Student Number ....

Barker College

2009
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Staff Involved:

AM WEDNESDAY 12 AUGUST

- **BTP***
- **WMD***
- **GDH**
- **MRB**

35 copies

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages of your answer sheets
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value
- ALL necessary working should be shown in every question
- Start each question on a NEW page
- Write on one side of each answer page
- Marks may be deducted for careless or badly arranged work

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Total marks – 120
Attempt Questions 1–8
ALL questions are of equal value

Answer each question on a **SEPARATE** sheet of paper

Marks

Question 1 (15 marks) **[START A NEW PAGE]**

(a) Using the substitution $u = e^x + 1$ or otherwise,

evaluate $\displaystyle\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$. **3**

(b) Find $\displaystyle\int \frac{1}{x \ln x}\,dx$. **1**

(c) (i) Find a, b, and c, such that

$$\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}.$$ **2**

(ii) Find $\displaystyle\int \frac{16}{(x^2+4)(2-x)}\,dx$. **2**

(d) Using integration BY PARTS ONLY, evaluate

$\displaystyle\int_0^1 \sin^{-1} x\,dx$. **3**

(e) Use the substitution $t = \tan\frac{\theta}{2}$ to show that:

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right).$$ **4**

End of Question 1

Marks

Question 2 (15 marks) **[START A NEW PAGE]**

(a) Given $z = \dfrac{\sqrt{3} + i}{1 + i}$,

(i) Find the argument and modulus of z. **2**

(ii) Find the smallest positive integer n such that z^n is real. **1**

(b) The complex number z moves such that $\text{Im}\left[\dfrac{1}{\bar{z} - i}\right] = 2$.

Show that the locus of z is a circle and find its centre and radius. **3**

(c) Sketch the region in the complex plane where the inequalities

$|z + 1 - i| < 2$ and $0 < \arg(z + 1 - i) < \dfrac{3\pi}{4}$ hold simultaneously. **3**

(d) Find the three different values of z for which

$z^3 = \dfrac{1 + i}{\sqrt{2}}$. **3**

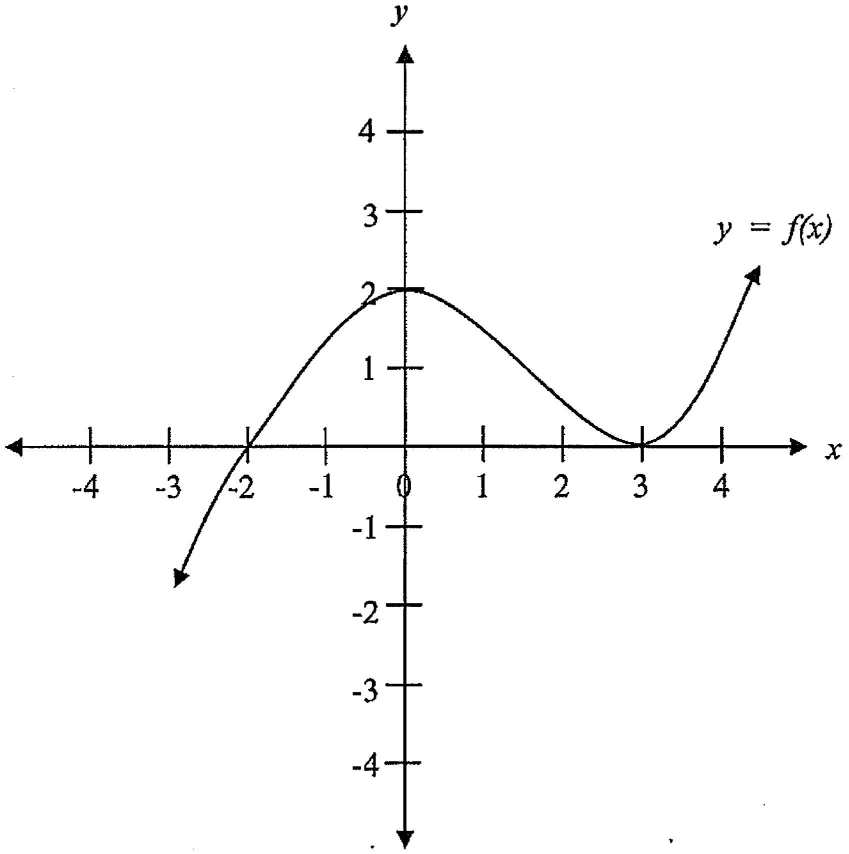
(e) By applying De Moivre's theorem and also expanding $(\cos\theta + i\sin\theta)^3$,

express $\cos 3\theta$ as a polynomial in $\cos\theta$. **3**

End of Question 2

Question 3 (15 marks) **[START A NEW PAGE]**

(a)



Given the above graph $y = f(x)$, draw **separate** sketches of the following graphs showing all critical points.

