Student Number ....

Barker College

2006
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Staff Involved:

- BHC*
- BTP
- MRB
- JM

50 copies

AM FRIDAY 4 AUGUST

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- Write your Barker Student Number on ALL pages of your answer sheets
- A table of standard integrals is provided on page 12

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value
- ALL necessary working should be shown in every question
- Start each question on a NEW page
- Write on one side of each answer page
- Marks may be deducted for careless or badly arranged work

Total marks – 120
Attempt Questions 1 – 8
ALL questions are of equal value
Answer each question on a SEPARATE sheet of paper

Marks

Question 1 (15 marks) **[START A NEW PAGE]**

(a) Find

(i) $\displaystyle\int \frac{2x - 6}{\sqrt{x^2 - 6x}}\, dx$ **1**

(ii) $\displaystyle\int \frac{dx}{\sqrt{x^2 - 6x}}$ **2**

(iii) $\displaystyle\int \frac{dx}{\sqrt{6x - x^2}}$ **2**

(b) Show that $\displaystyle\int_{-1}^{0} \frac{4\, dx}{(1 - x)(x - 3)} = \ln\left(\frac{4}{9}\right)$ **3**

(c) Using the substitution $x = 3\sec\theta + 3$, find $\displaystyle\int \frac{dx}{(x - 3)\sqrt{x^2 - 6x}}$ **3**

(d) Let $\displaystyle I_n = \int_0^2 x^n e^{-2x}\, dx$, where n is a non-negative integer.

(i) Show that $\displaystyle I_n = \frac{n}{2} I_{n-1} - \frac{2^{n-1}}{e^4}$ **2**

(ii) Evaluate I_2 **2**

Marks

Question 2 (15 marks) **[START A NEW PAGE]**

(a) (i) Find the square root of $45 - 28i$ **3**

(ii) Given that $Z = \dfrac{-3 - 2i + \sqrt{45 - 28i}}{5 + 5i}$ and that Z is purely imaginary, find Z^6 **3**

(b) (i) Find the modulus and argument of the complex numbers $(-1 + i)$ and $(1 + \sqrt{3}i)$, and hence express $z = \dfrac{-1 + i}{1 + \sqrt{3}i}$ in modulus/argument form. **3**

(ii) Hence, express $\cos\left(\dfrac{5\pi}{12}\right)$ in surd form. **2**

(c) The locus of w is described by the equation $|w - 4| = |w + 2 + 6i|$

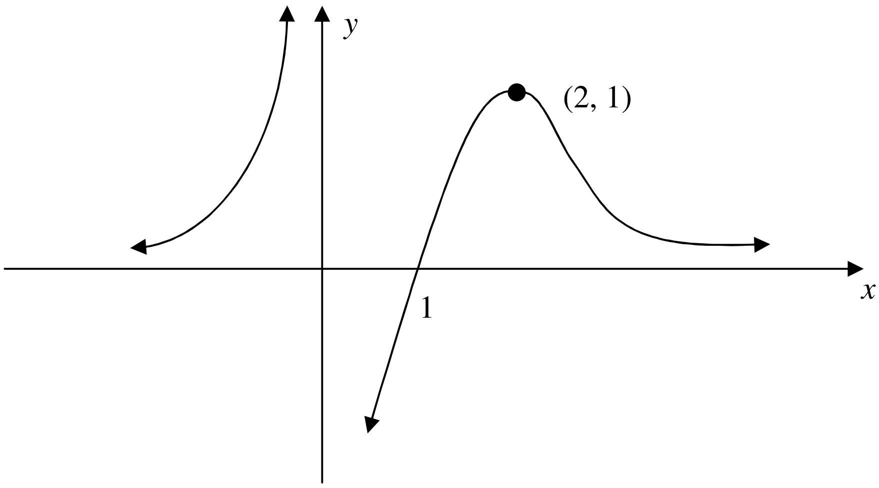
(i) Sketch on an Argand diagram the locus of w. **2**

(ii) Find the Cartesian equation of the locus of w. **2**

Marks

Question 3 (15 marks) **[START A NEW PAGE]**

(a) The graph of $y = f(x)$ is sketched below.



