Student Number ....

Barker College

2008
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Staff Involved:

- BHC*
- BTP*
- JM
- WMD

35 copies

PM MONDAY 11 AUGUST

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages of your answer sheets
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value
- ALL necessary working should be shown in every question
- Start each question on a NEW page
- Write on one side of each answer page
- Marks may be deducted for careless or badly arranged work

Total marks – 120
Attempt Questions 1–8
ALL questions are of equal value

Answer each question on a SEPARATE sheet of paper

Marks

Question 1 (15 marks) **[START A NEW PAGE]**

(a) If $(\sqrt{3}+i)z = 4\sqrt{3}-4i$, find

(i) z in the form $a+bi$ **1**

(ii) $|z|$ **1**

(iii) $\arg(z)$ **1**

(iv) z^8 in the form $a+bi$ **1**

(b) Find $\sqrt{-5-12i}$ in the form $a+bi$, and hence solve the equation

$$z^2+(1-2i)z+(\frac{1}{2}+2i)=0$$ **4**

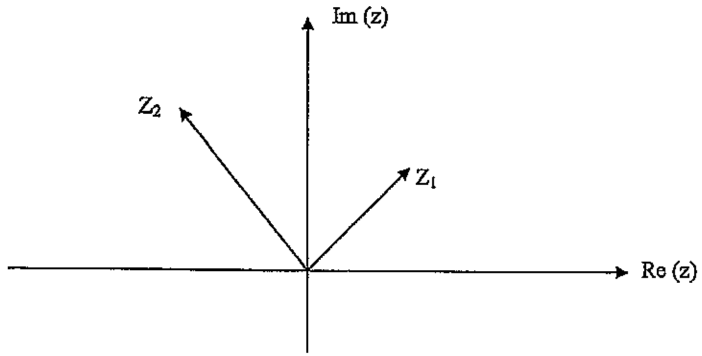
(c) If w is a complex cube root of unity, show that $1+w+w^2=0$, and hence prove that $(1+w)(1+2w)(1+3w)(1+7w)=31+2w$. **3**

Question 1 continues on page 3

Marks

Question 1 (continued)

(d) Let z_1 and z_2 be two given complex numbers as shown on the Argand diagram below.



Let z be a variable complex number.

Sketch and describe the locus of z on an Argand diagram if:

(i) $|z-z_1|=|z-z_2|$ **2**

(ii) $\arg\left(\frac{z-z_1}{z-z_2}\right)=\alpha$, where $0<\alpha<\pi$ **2**

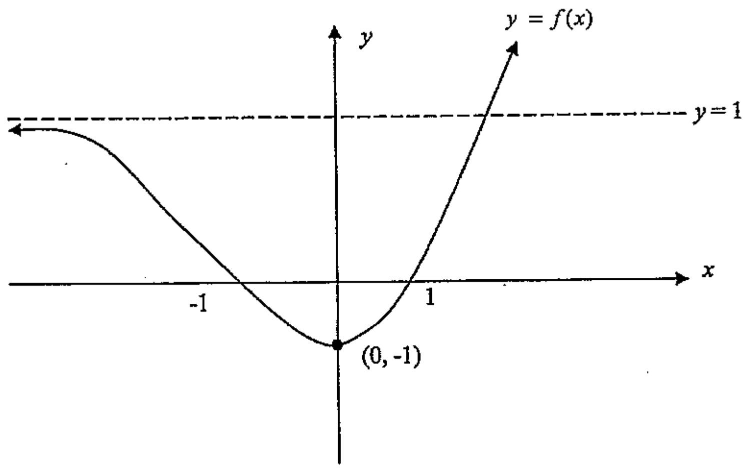
End of Question 1

Marks

Question 2 (15 marks) **[START A NEW PAGE]**

(a) The diagram below shows the graph of $y = f(x)$.

There is a minimum turning point at (0, -1).



On separate diagrams, draw the graph of

(i) $y = f(|x|)$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \sin^{-1}[f(x)]$ **2**

(iv) $y = ln[f(x)]$ **2**

Marks

Question 2 (continued)

b) The equation of a curve is given by $xy^2 + x^2 = 1$

(i) Explain why $x = 0$ is not in the domain of the curve. **1**

(ii) Find the x – intercepts of the curve. **1**

(iii) Re-write the equation of the curve making y the subject, and hence find the domain of the curve. **2**

(iv) The curve has two asymptotes. Write down the equations of both asymptotes. **2**

(v) Hence sketch the curve $xy^2 + x^2 = 1$ **1**

End of Question 2

Marks

Question 3 (15 marks) **[START A NEW PAGE]**

(a) Find

(i) $\displaystyle\int \frac{(x+3)\,dx}{\sqrt[3]{x^2+6x}}$ using the substitution $u = x^2 + 6x$ **2**

(ii) $\displaystyle\int \frac{dy}{y^2+10y+30}$ **2**

(iii) $\displaystyle\int \frac{dx}{2+\cos x}$ using the "t-results" **3**

(iv) $\displaystyle\int x^2 e^{2x}\,dx$ **3**

(b) Factorise $x^3 + x^2 - 6x$ and then find the values of A, B and C such that

$$\frac{x+1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$$

Hence find $\displaystyle\int \frac{(x+1)\,dx}{x^3+x^2-6x}$ **5**

End of Question 3

Marks

Question 4 (15 marks) **[START A NEW PAGE]**

(a) Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,dx$

Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$

Deduce the value of I_5. **5**

(b) Find the volume of the torus generated by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$. **4**

(c) (i) Show that

$$\frac{d}{dx}\left\{\frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right\} = \sqrt{a^2-x^2}$$ **3**

(ii) The base of a solid is the circle $x^2 + y^2 = 16x$. Every slice of this solid taken perpendicular to the x axis is a rectangle of height 6 units. Using the result from part (i) above, find the volume of this solid. **3**

End of Question 4

Marks

Question 5 (15 marks) **[START A NEW PAGE]**

(a) Consider the polynomial

$p(x) = ax^4 + bx^3 + cx^2 + d$

where a, b, c and d are integers.