Your answers should be superimposed on the sketches supplied at the back of this paper and then removed and attached to rest of your answers to question 3.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = |f(|x|)|$ **2**

(iii) $|y| = f(x)$ **2**

(iv) $y = \ln[f(x)]$ **2**

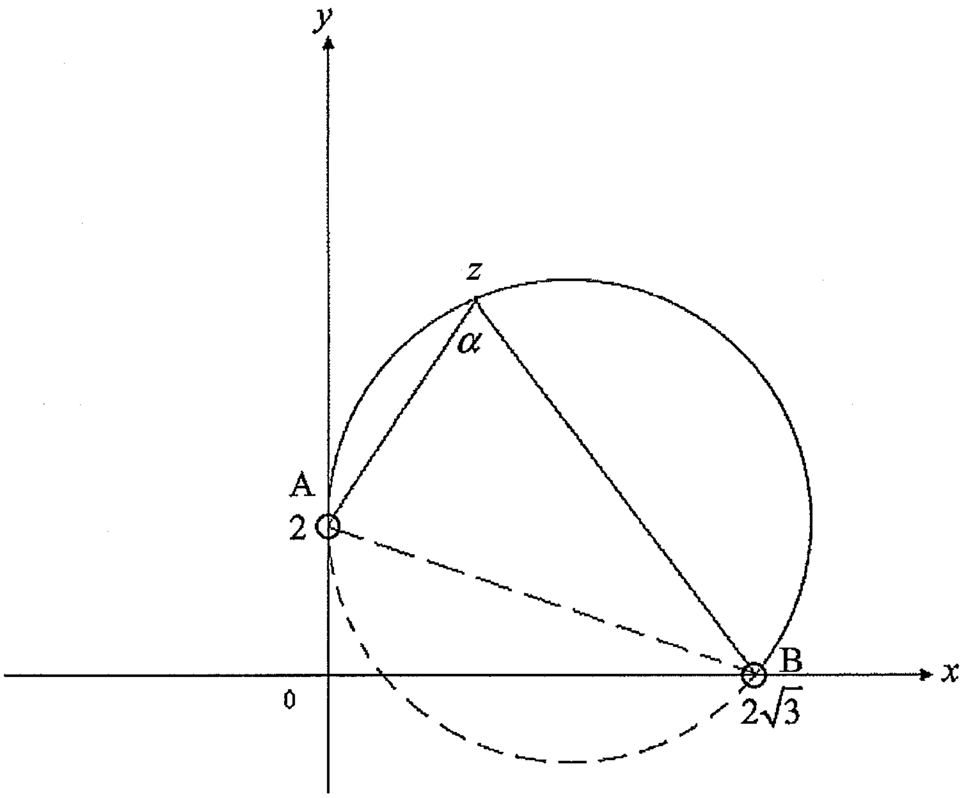
(v) $y = f(2(x+1))$ **2**

Question 3 continues on page 6

Question 3 (continued)

(b) The locus of the complex number z, moving in the complex plane such that $Arg(z - 2\sqrt{3}) - Arg(z - 2i) = \frac{\pi}{3}$, is a part of a circle.

The angle between the lines from $2i$ to z and then from $2\sqrt{3}$ to z is α, as shown in the diagram below.



(i) Show that $\alpha = \frac{\pi}{3}$. **2**

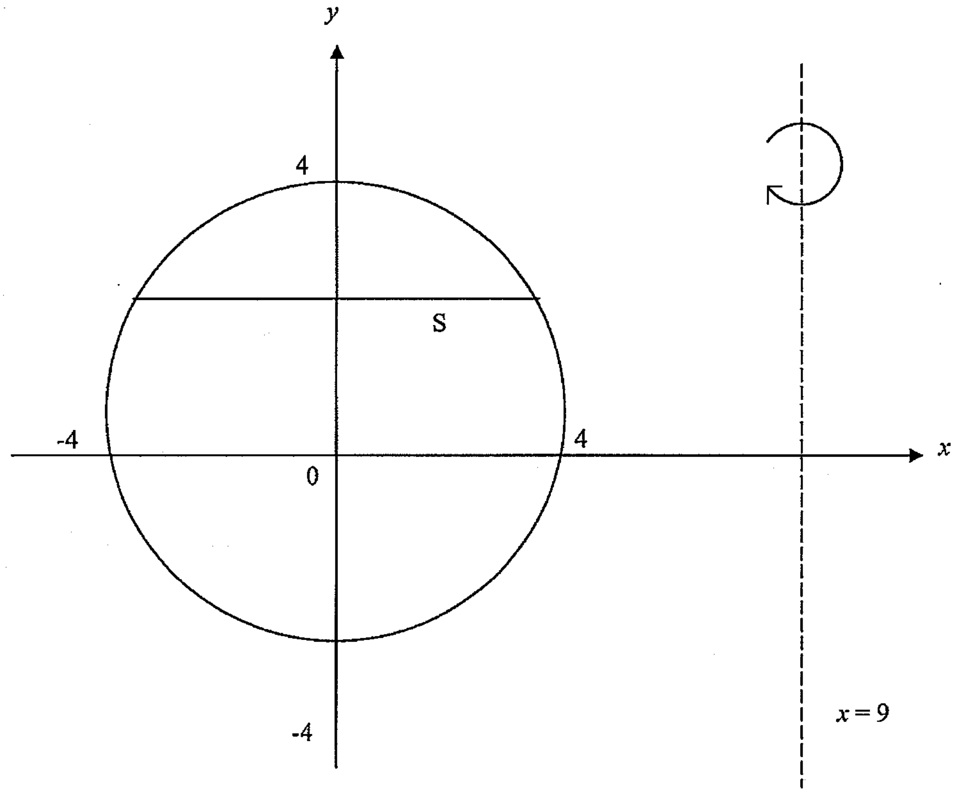
(ii) Find the centre and the radius of the circle. **3**

End of Question 3

Question 4 (15 marks) **[START A NEW PAGE]**

(a) The circle $x^2 + y^2 = 16$ is rotated about the line $x = 9$ to form a ring, i.e. a torus.

When the circle is rotated, the line segment S at height y sweeps out an annulus.