Draw separate sketches of:

(i) $y = f(|x|)$ **2**

(ii) $y = |f(x)|$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = \log_e(f(x))$ **2**

(v) $y = e^{f(x)}$ **2**

Question 3 continues on page 5

Marks

Question 3 (continued)

(b) (i) Sketch the curve $y = \dfrac{x}{2 + x^2}$

(Do **not** use calculus. Do **not** attempt to find the coordinates of any turning points). **2**

(ii) Show that the equation $kx^3 + (2k - 1)x = 0$ can be written in the form $\dfrac{x}{2 + x^2} = kx$ **1**

(iii) Using a graphical approach based on the curve $y = \dfrac{x}{2 + x^2}$, or otherwise, find the values of k for which the equation $kx^3 + (2k - 1)x = 0$ has **exactly** one real root. **2**

End of Question 3

Marks

Question 4 (15 marks) **[START A NEW PAGE]**

(a) The region between the positive x-axis, the y-axis and the curve $y = \cos^{-1} x$ is rotated about the y-axis.

By taking slices perpendicular to the y-axis, find the volume of this solid. **4**

(b) The base of a particular solid is the circle $x^2 + y^2 = 1$.

All cross-sections of the solid perpendicular to the x-axis are equilateral triangles.

Find the volume of this solid. **4**

(c) Use the method of cylindrical shells to calculate the volume of the solid generated when that part of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $x \geq 0$ and $y \geq 0$, is rotated about the line $x = a$. **7**

Marks

Question 5 (15 marks) **[START A NEW PAGE]**

(a) Show that $2 + i$ is a root of $P(x) = x^3 + x^2 - 15x + 25$ and hence completely factorise $P(x)$ over the field of complex numbers. **3**

(b) Prove that if the polynomial $H(x)$ has a root α of multiplicity m, then $H'(x)$ has a root α of multiplicity $m - 1$ **3**

(c) Consider any equation of the form $px^3 + qx^2 + r = 0$ where p, q and r are not equal to zero.

(i) Explain why such an equation cannot have a triple root. **2**

(ii) Assuming that the equation has a double root, find an expression for r in terms of p and q. **2**

(d) If α, β and γ are the roots of $x^3 - 2x^2 + 3x - 2 = 0$, find:

(i) $\alpha^3 + \beta^3 + \gamma^3$ **3**

(ii) the equation whose roots are $\dfrac{\alpha\beta}{\gamma}, \dfrac{\beta\gamma}{\alpha}, \dfrac{\alpha\gamma}{\beta}$ **2**

Marks

Question 6 (15 marks) **[START A NEW PAGE]**

(a) The hyperbola *H* has equation $x^2 - 2y^2 = 4$

(i) Find the foci, asymptotes and vertices of *H*. **3**

(ii) Show that the equation of the normal to *H* at $P\left(2\sqrt{2}, \sqrt{2}\right)$ is $x + y = 3\sqrt{2}$ **3**

(iii) Find the equation of the circle that is tangent to *H* at *P* and $Q\left(-2\sqrt{2}, \sqrt{2}\right)$ **3**

(b) The points $R(a\cos\theta, b\sin\theta)$ and $S(-a\sin\theta, b\cos\theta)$ lie on the ellipse *E* with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(i) The equation of the tangent at *R* is given by $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

Find the equation of the tangent at *S* and hence show that the coordinates of *T*, their point of intersection, are given by $T\left(a(\cos\theta - \sin\theta),\ \ b(\sin\theta + \cos\theta)\right)$ **4**

(ii) Show that the locus of *T* is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$ **2**

Marks

Question 7 (15 marks) **[START A NEW PAGE]**

Mr Mock's mathematics class bought him a ride in a hot air balloon. When Mr Mock's hands were h metres above the Earth's surface, he dropped overboard a bag of prawn crackers of mass m kg. The bag encounters air resistance proportional to its velocity, i.e. the resistive force is equal to mkv.

