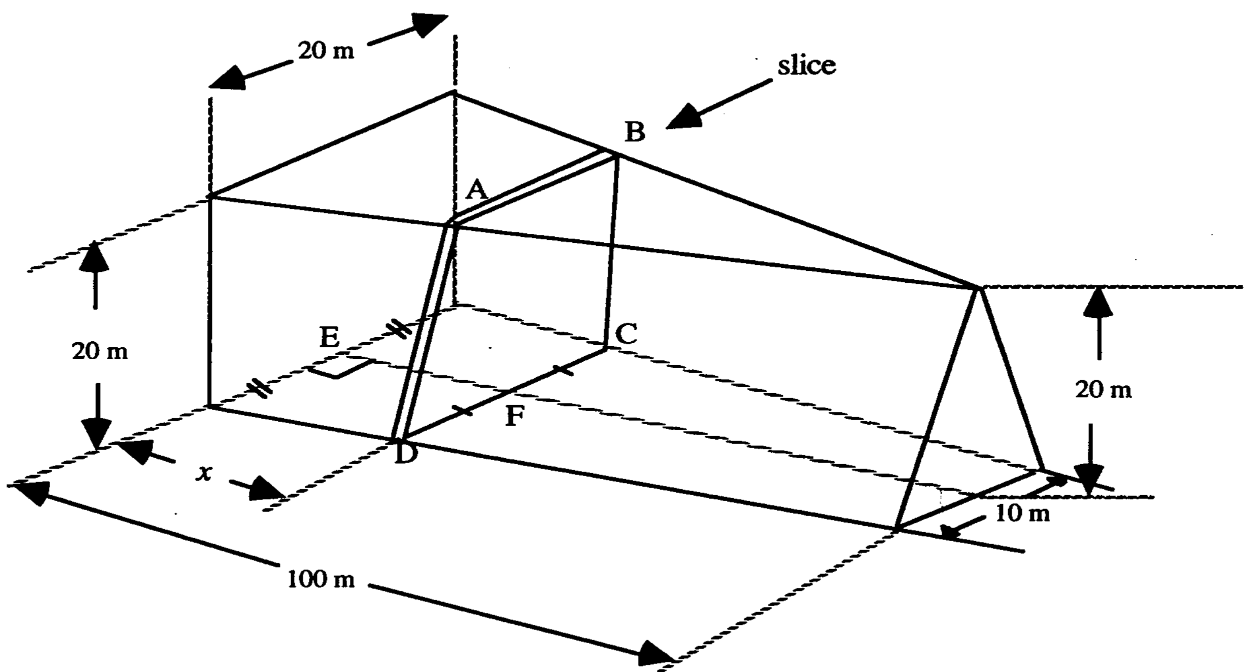
(ii) If the tangents at P and Q meet at R, find the locus of R. **4**

Question 5 (15 marks) **[BEGIN A NEW PAGE]**

(a) (i) Find the volume generated when the area bounded by $y = \sin x$ and the x–axis, where $0 \leq x \leq \pi$ is rotated about the x–axis. **3**

(ii) This same area is rotated about the line $x = 2\pi$. Find the volume of the solid formed. **4**

(b) A boat showroom is built on level ground. The length of the showroom is 100 m. At one end of the showroom the shape is a square measuring 20 m by 20 m and at the other end an isosceles triangle of height 20 m and base 10 m.



(i) If EF is x m in length, show that the length of DC is $\left(20 - \frac{x}{10}\right)$ m. **2**

(ii) By considering trapezoidal slices parallel to the ends of the showroom, find the volume enclosed by the showroom in m^3. **6**

Marks

Question 6 (15 marks) **[BEGIN A NEW PAGE]**

(a) A particle is projected upwards with a speed V and an angle of elevation α from level ground.

(i) If the point of projection is the origin, show that $y = x\tan\alpha - \dfrac{gx^2\sec^2\alpha}{2V^2}$ is the cartesian equation of the particle's path. **3**

(ii) Hence, or otherwise, deduce that the range on the horizonal plane is

$$\frac{V^2\sin 2\alpha}{g}$$ **1**

(iii) A vertical wall of "unlimited" height is a distance d from the origin, and the plane of the wall is perpendicular to the plane of the particle's trajectory.