The x coordinates of the end-points of S are x_1 and $-x_1$, where $x_1 = \sqrt{16 - y^2}$.

(i) Show that the area of the annulus is equal to $36\pi\sqrt{16 - y^2}$. **3**

(ii) Hence find the volume of the ring. **3**

(b) (i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. **1**

(ii) Hence or otherwise, find all solutions of

$\sin x + \sin 2x + \sin 3x = 0$, for $0 \le x < 2\pi$. **3**

Question 4 continues on page 8

Marks

Question 4 (continued)

(c) A certain solid has a circular base of radius 4. The centre of the base is the origin. The cross-sections, at right angles to the x-axis, are isosceles triangles. If the height h of each of these triangles is given by $h = 16 - x^2$,

(i) Show that the volume of the solid is given by $V = 2\int_0^4 (16 - x^2)^{\frac{3}{2}}\,dx$ **2**

(ii) Hence, find the volume V. **3**

End of Question 4

Marks

Question 5 (15 marks) **[START A NEW PAGE]**

(a) Given that 1, w, w^2 are the cube roots of unity, i.e. the roots of $z^3 = 1$, simplify $(1-w)\,(1-w^2)\,(1-w^7)(1-w^{11})$. **2**

(b) Find the coordinates of the point on the curve, $x^2y + xy^2 + 16 = 0$ at which the tangent is parallel to the x-axis. **4**

(c) A particle of unit mass moves in a straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$.

(i) Explain why $\ddot{x} = -(v + v^3)$. **1**

(ii) Show that v is related to the displacement x by the formula

$$x = \tan^{-1}\left[\frac{Q-v}{1+Qv}\right].$$ **2**

(iii) Show that the time t which has elapsed when the particle is travelling with velocity V is given by $t = \frac{1}{2}\log_e\left[\frac{Q^2(1+V^2)}{V^2(1+Q^2)}\right]$. **2**

(iv) Find V^2 as a function of t. **2**

(v) Find the limiting values of v and x *as* $t \to \infty$. **2**

End of Question 5

Marks

Question 6 (15 marks) **[START A NEW PAGE]**

(a) Consider the polynomial equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

where *a, b, c,* and *d* are all integers. Suppose the equation has a root of the form ki, where k is real, and $k \neq 0$.

(i) State why the conjugate $-ki$ is also a root. **1**

(ii) Show that $c = k^2 a$. **2**

(iii) Show that $c^2 + a^2 d = abc$. **2**

(iv) If 2 is also a root of the equation, and $b = 0$, show that d and c are both even. **2**

(b) (i) Solve $z^5 + 1 = 0$ by De Moivre's Theorem, leaving your solutions in modulus-argument form. **2**

(ii) Prove that the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^5 + 1 = 0$. **1**

(iii) Show that if $z^4 - z^3 + z^2 - z + 1 = 0$ where $z = cis\,\theta$ then $4\cos^2\theta - 2\cos\theta - 1 = 0$. **3**

Hint: $z^4 - z^3 + z^2 - z + 1 = 0 \Rightarrow z^2 - z + 1 - \dfrac{1}{z} + \dfrac{1}{z^2} = 0$

(iv) Hence find the exact value of $\sec\dfrac{3\pi}{5}$. **2**

End of Question 6

Marks

Question 7 (15 marks) **[START A NEW PAGE]**

(a) (i) Determine the real values of λ for which the equation

$$\frac{x^2}{4-\lambda}+\frac{y^2}{2-\lambda}=1 \text{ defines}$$

(α) an ellipse **1**

(β) a hyperbola **1**

(ii) Sketch the curve corresponding to the value $\lambda = 1$, indicating the positions of the foci and directrices and stating their coordinates and equations respectively. Also mark any axes intercepts on your diagram. **3**

(iii) Describe how the shape of this curve changes as λ increases from 1 towards 2. What is the limiting position of the curve as 2 is approached? **3**

(b) (i) Show that the equation of the normal to the hyperbola $xy = c^2$ at $P(cp, \frac{c}{p})$ is $p^3x - py = c(p^4 - 1)$. **2**

(ii) The normal at $P(cp, \frac{c}{p})$ meets the hyperbola $xy = c^2$ again at $Q(cq, \frac{c}{q})$. Prove that $p^3 q = -1$. **2**

(iii) Hence, show that the locus of the midpoint R of PQ is given by $c^2(x^2 - y^2)^2 + 4x^3y^3 = 0$. **3**

End of Question 7

Marks

Question 8 (15 marks) **[START A NEW PAGE]**

(a) If $I_n = \int \sec^n x \, dx$

(i) Show that $I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$. **4**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \sec^6 x \, dx$. **3**

(b) A marksman finds that on average he hits the target 9 times out of every 10 and scores a bull's eye on average once every 5 rounds. He fires 4 rounds. What is the probability that:

(i) He hits the target each time? **1**

(ii) He scores at least 2 bull's eyes? **1**

(iii) He scores at least 2 bull's eyes and he has hit the target on each of the four rounds? **2**

(c) A vertical rectangular target faces due south on a horizontal plane. The area of the shadow is $1\frac{1}{4}$ times the area of the target when the sun's altitude is α. Find the bearing of the sun. **4**