Suppose that α is an integer such that $p(\alpha) = 0$

(i) Prove that d is a multiple of α **2**

(ii) Prove that the polynomial $q(x) = 5x^4 - x^3 + 3x^2 - 3$

does not have an integer root. **2**

(b) Let $P(x) = x^3 - 11x - 14$

Factorise $P(x)$ over the reals and hence find the three roots of $P(x) = 0$ **3**

(c) Find the roots of $q(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$

given that it has a root of multiplicity 3. **2**

(d) Let α, β and γ be the roots of the equation $x^3 - 5x^2 + 5 = 0$

(i) Find the value of $(\alpha-1)(\beta-1)(\gamma-1)$ **2**

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$ **2**

(iii) Find a polynomial equation with integer coefficients whose roots are

α^2, β^2 and γ^2 **2**

End of Question 5

Marks

Question 6 (15 marks) **[START A NEW PAGE]**

(a) For the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$, find

(i) the lengths of the axes **1**

(ii) the eccentricity **1**

(iii) the co-ordinates of the foci **1**

(iv) the equations of the directrices **1**

(b) Let $P(2\sec\theta, \sqrt{5}\tan\theta)$ be a variable point on the hyperbola $5x^2 - 4y^2 = 20$

The tangent at P meets the directrix at T. Show that PT subtends a right angle at the corresponding focus. **6**

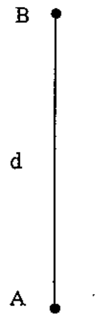
(c) (i) If the line $y = mx + b$ is a tangent to the ellipse $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$,

show that $b^2 = 6m^2 + 3$ **2**

(ii) The tangents to this ellipse from a point $P(X, Y)$ meet at right angles. Prove that the locus of P is the circle $x^2 + y^2 = 9$. **3**

End of Question 6

Question 7 (15 marks) **[START A NEW PAGE]** **Marks**



A and B are two points d units apart in a vertical line. B is directly above A. Two identical particles are projected from A and B towards each other with the same velocity, u.

The resistance of the medium is kv per unit mass.

(i) Draw a diagram indicating all forces acting on the particles. **1**

(ii) Consider the particle moving upward from A. By writing an expression for $\frac{dv}{dt}$,

(α) show that $t = \frac{1}{k} \ln \left(\frac{g + ku}{g + kv} \right)$ **2**

(β) Hence, find v in terms of t **2**

(γ) Hence, find x in terms of t **2**

(iii) Consider the particle moving downward from B. Given that $\frac{dv}{dt} = g - kv$,

(α) find t in terms of v. **2**

(β) Find v in terms of t **2**

(γ) Find x in terms of t. **2**

(iv) Hence, prove that the particles meet after a time of $\frac{1}{k} \ln \left(\frac{2u}{2u - kd} \right)$ **2**

End of Question 7

Marks

Question 8 (15 marks) **[START A NEW PAGE]**

(a) A particle is moving along the x axis. Its acceleration is given by

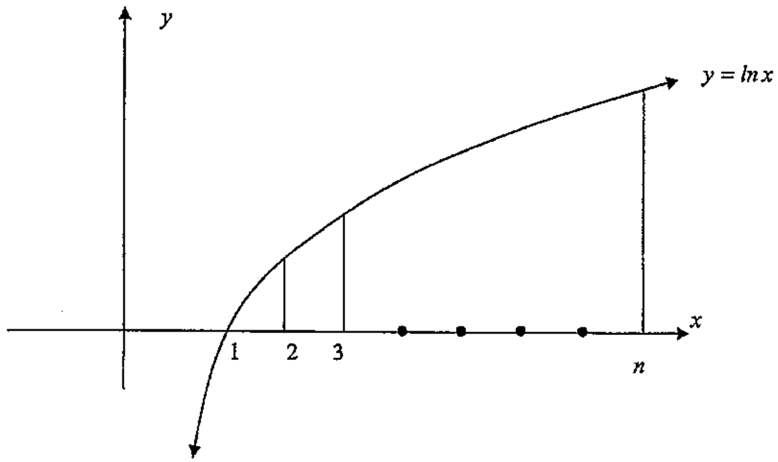
$\frac{d^2 x}{dt^2} = \frac{12 - 4x}{x^3}$. The particle starts from rest at the point $x = 6$.

(i) Show that the particle starts moving in the negative x direction. **1**

(ii) Find an expression for velocity, v, in terms of x. **3**

(iii) The path along which the particle moves is bounded. What part of the x axis is the path of the particle? **1**

(b) Consider the area under the curve $y = \ln x$ between $x = 1$ and $x = n$.



(i) Show that this area is exactly equal to $\ln\left(\frac{n^n}{e^{n-1}}\right)$ **2**

Question 8 continues on page 12

Marks

Question 8 (continued)

(ii) Use the Trapezoidal Rule to find an expression which approximates this area. **2**

(iii) Hence show that $n^n > \sqrt{n}\,(n-1)!\,e^{n-1}$ **1**

(c) Given that $\sin^{-1}2x$, $\cos^{-1}2x$ and $\sin^{-1}(1-2x)$ are all acute,

(i) Show that $\sin[\cos^{-1}2x - \sin^{-1}2x] = 1-8x^2$ **3**

(ii) Solve the equation $\cos^{-1}2x - \sin^{-1}2x = \sin^{-1}(1-2x)$ **2**

End of Paper

Extension 2 Mathematics

Trial Examination Term 3 2008.

Question 1:

(a) $(\sqrt{3}+i)z = 4\sqrt{3}-4i$

(i) $z = \dfrac{4\sqrt{3}-4i}{\sqrt{3}+i} \times \dfrac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \dfrac{12-4\sqrt{3}i-4\sqrt{3}i-4}{4}$

$= 2-2\sqrt{3}\,i$

(ii) $|z| = \sqrt{2^2+(2\sqrt{3})^2}$

$= \sqrt{16}$

$= 4$

(iii) $\arg(z) = -\dfrac{\pi}{3}$

(iv) $z^8 = \left[4\,\text{cis}\left(-\frac{\pi}{3}\right)\right]^8$

$= 2^{16}\,\text{cis}\left(-\frac{8\pi}{3}\right)$

$= 2^{16}\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

$= 2^{16}\left\{-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right\}$

$= -2^{15} - 2^{15}\sqrt{3}\,i$

1 (b) Let $a + bi = \sqrt{-5-12i}$ then

$$a^2 - b^2 = -5 \quad \text{and} \quad 2abi = -12i$$
$$b = \frac{-6}{a}.$$

So $a^2 - \left(\frac{-6}{a}\right)^2 = -5$

$$a^4 + 5a^2 - 36 = 0$$
$$(a^2-4)(a^2+9) = 0$$
$$\therefore a = \pm 2$$

when $a = 2$, $b = -3$.
when $a = -2$, $b = 3$.
Hence $\sqrt{-5-12i} = \pm(2-3i)$.

Solving $z^2 + (1-2i)z + (\frac{1}{2}+2i) = 0$ we have

$$z = \frac{-(1-2i) \pm \sqrt{(1-2i)^2 - 4(\frac{1}{2}+2i)}}{2}$$

$$z = \frac{-1+2i \pm \sqrt{-5-12i}}{2}$$

$$z = \frac{-1+2i \pm (2-3i)}{2}$$

$$z = \frac{1-i}{2} \quad \text{and} \quad z = \frac{-3+5i}{2}.$$

1 (c) If w is a complex cube root of unity, then

$$w^3 - 1 = 0$$

ie, $(w-1)(w^2+w+1) = 0$.

But $w \neq 1$, hence $1 + w + w^2 = 0$, as required.