Show that if $d < \dfrac{V^2}{g}$, the particle will strike the wall provided that

$$\beta < \alpha < \frac{\pi}{2} - \beta \quad \text{where} \quad \beta = \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right)$$ **3**

(b) A particle of unit mass moves in a straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$.

(i) Show that v is related to the displacement x by the formula

$$x = \tan^{-1}\left[\frac{Q - v}{1 + Qv}\right]$$ **2**

(ii) Show that the time t which has elapsed when the particle is travelling with velocity v is given by $t = \dfrac{1}{2}\log_e\left[\dfrac{Q^2\left(1 + v^2\right)}{v^2\left(1 + Q^2\right)}\right]$ **2**

(iii) Find v^2 as a function of t. **2**

(iv) Find the limiting values of v and x as $t \to \infty$. **2**

Marks

Question 7 (15 marks) **[BEGIN A NEW PAGE]**

(a) A fair coin is tossed $2n$ times. The probability of observing k heads and $(2n - k)$ tails is given by

$$P_k = \binom{2n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{2n-k}$$

(i) Show that the most likely outcome is $k = n$, i.e. show that P_k is greatest when $k = n$ **2**

(ii) Show that $P_n = \dfrac{(2n)!}{2^{2n}(n!)^2}$ **1**

(iii) If $\dfrac{\pi}{2}\left(\dfrac{2n}{2n+1}\right) < \dfrac{2^2.4^2 \ldots\ldots (2n)^2}{1.3^2.5^2 \ldots\ldots (2n-1)^2(2n+1)} < \dfrac{\pi}{2}$

then show that $\dfrac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < P_n < \dfrac{1}{\sqrt{\pi n}}$ **3**

(b) A polynomial $P(x)$ is given by $P(x) = x^7 - 1$.
Let p be the complex root of $P(x) = 0$ which has the smallest positive argument. (N.B. $p \neq 1$)

(i) Show that $1 + p + p^2 + p^3 + p^4 + p^5 + p^6 = 0$ **1**

(ii) Let $\theta = p + p^2 + p^4$ and $Q = p^3 + p^5 + p^6$

I. Prove that $\theta + Q = -1$ and $\theta Q = 2$ **3**

II. Hence show that $\theta = \dfrac{-1 + i\sqrt{7}}{2}$ and $Q = \dfrac{-1 - i\sqrt{7}}{2}$ **2**

(iii) Given that $T(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$

(i) Explain why $T(x) = (x-p)(x-p^2)(x-p^4)(x-p^3)(x-p^5)(x-p^6)$ **1**

(ii) Express $T(x)$ as a product of two cubics with coefficients involving θ, Q and rational numbers. **2**

Marks

Question 8 (15 marks) **[BEGIN A NEW PAGE]**

(a) (i) By considering the expansion of $(\cos\theta + i\sin\theta)^7$ and by using De Moivre's theorem show that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$ **2**

(ii) Hence find all six roots of the equation

$$64x^6 - 112x^4 + 56x^2 - 7 = 0$$ **2**

(iii) Hence show that $\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$ **2**

(iv) Find the exact value of $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}$ **2**

(b) (i) Sketch the curve $y = \log_e x$ where $x > 0$ **1**

(ii) Prove that the curve $y = \log_e x$ is concave down, for all $x > 0$ **1**

(iii) Find the approximate area under $y = \log_e x$ bounded by the x–axis, $x = 1$ and $x = n$ by using the areas of trapezia drawn under the curve, each of unit width (where n is an integer). **1**

(iv) Show that the area under $y = \log_e x$ bounded by the x–axis, $x = 1$ and $x = n$ is equal to $(1 - n + n\log_e n)$ **2**

(v) Hence show that $n! < \dfrac{en^{n+\frac{1}{2}}}{e^n}$ **2**

End of Paper