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

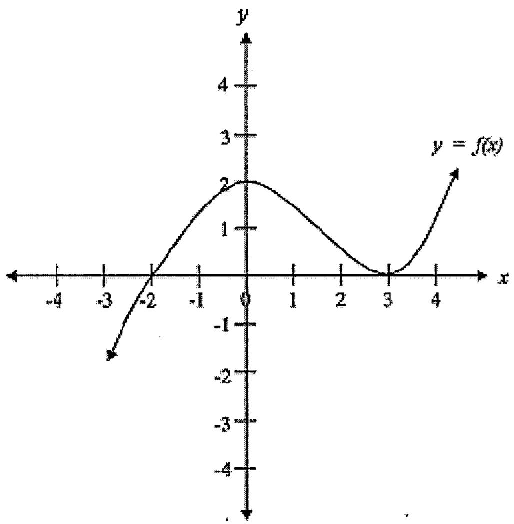
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

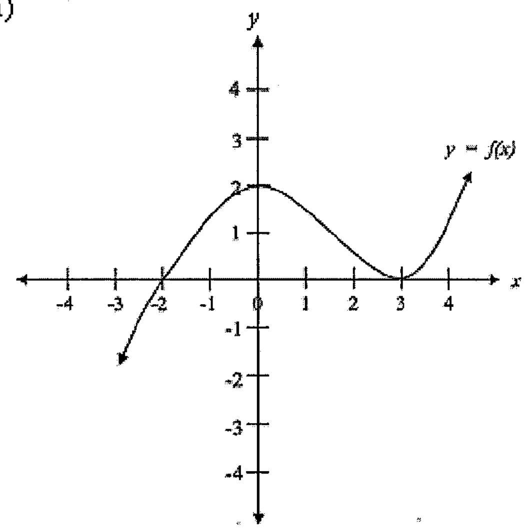
NOTE: $\ln x = \log_e x, \quad x > 0$

Question 3 (a) ANSWER SHEET – ATTACH THIS SHEET TO THE REMAINDER OF YOUR QUESTION 3 SOLUTIONS.
Superimpose your answers to Question 3 (a) on the graphs below.

(i)



(ii)



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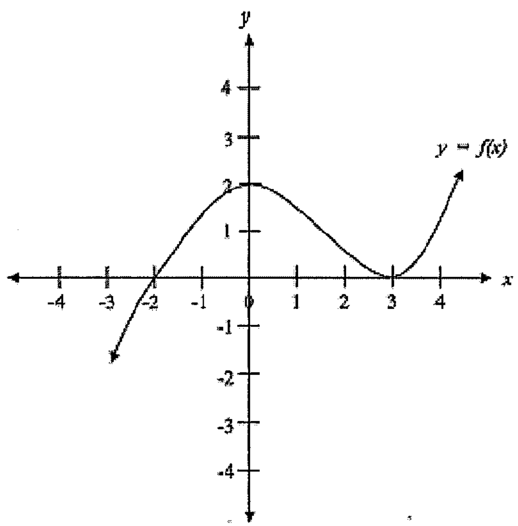
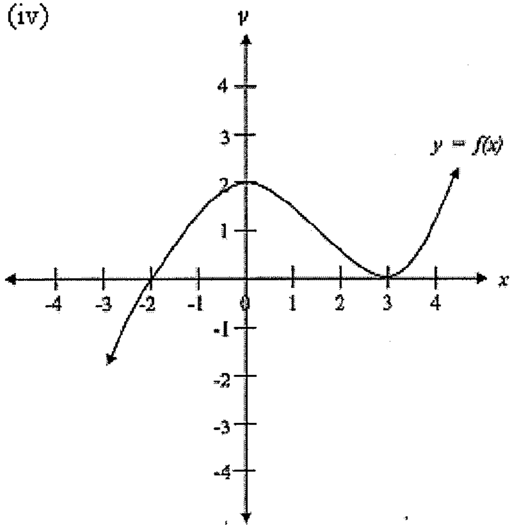
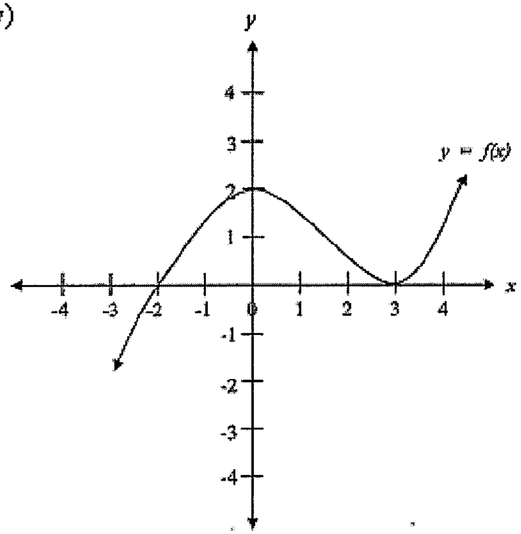


(iv)



(v)



Question 1

(a) $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$

$= \int_0^1 e^x(1+e^x)^{-2}\,dx$

$= \left[\frac{(e^x+1)^{-1}}{-1}\right]_0^1$

$= \left[\frac{-1}{e^x+1}\right]_0^1$

$= \frac{1}{2} - \frac{1}{e+1}$

(b) $\int \frac{1}{x\ln x}\,dx = \ln[\ln x] + C$

(c) $16 = (ax+b)(2-x) + c(x^2+4)$

(i) $x = 2 \Rightarrow 16 = 8c \Rightarrow c = 2$

$x=1 \Rightarrow 16 = a+b+10 \Rightarrow a+b=6$

$x=0 \Rightarrow 16 = 2b+8 \Rightarrow b=4$

$a = 2$

(ii) $\int \frac{2x+4}{x^2+4}\,dx + \int \frac{2}{2-x}\,dx$

$= \int \frac{2x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx - \int \frac{2}{x-2}\,dx$