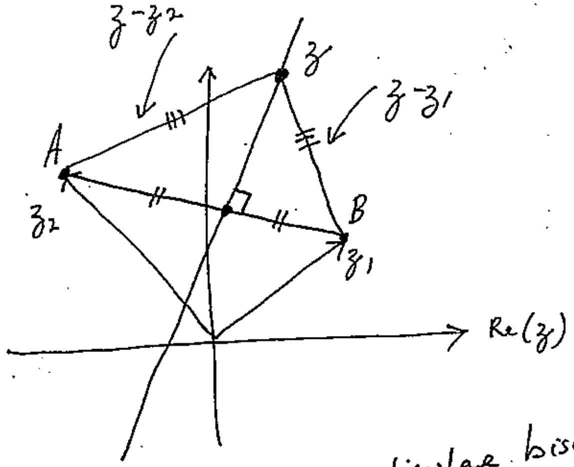
RTP that

$$(1+w)(1+2w)(1+3w)(1+7w) = 31 + 2w.$$

$$\begin{aligned}
\text{LHS} &= (1+3w+2w^2)(1+10w+21w^2) \\
&= 1 + 10w + 21w^2 + 3w + 30w^2 + 63w^3 + 2w^2 + 20w^3 + 42w^4 \\
&= 84 + 13w + 53w^2 + 42w^4 && (\text{because } w^3 = 1) \\
&= 84 + 55w + 53w^2 && (\text{because } w^4 = w) \\
&= (53 + 53w + 53w^2) + (31 + 2w) \\
&= 53(1 + w + w^2) + (31 + 2w) \\
&= 31 + 2w && (\text{because } 1 + w + w^2 = 0) \\
&= \text{RHS}.
\end{aligned}$$

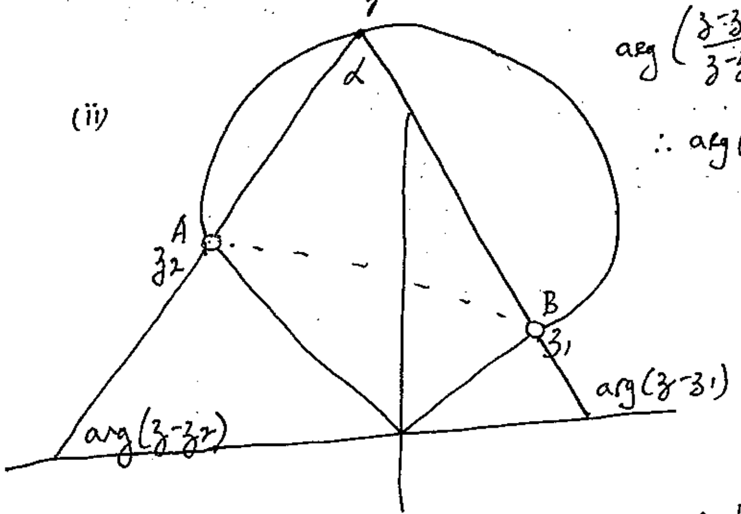
1 (d)

(i)



The locus of z is the perpendicular bisector of AB.

(ii)



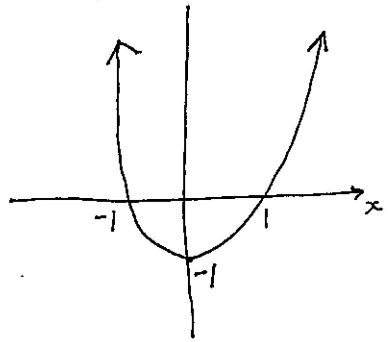
$\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha$

$\therefore \arg(z-z_1) = \alpha + \arg(z-z_2)$

The locus of z is the upper part of the circle above the chord AB.

Question 2: (a)

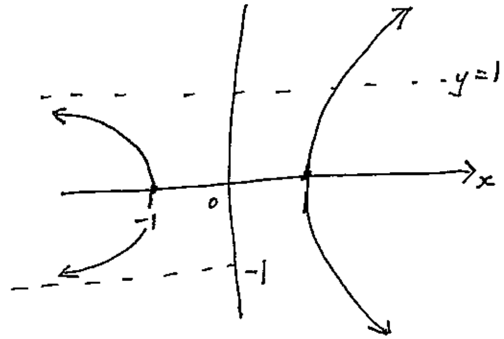
(i) $y = f(|x|)$



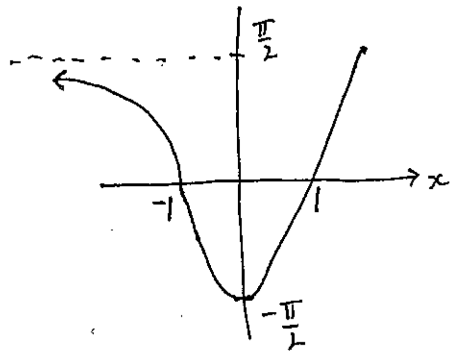
The left branch is the reflection of the right branch in the y axis.

(ii) $y^2 = f(x)$

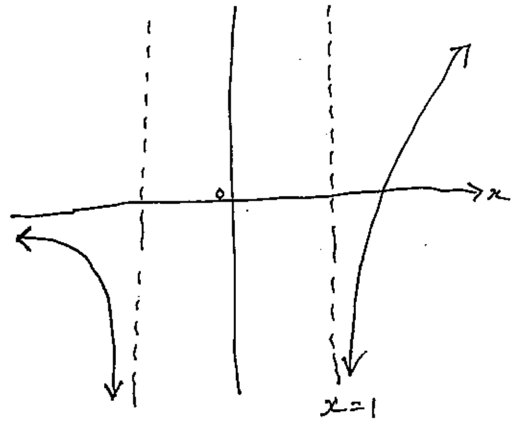
ie, $y = \pm\sqrt{f(x)}$



(iii) $y = \sin^{-1}[f(x)]$



(iv) $y = \ln[f(x)]$



2 (b)

(i) Suppose that $x=0$ were in the domain. Then, by substitution, into the equation of the curve, we would have

$$0 \times y^2 + 0^2 = 1$$

ie, $0 = 1$, which is false.

Hence $x = 0$ is not in the domain.

(ii) Substitute $y=0$ and solve $x^2 = 1$.

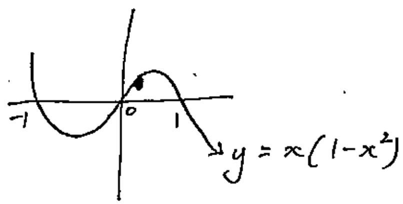
The x intercepts are $(1, 0)$ and $(-1, 0)$.

(iii) $$y^2 = \frac{1-x^2}{x}$$

$$y = \pm\sqrt{\frac{1-x^2}{x}}$$

Need $\frac{1-x^2}{x} \geq 0$

ie, $x(1-x^2) \geq 0$



∴ The domain is

$x \leq -1$ OR $0 < x \leq 1$.