Taking Mr Mock's hands as the origin and downwards displacement as positive:

(i) Show that the equation of motion of the bag of prawn crackers is $\ddot{x} = k(V - v)$, where V m/sec is the terminal velocity of the bag. **3**

(ii) Show that the displacement, x metres, of the bag from Mr Mock's hands is given by $x = \frac{-v}{k} - \frac{g}{k^2} \ln\left(\frac{g - kv}{g}\right)$ **4**

(iii) If the bag reaches the Earth's surface with a velocity of u m/sec, show that $\ln\left(1 - \frac{uk}{g}\right) + \frac{uk}{g} + \frac{k^2 h}{g} = 0$ **2**

(iv) Find the time T seconds for the bag of prawn crackers to attain 50% of its terminal velocity, and the distance fallen in this time. **6**

Marks

Question 8 (15 marks) **[START A NEW PAGE]**

(a) Consider the infinitely nested radical α, where $\alpha = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$

I claim that α has a limiting value of $\frac{1}{2}\left(1 + \sqrt{5}\right)$.

This claim can be proved as follows:

Now $\alpha = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$

so $\alpha = \sqrt{1 + \alpha}$

so $\alpha^2 = 1 + \alpha$

so $\alpha^2 - \alpha - 1 = 0$

so $\alpha = \dfrac{1 \pm \sqrt{5}}{2}$

But $\alpha > 0$ so $\alpha = \frac{1}{2}\left(1 + \sqrt{5}\right)$

Use this technique to express in surd form the limiting value of β, where

$$\beta = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \ldots}}}$$ **2**

(b) Let $t = \tan^{-1}\left(\frac{1}{5}\right)$

(i) Use trigonometric identities to show that

$\tan 2t = \dfrac{5}{12}$

$\tan 4t = \dfrac{120}{119}$

$\tan\left(4t - \dfrac{\pi}{4}\right) = \dfrac{1}{239}$ **3**

(ii) Hence show that $\dfrac{\pi}{4} = 4\tan^{-1}\left(\dfrac{1}{5}\right) - \tan^{-1}\left(\dfrac{1}{239}\right)$ **2**

Question 8 continues on page 11

Marks

Question 8 (continued)

(c) Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

(i) Show that $f'(x) = \frac{1}{1 + x}$, assuming that $-1 < x < 1$ **2**

(ii) Show that $f(x) = \ln(1 + x)$, assuming that $-1 < x < 1$ **2**

(iii) Hence, derive the formula $\frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$ **3**

(iv) By an appropriate substitution into the formula in (iii) above,

show that $\ln\left(\frac{3}{2}\right) \approx 0 \cdot 405$, correct to 3 decimal places. **1**

End of Paper

LIST OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1\,;\; x \neq 0, \text{ if } n < 0.$$

$$\int \frac{1}{x}\, dx = \log_e x, \; x > 0.$$

$$\int e^{ax}\, dx = \frac{1}{a} e^{ax}, a \neq 0.$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax, a \neq 0.$$

$$\int \sin ax\, dx = \frac{-1}{a} \cos ax, \; a \neq 0.$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax, a \neq 0.$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax, a \neq 0.$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, a \neq 0.$$

$$\int \frac{1}{\sqrt{(a^2 - x^2)}} dx = \sin^{-1} \frac{x}{a}, a > 0, -a < x < a.$$

$$\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log_e \left\{x + \sqrt{(x^2 - a^2)}\right\} |x| > |a|.$$

$$\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log_e \left\{x + \sqrt{(x^2 + a^2)}\right\}$$