$= \ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) - 2\ln(x-2) + c$

(d) $\int_0^1 \sin^{-1}x\,dx = \int_0^1 1\times\sin^{-1}x\,dx$

$= [x\sin^{-1}x]_0^1 - \int_0^1 x \times \frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{\pi}{2} - 0 + \frac{1}{2}\int (-2x)(1-x^2)^{-\frac{1}{2}}\,dx$

$= \frac{\pi}{2} + \frac{1}{2}\left[\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^1$

$= \frac{\pi}{2} + (0-1) = \frac{\pi}{2} - 1$

(e) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6}$

$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{4\times\frac{2t}{1+t^2} - 2\times\frac{1-t^2}{1+t^2} + 6}$

$= \int_0^1 \frac{2}{8t^2+8t+4}\,dt$

$= \frac{1}{4}\int_0^1 \frac{dt}{(t+\frac{1}{2})^2+\frac{1}{4}}$

$= \frac{1}{4}\left[2\tan^{-1}(2t+1)\right]_0^1$

$= \frac{1}{2}\left[\tan^{-1}3 - \tan^{-1}1\right]$

$= \frac{1}{2}\left[\tan^{-1}\left(\frac{3-1}{1+3\times1}\right)\right]$

$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$

Question 2

(a)(i) $z = \frac{\sqrt{3}+i}{1+i}$

$\arg z = \arg(\sqrt{3}+i) - \arg(1+i)$

$= \frac{\pi}{6} - \frac{\pi}{4} = \frac{-\pi}{12}$

modulus $z = \frac{2}{\sqrt{2}} = \sqrt{2}$

(ii) $z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$

$z^{12} = (\sqrt{2})^{12}\,\text{cis}(-\pi)$

$= -2^6 \quad \therefore n = 12$

(b) $\text{Im}\left[\frac{1}{\bar{z}-i}\right] = 2$

Let $z = x+iy$

$\frac{1}{\bar{z}-i} = \frac{1}{x-iy-i}$

$= \frac{1}{x-i(y+1)} \times \frac{x+i(y+1)}{x+i(y+1)}$

$= \frac{x+i(y+1)}{x^2+(y+1)^2}$

Question 2 (continued)

(b) $\text{Im}\left[\frac{1}{\bar{z}-i}\right] = 2 \Rightarrow$

$\frac{y+1}{x^2+(y+1)^2} = 2$

$y+1 = 2x^2 + 2(y+1)^2$

$0 = 2x^2 + 2y^2 + 4y + 2 - y - 1$

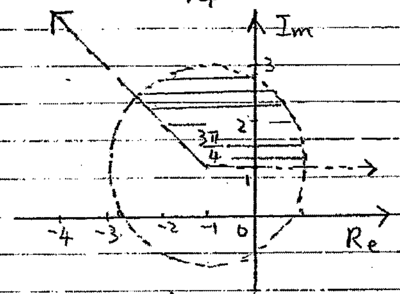
$0 = 2x^2 + 2\left(y^2 + \frac{3y}{2} + \frac{9}{16}\right) - \frac{1}{8}$

$x^2 + \left(y+\frac{3}{4}\right)^2 = \frac{1}{16}$

centre $\left(0, \frac{-3i}{4}\right)$,

radius $= \frac{1}{4}$ unit

(c)



$|z-(-1+i)| < 2$

$0 < \arg[z-(-1+i)] < \frac{3\pi}{4}$

(d) $z^3 = \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

$= 1\,\text{cis}\left(\frac{\pi}{4}+2k\pi\right), \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$

$z = \text{cis}\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right)$

$k=0, \; z = \text{cis}\,\frac{\pi}{12}$,

$k=1, \; z = \text{cis}\left(\frac{3\pi}{4}\right)$,

$k=2 \quad z = \text{cis}\,\frac{17\pi}{12} = \text{cis}\left(\frac{-7\pi}{12}\right)$

(e) $(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$(\cos\theta + i\sin\theta)^3 = \cos3\theta + i\sin3\theta$

Equating real part.

$\cos3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

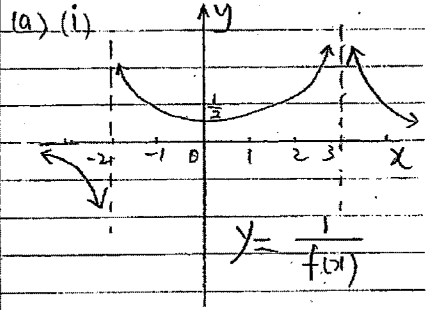
$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

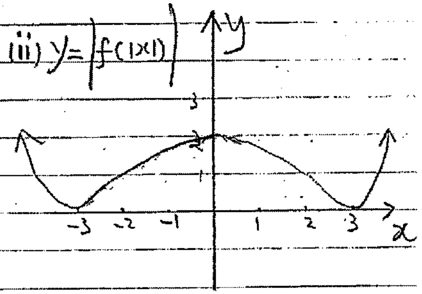
$= 4\cos^3\theta - 3\cos\theta$

Question 3

(a) (i)