2b

(iv) $$y = \pm\sqrt{\frac{1-x^2}{x}}$$

ie, $$y = \pm\sqrt{\frac{1}{x} - x}$$

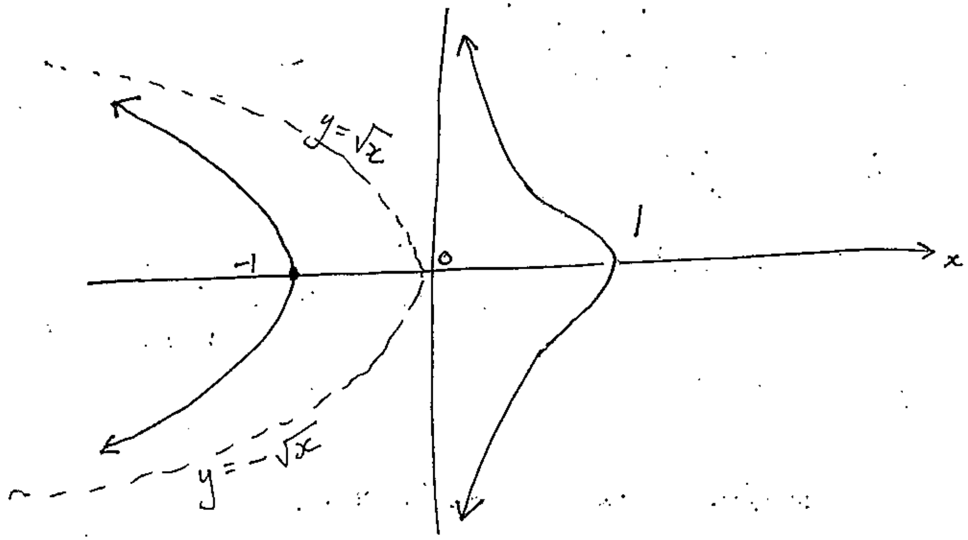
as $x \to -\infty$, $\frac{1}{x} \to 0$

Hence $y \to \pm\sqrt{-x}$. $[y^2 \to -x]$

Hence the asymptotes are $x = 0$ and $y = \pm\sqrt{-x}$

i.e. $x = 0$ and $y^2 = -x$.

(v) $xy^2 + x^2 = 1$



Question 3: (a)

(i) $\int \frac{(x+3)\,dx}{\sqrt[3]{x^2+6x}}$

Let $u = x^2+6x$

$du = (2x+6)\,dx$

$\frac{1}{2}du = (x+3)\,dx$

$I = \int \frac{\frac{1}{2}du}{u^{1/3}}$

$= \frac{1}{2}\int u^{-1/3}\,du$

$= \frac{3}{4}u^{2/3}$

$= \frac{3}{4}(x^2+6x)^{2/3} + C.$

(ii) $\int \frac{dy}{y^2+10y+30}$

$= \int \frac{dy}{5+(y+5)^2}$

$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{y+5}{\sqrt{5}}\right) + C.$

(iii) $\int \frac{dx}{2+\cos x}$

Let $t = \tan\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

$= \frac{1}{2}(1+t^2)$

$\therefore dx = \frac{2\,dt}{1+t^2}.$

And $2+\cos x = 2+\frac{1-t^2}{1+t^2}$

$= \frac{2+2t^2+1-t^2}{1+t^2} = \frac{3+t^2}{1+t^2}$

Hence $I = \int \frac{\frac{2dt}{1+t^2}}{\frac{3+t^2}{1+t^2}}$

$= \int \frac{2\,dt}{3+t^2}$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{3}}\right) + C.$

3(a)

(iv) $\int x^2 e^{2x}\,dx$

$= \frac{1}{2}e^{2x}.x^2 - \int \frac{1}{2}e^{2x}.(2x)\,dx$

$= \frac{1}{2}x^2e^{2x} - \int x e^{2x}\,dx$

$= \frac{1}{2}x^2e^{2x} - \left\{\frac{1}{2}e^{2x}.x - \int \frac{1}{2}e^{2x}(1)\,dx\right\}$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}\,dx$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C.$

3 (b) $x^3 + x^2 - 6x$

$= x(x^2 + x - 6)$

$= x(x+3)(x-2).$

Let $\dfrac{x+1}{x^3+x^2-6x} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+3}$, then

$x + 1 = A(x+3)(x-2) + Bx(x+3) + Cx(x-2).$

Let $x = 0$. Then $1 = -6A \Rightarrow A = \frac{-1}{6}$.

Let $x = 2$. Then $3 = 10B \Rightarrow B = \frac{3}{10}$.

Let $x = -3$. Then $-2 = 15C \Rightarrow C = \frac{-2}{15}$.

Hence $\displaystyle\int \frac{(x+1)\,dx}{x^3+x^2-6x}$

$\displaystyle = \int \left(\frac{-1/6}{x} + \frac{3/10}{x-2} - \frac{2/15}{x+3} \right) dx$

$= -\frac{1}{6}\ln(x) + \frac{3}{10}\ln(x-2) - \frac{2}{15}\ln(x+3) + C.$

Question 4:

(a) $\displaystyle I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$

$\displaystyle = \int_0^{\pi/4} \tan^{n-2} x \,.\, (\sec^2 x - 1)\, dx$

$\displaystyle = \int_0^{\pi/4} \tan^{n-2} x \,.\, \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$

$\displaystyle = \left[\frac{1}{n-1} \tan^{n-1} x \right]_0^{\pi/4} - I_{n-2}$

$\therefore I_n + I_{n-2} = \frac{1}{n-1} \left(\tan \frac{\pi}{4}\right)^{n-1}$

$\therefore I_n + I_{n-2} = \frac{1}{n-1}$, as required.