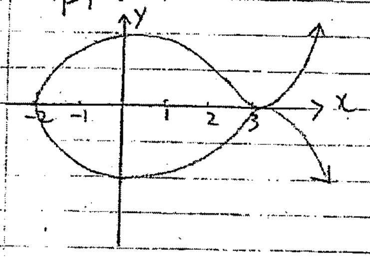


(iii) $y = |f(|x|)|$

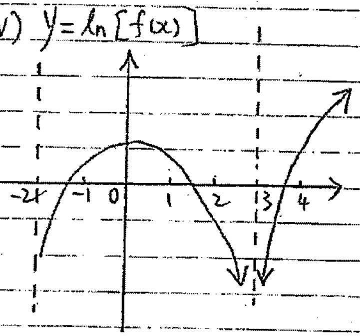


Question 3 (continued)

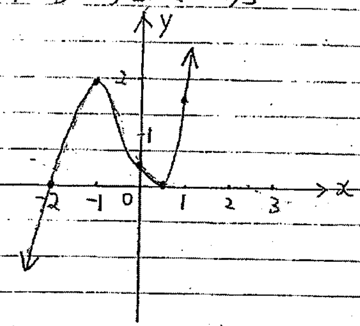
(a) (iii) $|y| = f(x)$



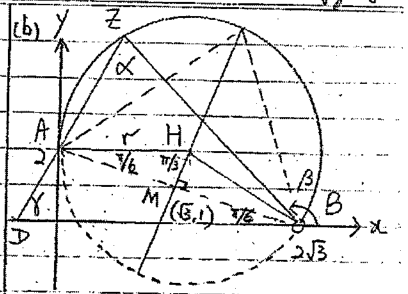
(iv) $y = \ln[f(x)]$



(v) $y = f[2(x+1)]$



(b)



(i)

$\arg(z - 2\sqrt{3}) = \beta$

$\arg(z - 2i) = \gamma$

$\therefore \arg(z - 2\sqrt{3}) - \arg(z - 2i)$

$= \beta - \gamma$

$= \alpha = \frac{\pi}{3}$

(ii) Let H be the centre.

$BA^2 = 2^2 + (2\sqrt{3})^2$

$BA = 4$

$\frac{2}{r} = \cos\frac{\pi}{6}$

$r = \frac{2}{\sqrt{3}/2}$

$r = \frac{4}{\sqrt{3}}$

$\angle DBA = \frac{\pi}{6}$

$= \angle HAM$

$\therefore HA \parallel DB$

Centre $\left(\frac{4}{\sqrt{3}}, 2i\right)$

Question 4

(a) (i) Area $= \pi[(9+x_1)^2 - (9-x_1)^2]$

$= \pi(18 \times 2x_1)$

$= 36\pi\sqrt{16-y^2}$

(ii) $V \doteqdot \sum_{-4}^{4} 36\pi\sqrt{16-y^2}\,\delta y$

$= 2\int_0^4 36\pi\sqrt{16-y^2}\,dy$

$= 72\pi \times \frac{1}{4}\pi \times 4^2$

$= 288\pi^2 \quad u^3$

(b) (i) $\sin x + \sin 3x$

$= 2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$

$= 2\sin 2x \cos x$

OR LHS $= \sin(2x - x) + \sin(2x + x)$

$= \sin 2x \cos x - \cos 2x \sin x$

$+ \sin 2x \cos x + \cos 2x \sin x$

$= 2\sin 2x \cos x$

(ii) $\sin x + \sin 2x + \sin 3x = 0$

$2\sin 2x \cos x + \sin 2x = 0$

$\sin 2x(2\cos x + 1) = 0$

$\sin 2x = 0$ or $\cos x = -\frac{1}{2}$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

4(c)

$\delta V = \frac{1}{2} \times b \times h\,\delta x$

$= \frac{1}{2} \times 2y \times (16 - x^2)\,\delta x$

$= \sqrt{16-x^2} \times (16-x^2)\,\delta x$

$= (16-x^2)^{3/2}\,\delta x$

$V = \int_{-4}^{4} (16-x^2)^{3/2}\,dx$

$= 2\int_0^4 (16-x^2)^{3/2}\,dx$

Let $x = 4\sin\theta$, $\frac{dx}{d\theta} = 4\cos\theta$

$x = 0, \theta = 0$

$x = 4, \theta = \frac{\pi}{2}$

$16 - x^2 = 16 - 16\sin^2\theta$

$= 16\cos^2\theta$

$V = 2\int_0^{\pi/2} 256\cos^4\theta\,d\theta$

$= 512\int\left[\frac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$

$= 128\int_0^{\frac{\pi}{2}}(1 + 2\cos 2\theta + \cos^2 2\theta)\,d\theta$

$= 128\int\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta$

$= 128\left[\theta + \sin 2\theta + \frac{\theta}{2} + \frac{1}{8}\sin 4\theta\right]_0^{\frac{\pi}{2}}$

$= 128\left[\frac{3}{2}\theta + 0\right]_0^{\frac{\pi}{2}}$

$= 128 \times \frac{3}{2} \times \frac{\pi}{2}$

$= 96\pi \quad u^3$

Question 5

(a) $(1-w)(1-w^2)(1-w^7)(1-w^{11})$

$= (1-w)(1-w^2)(1-w)(1-w^2)$

$= (1-w)^2(1-w^2)^2$

$= (1-2w+w^2)(1-2w^2+w^4)$

$= (1+w+w^2-3w)(1-2w^2+w)$

$= (-3w)(-w^2-2w^2)$

$= (-3w)(-3w^2)$

$= 9$

(b) $x^2y+xy^2+16=0 \quad *$

$2xy+x^2\frac{dy}{dx}+x.2y\frac{dy}{dx}+y^2=0$

$\frac{dy}{dx}(x^2+2xy)=-y^2-2xy$

$\frac{dy}{dx}=0 \Rightarrow -y^2-2xy=0$

$\therefore y(y+2x)=0$

$\therefore y=0$, or $y=-2x$

Subst $y=0$ into $*$

$0+16=0$, impossible

Subst $y=-2x$ into $*$

$-2x^3+4x^3+16=0$

$x^3=-8$

$x=-2$

when $x=-2$, $y=4$

$\therefore (-2,4)$ is the only point.

(c) (i) $F=ma$, $m=1$, $a=\ddot{x}$

resistance has negative sign

$ma=-(v+v^3)$

$\ddot{x}=-(v+v^3)$

(ii) $v\frac{dv}{dx}=-(v+v^3)$

$\int dx=\int\frac{-v}{v+v^3}\,dv$

$x=\int\frac{-1}{1+v^2}\,dv$

$=-\tan^{-1}v+c$

$x=0$, $v=Q \Rightarrow c=\tan^{-1}Q$

$\therefore x=\tan^{-1}Q-\tan^{-1}v$

$\tan x=\frac{Q-v}{1+Qv}$

$x=\tan^{-1}\left(\frac{Q-v}{1+Qv}\right)$

(iii) $\frac{dv}{dt}=-(v+v^3)$

$\int dt=-\int\frac{dv}{v+v^3}$

$t=-\int\left(\frac{1}{v}-\frac{v}{1+v^2}\right)dv$

$=-\ln v+\frac{1}{2}\ln(1+v^2)+c$

$t=0, v=Q \Rightarrow c=\ln Q-\frac{1}{2}\ln(1+Q^2)$

$t=-\ln v+\frac{1}{2}\ln(1+v^2)+\ln Q+\frac{1}{2}\ln(1+Q^2)$

$=\frac{1}{2}\ln\left[\frac{Q^2(1+v^2)}{v^2(1+Q^2)}\right]$

Question 5 (continued)

(c)(iv) $e^{2t}=\frac{Q^2(1+v^2)}{v^2(1+Q^2)}$

$v^2(1+Q^2)e^{2t}=Q^2+Q^2v^2$

$v^2[(1+Q^2)e^{2t}-Q^2]=Q^2$

$v^2=\frac{Q^2}{(1+Q^2)e^{2t}-Q^2}$

(v) As $t\to\infty$, $v\to 0$

As $t\to\infty$, $x=\tan^{-1}Q$

Question 6

(a) (i) The coefficients are real

$\therefore -ki$ is also a root.

(ii) $(ki)^4+a(ki)^3+b(ki)^2+cki+d=0$

$k^4-ak^3i-bk^2+kci+d=0$

Equating imaginary part,

$-ak^3+kc=0$

$\therefore c=ak^2$

(iii) Equating real part,

$k^4-bk^2+d=0$

$\left(\frac{c}{a}\right)^2-b\times\frac{c}{a}+d=0$

$c^2-abc+a^2d=0$

$c^2+a^2d=abc$

(iv) subst $x=2$,

$16+8a+2c+d=0$

$d=2(-c-4a-8)$ $\therefore$ even.

subst $b=0$ into $c^2+a^2d=abc$

$c^2+a^2d=0$

Since d is even, c^2 is even

Since c^2 is even, c is even

(b) (i) $z^5=-1$

$z^5=\text{cis}(\pi+2k\pi)$

$z=\text{cis}\left(\frac{\pi(2k+1)}{5}\right)$

$=\text{cis}\frac{\pi}{5}, \text{cis}\frac{3\pi}{5}, \text{cis}\pi, \text{cis}\frac{-\pi}{5}, \text{cis}\frac{-3\pi}{5}$

$\text{cis}\pi = -1$

(ii) $z^5+1=(z+1)(z^4-z^3+z^2-z+1)$

The only real root $z=-1$

All the 4 roots are non real.

(iii) $z^4-z^3+z^2-z+1=0$

$\div z^2$, $z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=0$

$z^2+\frac{1}{z^2}-\left(z+\frac{1}{z}\right)+1=0$

$2\cos2\theta-2\cos\theta+1=0$

$2(2\cos^2\theta-1)-2\cos\theta+1=0$

$4\cos^2\theta-2\cos\theta-1=0$

where $z=\cos\theta+i\sin\theta$

(iv) $z=\text{cis}\frac{3\pi}{5}$ a solution of $z^4-z^3+z^2-z+1=0$, so $\theta=\frac{3\pi}{5}$ is a soln of $4\cos^2\theta-2\cos\theta-1=0$

$\cos\theta=\frac{2\pm\sqrt{4-4(4)(-1)}}{8}=\frac{1-\sqrt{5}}{4}$

$\cos\frac{3\pi}{5}<0$

$\therefore \sec\frac{3\pi}{5}=\frac{4}{1-\sqrt{5}}$

$=-(1+\sqrt{5})$

Question 7

(a) $\dfrac{x^2}{4-\lambda} + \dfrac{y^2}{2-\lambda} = 1$

(i)(α) ellipse: $4-\lambda > 0$ & $2-\lambda > 0$

$\lambda < 4$ and $\lambda < 2$

Hence $\lambda < 2$

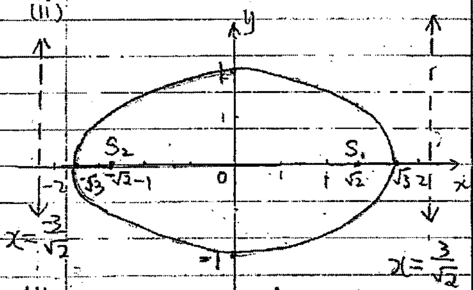
(β) hyperbola: $4-\lambda > 0$ and $2-\lambda < 0$

or $4-\lambda < 0$ and $2-\lambda > 0$

Hence, $2 < \lambda < 4$

(Not possible $\lambda < 2$ and $\lambda > 4$)