$I_5 + I_3 = \frac{1}{4}$

$I_3 + I_1 = \frac{1}{2}$

$\displaystyle I_1 = \int_0^{\pi/4} \tan x \, dx$

$= \left[-\ln(\cos x) \right]_0^{\pi/4}$

$= -\ln(1/\sqrt{2})$

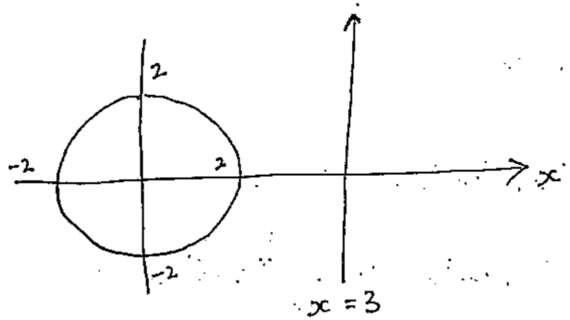
$\therefore I_3 = \frac{1}{2} + \ln\left(\frac{1}{\sqrt{2}}\right)$

$\therefore I_5 = \frac{1}{4} - \left(\frac{1}{2} + \ln\left(\frac{1}{\sqrt{2}}\right)\right)$

$\therefore I_5 = -\frac{1}{4} - \ln\left(\frac{1}{\sqrt{2}}\right)$

$\therefore I_5 = \ln\sqrt{2} - \frac{1}{4}.$

4 (b)



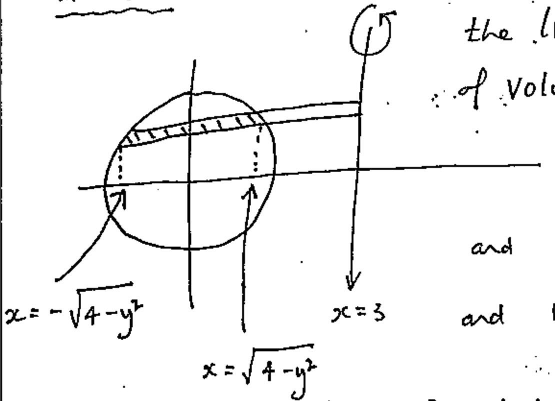
<u>Method 1:</u>

Rotation of the shaded strip about the line $x = 3$ produces a "washer" of volume ΔV, where

$$\Delta V = \pi\,(R^2 - r^2)\,.\,\Delta y$$

and $R = 3 + \sqrt{4 - y^2}$

and $r = 3 - \sqrt{4 - y^2}$.



Hence
$$\Delta V = \pi\,(R^2 - r^2)\,\Delta y$$
$$= \pi\left\{\left(3 + \sqrt{4 - y^2}\right)^2 - \left(3 - \sqrt{4 - y^2}\right)^2\right\}\Delta y$$
$$= \pi\left\{6\left(2\sqrt{4 - y^2}\right)\right\}\Delta y$$
$$= 12\pi\sqrt{4 - y^2}\,\Delta y$$

4(b)

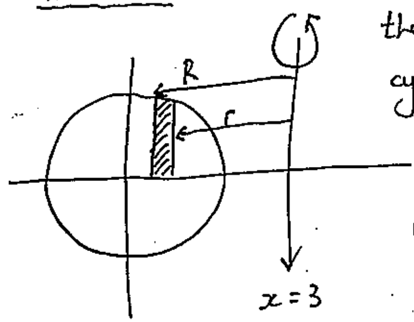
Hence the required volume is V, where

$$V = 12\pi\int_{-2}^{2}\sqrt{4 - y^2}\,dy$$
$$= 24\pi\int_{0}^{2}\sqrt{4 - y^2}\,dy$$
$$= 24\pi\left(\tfrac{1}{4}\pi \times 2^2\right)$$
$$= 24\pi^2 \text{ units}^3$$

<u>Method 2:</u>

Rotation of the shaded strip about the line $x = 3$ produces a cylindrical shell of volume ΔV, where

$$\Delta V = \pi\,(R^2 - r^2) \times \text{height}.$$



Now $R = (3 - x)$

$r = (3 - (x + \Delta x))$

height $= \sqrt{4 - x^2}$.

Hence
$$\Delta V = \pi\left\{(3 - x)^2 - (3 - (x + \Delta x))^2\right\} \times \sqrt{4 - x^2}$$
$$= \pi\left\{(6 - 2x - \Delta x)(\Delta x)\right\}\sqrt{4 - x^2}$$
$$= 2\pi\,(3 - x)\sqrt{4 - x^2}\,.\,\Delta x$$

Hence the required volume is V, where

$$V = 2 \times 2\pi \int_{-2}^{2} (3-x)\sqrt{4-x^2}\, dx$$

$$= 4\pi \int_{-2}^{2} 3\sqrt{4-x^2}\, dx - 4\pi \int_{-2}^{2} x\sqrt{4-x^2}\, dx$$

$$= 12\pi\left(\tfrac{1}{2} \times \pi \times 2^2\right) - 4\pi\left[\tfrac{-1}{2}\cdot\tfrac{2}{3}(4-x^2)^{\frac{3}{2}}\right]_{-2}^{2}$$

$$= 24\pi^2 - 0$$

$$= 24\pi^2 \text{ units}^3.$$

4(c) (i) $\frac{d}{dx}\left\{\frac{1}{2}x(a^2-x^2)^{\frac{1}{2}} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right\}$

$$= \tfrac{1}{2}x\cdot\tfrac{1}{2}(a^2-x^2)^{-\frac{1}{2}}(-2x) + \tfrac{1}{2}(a^2-x^2)^{\frac{1}{2}} + \tfrac{1}{2}a^2\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}$$

$$= \frac{-x^2}{2\sqrt{a^2-x^2}} + \frac{\sqrt{a^2-x^2}}{2} + \frac{a^2}{2\sqrt{a^2-x^2}}$$

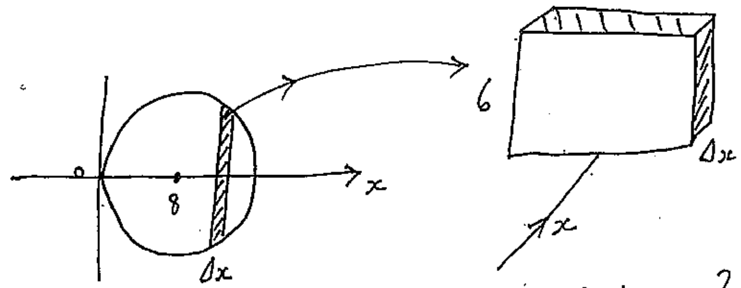
$$= \frac{a^2-x^2}{2\sqrt{a^2-x^2}} + \frac{\sqrt{a^2-x^2}}{2}$$

$$= \frac{a^2-x^2}{2\sqrt{a^2-x^2}} + \frac{a^2-x^2}{2\sqrt{a^2-x^2}} = \frac{a^2-x^2}{\sqrt{a^2-x^2}} = \sqrt{a^2-x^2}.$$

4(c) (ii) $x^2+y^2 = 16x$

$$x^2 - 16x + 64 + y^2 = 64$$

$$(x-8)^2 + y^2 = 64. \rightarrow y = \pm\sqrt{64-(x-8)^2}$$



length of slice $= 2\sqrt{64-(x-8)^2}$

The volume of a slice is ΔV, where

$$\Delta V = 6 \times 2\sqrt{64-(x-8)^2} \times \Delta x$$

$$= 12\sqrt{64-(x-8)^2}\,\Delta x.$$

Hence the volume, V, of the required solid is

$$V = 12\int_{0}^{16}\sqrt{64-(x-8)^2}\, dx$$

$$= 12\left\{\left[\tfrac{1}{2}(x-8)\sqrt{64-(x-8)^2} + \tfrac{1}{2}\times 64 \times \sin^{-1}\left(\frac{x-8}{8}\right)\right]_0^{16}\right\}$$

$$= 12\left\{\left(0+32\left(\tfrac{\pi}{2}\right)\right) - \left(0+32\left(\tfrac{-\pi}{2}\right)\right)\right\}$$

$$= 12 \times 32\pi$$

$$= 384\pi \text{ units}^3.$$

Question 5:

(a) (i) $p(x) = ax^4 + bx^3 + cx^2 + d$

$p(\alpha) = a\alpha^4 + b\alpha^3 + c\alpha^2 + d = 0$

$\therefore \quad d = -a\alpha^4 - b\alpha^3 - c\alpha^2$

$\therefore \quad d = -\alpha(a\alpha^3 + b\alpha^2 + c\alpha)$,

which is divisible by α.