(ii)



(iii) As λ increases from 1 to 2, $4-\lambda$ decreases from 3 to 2, while $2-\lambda$ decreases from 1 to 0. The curve remains an ellipse, with the semi-major axis reducing from $\sqrt{3}$ to $\sqrt{2}$, and the semi-minor axis from 1 to 0.
As 2 is approached, $b \to 0$, the ellipse becomes a line segment joining $(-\sqrt{2}, 0)$, to $(\sqrt{2}, 0)$

(b) $y = c^2 x^{-1}$

$\dfrac{dy}{dx} = \dfrac{-c^2}{x^2}$

(i) $= \dfrac{-c^2}{c^2p^2}$ gradient of normal $= p^2$

$= -\dfrac{1}{p^2}$

$y - \frac{c}{p} = p^2(x - cp)$

$p^3x - yp = c(p^4 - 1)$

(ii) grad $PQ = \dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq}$

$= \dfrac{-1}{pq}$

Hence $\dfrac{-1}{pq} = p^2$

$p^3q = -1$

(iii) $R\left(\frac{c}{2}(p+q), \frac{c}{2}\left(\frac{p+q}{pq}\right)\right)$ [x, y]

$\dfrac{x}{y} = \frac{c}{2}(p+q) \times \dfrac{2pq}{c(p+q)}$

$= pq$

From (ii), $pq = \dfrac{-1}{p^2}$

$\therefore \dfrac{x}{y} = -\dfrac{1}{p^2}$

using equation of normal

$p^2x - y = \frac{c}{p}(p^4 - 1)$

$-\left(\frac{y}{x}\right)x - y = \frac{c}{p}\left(\left(\frac{-y}{x}\right)^2 - 1\right)$

$-2y = \frac{c}{p}\left(\frac{y^2 - x^2}{x^2}\right)$

$2y = \frac{c}{p}\left(\frac{x^2 - y^2}{x^2}\right)$

Question 7 (continued)

(b)(iii) squaring both sides,

$4y^2 = c^2 \times \left(-\frac{x}{y}\right)\left(\frac{x^2 - y^2}{x^2}\right)^2$

$4y^3x^3 = -c^2(x^2 - y^2)^2$

$4y^3x^3 + c^2(x^2 - y^2)^2 = 0$

Question 8

(a) $I_n = \int \sec^n x \, dx$

(i) $= \int \sec^2 x . \sec^{n-2} x \, dx$

$= \tan x . \sec^{n-2} x - \int \tan x (n-2) \sec^{n-3} x \times \sec x \tan x \, dx$

$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x (\sec^2 x - 1) dx$

$= \sec^{n-2} x \tan x - (n-2)\int(\sec^n x - \sec^{n-2} x) dx$

$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$I_n (n-1) = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$I_n = \dfrac{1}{n-1} \sec^{n-2} x \tan x + \dfrac{n-2}{n-1} I_{n-2}$

(ii) $\int_0^{\frac{\pi}{4}} \sec^6 x \, dx$

$I_2 = \int_0^{\frac{\pi}{4}} \sec^2 x \, dx$

$= [\tan x]_0^{\frac{\pi}{4}} = 1$

$I_4 = \frac{1}{3} \times \sec^2 x + \frac{4-2}{4-1} \times I_2$

$= \frac{1}{3} \times \sqrt{2}^2 + \frac{2}{3} \times 1$

$= 4/3$

$I_6 = \frac{1}{5} \times (\sqrt{2})^4 + \frac{4}{5} \times \frac{4}{3} = \frac{28}{15}$

Q8 (b)

(i) $\left(\frac{9}{10}\right)^4 = \dfrac{6561}{10000}$

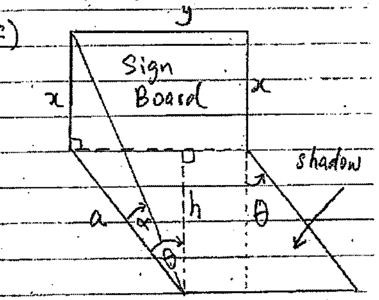
(ii) ${}^4C_2\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2 + {}^4C_3\left(\frac{4}{5}\right)^1\left(\frac{1}{5}\right)^3 + {}^4C_4\left(\frac{1}{5}\right)^4$

$= \dfrac{113}{625}$

(iii) ${}^4C_2\left(\frac{1}{5}\right)^2\left(\frac{7}{10}\right)^2 + {}^4C_3\left(\frac{1}{5}\right)^3\left(\frac{7}{10}\right)^1 + {}^4C_4\left(\frac{1}{5}\right)^4$

$= \dfrac{177}{1250}$

(c)



Let θ be the bearing of the sun and α be the altitude of the sun.

$a = \dfrac{x}{\tan\alpha}$, $h = a\cos\theta = \dfrac{x\cos\theta}{\tan\alpha}$

Area of shadow $= y \times h$

$= \dfrac{yx\cos\theta}{\tan\alpha}$

$\frac{5}{4}xy = \dfrac{yx\cos\theta}{\tan\alpha}$

$\theta = \cos^{-1}\left[\dfrac{5\tan\alpha}{4}\right]$ E or W of North. —END—