Hence d is a multiple of α.

(ii) Using the result from part (i), if $q(x)$ has an integer root, then that integer divides -3.

The divizors of -3 are $\pm 1, \pm 3$.

Now $q(1) = 5 - 1 + 3 - 3 \neq 0$.

And $q(-1) = 5 + 1 + 3 - 3 \neq 0$.

And $q(3) = 405 - 27 + 27 - 3 \neq 0$.

And $q(-3) = 405 + 27 + 27 - 3 \neq 0$.

Hence $q(x)$ does not have an integer root.

5(b) $P(x) = x^3 - 11x - 14$

Notice that $P(-2) = -8 + 22 - 14 = 0$.

Hence $(x+2)$ is a factor of $P(x)$.

$$
\begin{array}{r|l}
 & x^2 - 2x - 7 \\
x+2 & x^3 + 0x^2 - 11x - 14 \\
 & x^3 + 2x^2 \\
\hline
 & -2x^2 - 11x - 14 \\
 & -2x^2 - 4x \\
\hline
 & -7x - 14 \\
 & -7x - 14 \\
\hline
 & 0
\end{array}
$$

$\therefore \quad P(x) = (x+2)(x^2 - 2x - 7)$

Solving $x^2 - 2x - 7 = 0$

$$x = \frac{2 \pm \sqrt{4 - 4(-7)}}{2}$$

$$x = \frac{2 \pm \sqrt{32}}{2} = \frac{2 \pm 4\sqrt{2}}{2} = 1 \pm 2\sqrt{2}$$

The three roots of $P(x) = 0$ are

$x = -2$ and $1 \pm 2\sqrt{2}$.

5(c) $q(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$

$q'(x) = 4x^3 - 18x^2 + 24x - 10$

$q''(x) = 12x^2 - 36x + 24$

$= 12(x^2 - 3x + 2)$

$= 12(x-1)(x-2)$

Test $x = 1$:

$q'(1) = 4 - 18 + 24 - 10 = 0$

$q(1) = 1 - 6 + 12 - 10 + 3 = 0.$

Hence $x = 1$ is a root of multiplicity 3.

Hence $q(x) = (x-1)^3(x-3)$.

So the roots of $q(x) = 0$ are $x = 1, 1, 1, 3$.

(d) $x^3 - 5x^2 + 5 = 0$ $\quad \begin{cases} \alpha + \beta = 5 - \gamma \\ \alpha\beta + \alpha\gamma + \beta\gamma = 0 \\ \alpha\beta\gamma = -5 \end{cases}$

(i) $(\alpha - 1)(\beta - 1)(\gamma - 1)$

$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1$

$= -5 + 5 - 1$

$= -1$

5 (d) (ii) $x^3 - 5x^2 + 5 = 0$

So $\alpha^3 - 5\alpha^2 + 5 = 0$

and $\beta^3 - 5\beta^2 + 5 = 0$

and $\gamma^3 - 5\gamma^2 + 5 = 0$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = 5(\alpha^2 + \beta^2 + \gamma^2) - 15.$

Now $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (5)^2 - 2(0)$

$= 25.$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = 5 \times 25 - 15 = 110.$

(iii) The equation whose roots are α^2, β^2 and γ^2 is

$(\sqrt{x})^3 - 5(\sqrt{x})^2 + 5 = 0$

$x\sqrt{x} - 5x + 5 = 0$

$x\sqrt{x} = 5x - 5$

$(x\sqrt{x})^2 = (5x-5)^2$

$x^3 = 25x^2 - 50x + 25$

ie, $x^3 - 25x^2 + 50x - 25 = 0.$

Question 6:

(a) $\frac{x^2}{36} + \frac{y^2}{9} = 1 \Rightarrow a = 6, \ b = 3.$

(i) Length of major axis is 12 units.
Length of minor axis is 6 units.

(ii) $b^2 = a^2(1-e^2)$

$9 = 36(1-e^2)$

$e^2 = \frac{3}{4} \Rightarrow e = \frac{\sqrt{3}}{2}.$

(iii) The foci are at $(\pm ae, 0)$.

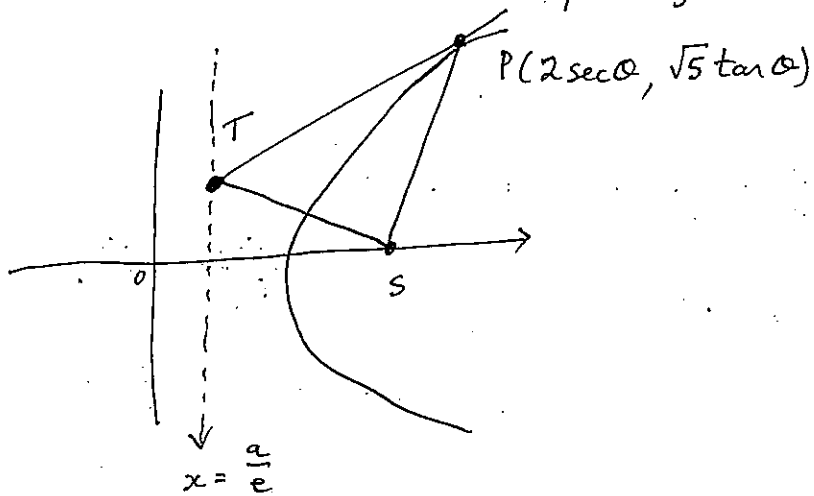
The foci are at $(\pm 3\sqrt{3}, 0)$.

(iv) The directrices have equations $x = \pm \frac{a}{e}$.

The directrices have equations $x = \pm \frac{6}{\sqrt{3}/2} = \pm \frac{12}{\sqrt{3}}$

ie, $x = \pm 4\sqrt{3}$.

6(b) $5x^2 - 4y^2 = 20$, ie, $\frac{x^2}{4} - \frac{y^2}{5} = 1$.



To find the equation of the tangent at P:

$\frac{2x}{4} - \frac{2y}{5} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2x}{4} \cdot \frac{5}{2y} = \frac{5x}{4y}$

At $P(2\sec\theta, \sqrt{5}\tan\theta)$, $\frac{dy}{dx} = \frac{10\sec\theta}{4\sqrt{5}\tan\theta}$

Hence the equation of the tangent at P is

$y - \sqrt{5}\tan\theta = \frac{10\sec\theta}{4\sqrt{5}\tan\theta}(x - 2\sec\theta)$, etc.

... (1)

6(b) ctd

To find the equation of the directrix, first find the eccentricity:

$$5 = 4(e^2-1)$$

$$\frac{5}{4} + \frac{4}{4} = \frac{9}{4} = e^2 \quad \Rightarrow \quad e = \frac{3}{2}.$$

Hence the directrix has equation $x = \dfrac{\cancel{4}2}{3/2} = \dfrac{4}{3}$.

To find the y-coordinate of T: substitute $x = \frac{4}{3}$ in (1):

$$y - \sqrt{5}\tan\theta = \frac{10\sec\theta}{4\sqrt{5}\tan\theta}\left(\frac{4}{3} - 2\sec\theta\right)$$

$$\therefore \quad y = \frac{10\sec\theta}{3\sqrt{5}\tan\theta} - \frac{10\sec^2\theta}{2\sqrt{5}\tan\theta} + \sqrt{5}\tan\theta$$

$$\therefore \quad y = \frac{2\sqrt{5}}{3\sin\theta} - \frac{\sqrt{5}(1+\tan^2\theta)}{\tan\theta} + \sqrt{5}\tan\theta$$

$$\therefore \quad y = \frac{2\sqrt{5}}{3\sin\theta} - \frac{\sqrt{5}\cos\theta}{\sin\theta} = \frac{2\sqrt{5} - 3\sqrt{5}\cos\theta}{3\sin\theta}.$$

Hence $T = \left\{ \dfrac{4}{3}, \dfrac{2\sqrt{5} - 3\sqrt{5}\cos\theta}{3\sin\theta} \right\}$.

And $S = \{3, 0\}$.

6(c) (i) Substitute $y = mx + b$ into the equation of ellipse:

$$\frac{x^2}{6} + \frac{(mx+b)^2}{3} = 1$$

ie, $x^2 + 2(m^2x^2 + 2bmx + b^2) = 6$

ie, $x^2(2m^2+1) + 4bmx + (2b^2-6) = 0.$

We need $\Delta = 0$, ie,

$$(4bm)^2 - 4(2m^2+1)(2b^2-6) = 0$$

$$16b^2m^2 - 4(4b^2m^2 - 12m^2 + 2b^2 - 6) = 0$$

$$16b^2m^2 - 16b^2m^2 + 48m^2 - 8b^2 + 24 = 0$$

ie, $12m^2 - 2b^2 + 6 = 0$

ie, $6m^2 - b^2 + 3 = 0$

ie, $b^2 = 6m^2 + 3$, as required.

6c(i) ctd

Hence m of $PS = \dfrac{\sqrt{5}\tan\theta}{2\sec\theta - 3}$

and m of $ST = \dfrac{\dfrac{2\sqrt{5} - 3\sqrt{5}\cos\theta}{3\sin\theta}}{\frac{4}{3} - 3}$

$$= \frac{2\sqrt{5} - 3\sqrt{5}\cos\theta}{-5\sin\theta}$$

$$= \frac{3\sqrt{5}\cos\theta - 2\sqrt{5}}{5\sin\theta}.$$

Now $(m \text{ of } PS) \times (m \text{ of } ST)$

$$= \frac{\sqrt{5}\tan\theta}{2\sec\theta - 3} \times \frac{3\sqrt{5}\cos\theta - 2\sqrt{5}}{5\sin\theta}$$

$$= \frac{15\sin\theta - 10\tan\theta}{10\tan\theta - 15\sin\theta}$$

$$= \frac{-(10\tan\theta - 15\sin\theta)}{(10\tan\theta - 15\sin\theta)}$$

$$= -1.$$

Hence $\angle PST = 90°$, as required.

6(c) (ii) The tangent $y = mx + b$ passes through (X, Y) so

$$Y = mX + b$$

ie, $b = (Y - mX)$

ie, $b^2 = Y^2 - 2mXY + m^2X^2.$

But $b^2 = 6m^2 + 3$, so

$$6m^2 + 3 = Y^2 - 2mXY + m^2X^2$$

ie, $0 = m^2X^2 - 6m^2 - 2mXY + Y^2 - 3$

ie, $0 = m^2(X^2 - 6) - 2mXY + (Y^2 - 3).$

Now this is a quadratic in m with two roots, namely m and $\frac{-1}{m}$. Hence the product of the roots is -1. But the product of the roots is also equal to $\frac{c}{a}$, ie, $\dfrac{Y^2 - 3}{X^2 - 6}$

Hence $\dfrac{Y^2 - 3}{X^2 - 6} = -1$

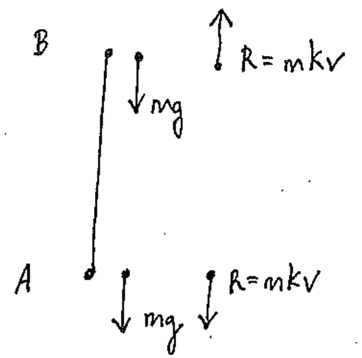
ie, $Y^2 - 3 = 6 - X^2$

ie, $X^2 + Y^2 = 9$

ie, the locus of $P(X, Y)$ is the circle $x^2 + y^2 = 9$.

Question 7:

(i)



(ii) $m\ddot{x} = -mg - mkv$

$\ddot{x} = -g - kv = \dfrac{dv}{dt}$

(α) $\dfrac{dt}{dv} = \dfrac{-1}{g+kv}$

$t = \displaystyle\int \frac{-dv}{g+kv}$

$t = -\frac{1}{k}\ln(g+kv) + C$

But when $t = 0$, $v = u$ so $C = \frac{1}{k}\ln(g+ku)$.

Hence $t = -\frac{1}{k}\ln(g+kv) + \frac{1}{k}\ln(g+ku)$

ie, $t = \dfrac{1}{k}\ln\left(\dfrac{g+ku}{g+kv}\right)$, as required.

(β) So $e^{kt} = \left(\dfrac{g+ku}{g+kv}\right)$

$g + kv = (g+ku)e^{-kt}$

So $v = \frac{1}{k}(g+ku)e^{-kt} - g/k$

7(ii)

(γ) So $\dfrac{dx}{dt} = \dfrac{1}{k}(g+ku)e^{-kt} - g/k$

$\therefore x = \dfrac{-1}{k^2}(g+ku)e^{-kt} - \dfrac{gt}{k} + F$

But when $t = 0$, $x = 0$ so $F = \dfrac{1}{k^2}(g+ku)$

$\therefore x = \dfrac{-1}{k^2}(g+ku)e^{-kt} - \dfrac{gt}{k} + \dfrac{1}{k^2}(g+ku)$

7 (iii) $\dfrac{dv}{dt} = g - kv$

(α) $\dfrac{dt}{dv} = \dfrac{1}{g-kv}$

$\therefore t = \displaystyle\int \frac{dv}{g-kv} = -\frac{1}{k}\ln(g-kv) + H.$

But when $t=0$, $v = u$ so $H = \frac{1}{k}\ln(g-ku)$

$\therefore t = -\frac{1}{k}\ln(g-kv) + \frac{1}{k}\ln(g-ku)$

ie, $t = \dfrac{1}{k}\ln\left(\dfrac{g-ku}{g-kv}\right)$

(β) So $e^{kt} = \left(\dfrac{g-ku}{g-kv}\right)$

$g - kv = (g-ku)e^{-kt}$

$\therefore \; kV = g - (g - ku)e^{-kt}$

$\therefore \; V = g/k - \frac{1}{k}(g - ku)e^{-kt}$

7(iii)

(γ) $\frac{dx}{dt} = g/k - \frac{1}{k}(g - ku)e^{-kt}$

$\therefore \; x = \frac{gt}{k} + \frac{1}{k^2}(g - ku)e^{-kt} + R$

But when $t = 0$, $x = 0$ so $R = \frac{-1}{k^2}(g - ku)$

$\therefore \; x = \frac{1}{k^2}(g - ku)e^{-kt} + \frac{gt}{k} - \frac{1}{k^2}(g - ku).$

(iv) Let the particles meet after t units of time. A will have travelled a distance of x_1, and B will have travelled a distance of x_2 such that

$x_1 + x_2 = d$, ie,

$$\left\{-\frac{1}{k^2}(g + ku)e^{-kt} - \frac{gt}{k} + \frac{1}{k^2}(g + ku) + \frac{1}{k^2}(g - ku)e^{-kt} + \frac{gt}{k} - \frac{1}{k^2}(g - ku)\right\} = d$$

ie, $\frac{2u}{k} - \frac{2u}{k}e^{-kt} = d$

ie, $\frac{kd}{2u} = 1 - e^{-kt}$ ie, $e^{-kt} = \frac{2u - kd}{2u}$

ie, $e^{kt} = \frac{2u}{2u - kd}$ ie, $t = \frac{1}{k}\ln\left(\frac{2u}{2u - kd}\right)$, as required.

PR28

Question 8: (a)

(ii) $\frac{d}{dx}\left(\frac{1}{2}V^2\right) = \frac{12 - 4x}{x^3} = \frac{12}{x^3} - \frac{4}{x^2}$

$\therefore \; \frac{1}{2}V^2 = \frac{-6}{x^2} + \frac{4}{x} + c$

But when $x = 6$, $V = 0$ so

$0 = -\frac{6}{36} + \frac{4}{6} + c \Rightarrow c = -\frac{1}{2}$

$\therefore \; V^2 = -\frac{12}{x^2} + \frac{8}{x} - 1$

$\therefore \; V^2 = \frac{8x - x^2 - 12}{x^2}$

$\therefore \; V = \pm\sqrt{\frac{8x - x^2 - 12}{x^2}} = \frac{\pm\sqrt{8x - x^2 - 12}}{x}$

(i) When $x = 6$, $V = 0$ and $\frac{d^2x}{dt^2} = \frac{12 - 24}{64} < 0$.

Since acceleration is negative, and $V = 0$, therefore it starts moving in the negative direction.

(iii) $V = \frac{\pm\sqrt{8x - x^2 - 12}}{x}$

For V to be real, we need $8x - x^2 - 12 \geq 0$, ie,

$(x-2)(6-x) \geq 0$,

ie, $2 \leq x \leq 6$.

The particle is restricted to $2 \leq x \leq 6$.

PR29

8 (b) (i) Exact area $= \int_1^n \log_e x \, dx$

$= \left[x \ln x - x \right]_1^n$

$= (n \ln n - n) - (0 - 1)$

$= n \ln n - n + 1$

$= (\ln n^n) - (n-1)$

$= (\ln n^n) - \log_e e^{(n-1)}$

$= \ln \left(\dfrac{n^n}{e^{n-1}} \right)$, as required.

(ii) Using the Trapezoidal Rule with strips of width 1 unit,

Area $\doteqdot \frac{1}{2} \{ 0 + \ln n + 2[\ln 2 + \ln 3 + \dots + \ln(n-1)] \}$

(iii) Area by Trapezoidal Rule < Exact area

$\frac{1}{2} \ln n + \ln (n-1)! < \ln \left(\dfrac{n^n}{e^{n-1}} \right)$

$\ln \sqrt{n}\,(n-1)! < \ln \left(\dfrac{n^n}{e^{n-1}} \right)$

$\sqrt{n}\,(n-1)! < \left(\dfrac{n^n}{e^{n-1}} \right)$

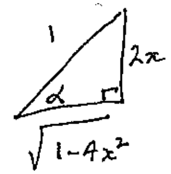
$\sqrt{n}\,(n-1)!\, e^{n-1} < n^n$

$\therefore \quad n^n > \sqrt{n}\,(n-1)!\, e^{n-1}$, as required.

8 (c) (i) Let $\alpha = \sin^{-1} 2x$

$\sin \alpha = 2x$

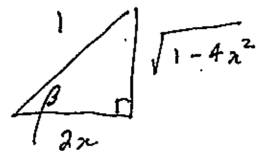
$\cos \alpha = \sqrt{1 - 4x^2}$.



Let $\beta = \cos^{-1} 2x$

$\cos \beta = 2x$

$\sin \beta = \sqrt{1 - 4x^2}$.



Now $\sin [\cos^{-1} 2x - \sin^{-1} 2x]$

$= \sin (\beta - \alpha)$

$= \sin \beta \cos \alpha - \cos \beta \sin \alpha$

$= \sqrt{1-4x^2} \times \sqrt{1-4x^2} - 2x \times 2x$

$= 1 - 4x^2 - 4x^2$

$= 1 - 8x^2$, as required.

(ii) To solve $\cos^{-1} 2x - \sin^{-1} 2x = \sin^{-1}(1-2x)$

$\sin [\cos^{-1} 2x - \sin^{-1} 2x] = \sin [\sin^{-1}(1-2x)]$

$1 - 8x^2 = 1 - 2x$, from part (i)

$\therefore \quad 0 = 8x^2 - 2x$

$0 = 2x(4x - 1)$

$\therefore x = 0$ or $\frac{1}{4}$

However $\sin^{-1} 2x$, $\cos^{-1} 2x$ & $\sin^{-1}(1-2x)$ are acute, so $x = \frac{1}{4}$ only.