Student Number ....

Barker College

Mathematics Extension 2

2014
TRIAL
HIGHER SCHOOL CERTIFICATE

PM Thursday 1st August

Section I – Multiple Choice

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

(A) ○ (B) ● (C) ○ (D) ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) ● (B) ~~●~~ (C) ○ (D) ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word ***correct*** and drawing an arrow as follows.

correct

(A) ~~●~~ (B) ~~●~~ (C) ○ (D) ○

Start Here →

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

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Student Number: ………………………………….

Barker College

2014
YEAR 12
TRIAL HSC
EXAMINATION

Mathematics Extension 2

Staff Involved:

PM 1st AUGUST 2014

- **BHC**
- **VAB**
- **KJL***
- **GDH***

Number of copies: 40

General Instructions
- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I **Pages 2-6**

10 marks
- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II **Pages 7-15**

90 marks
- Attempt Questions 11 - 16
- Allow about 2 hours 45 minutes for this section

Section I — Multiple Choice

10 marks

Attempt Questions 1-10. Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10.

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1. What is the eccentricity of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$?

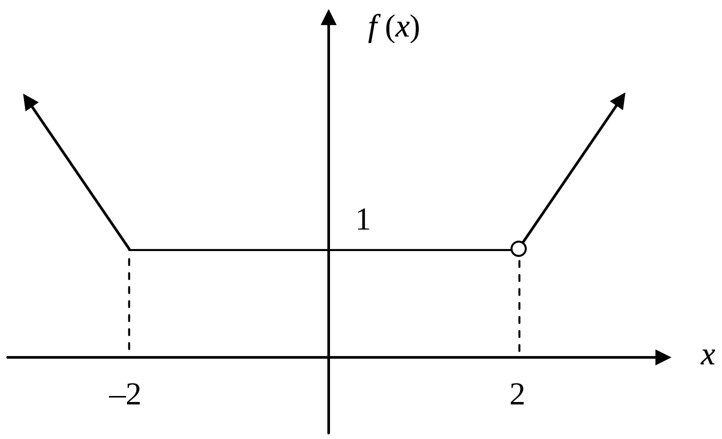
(A) $\frac{2}{3}$

(B) $\frac{3}{2}$

(C) $\frac{\sqrt{5}}{3}$

(D) $\frac{3}{\sqrt{5}}$

2. The graph of a function $y = f(x)$ is shown below.



Which of the following statements is true?

(A) $f(x)$ is not continuous at $x = 2$ and $f(x)$ is not differentiable for $-2 < x < 2$

(B) $f(x)$ is not continuous at $x = 2$ and $x = -2$, and $f(x)$ is not differentiable at $x = 2$ and $x = -2$

(C) $f(x)$ is not continuous at $x = 2$, but $f(x)$ is differentiable for all x

(D) $f(x)$ is not continuous at $x = 2$ and not differentiable at $x = 2$ and $x = -2$

3. The graph below shows a hyperbola, including the equations of the directrices and the coordinates of the foci.

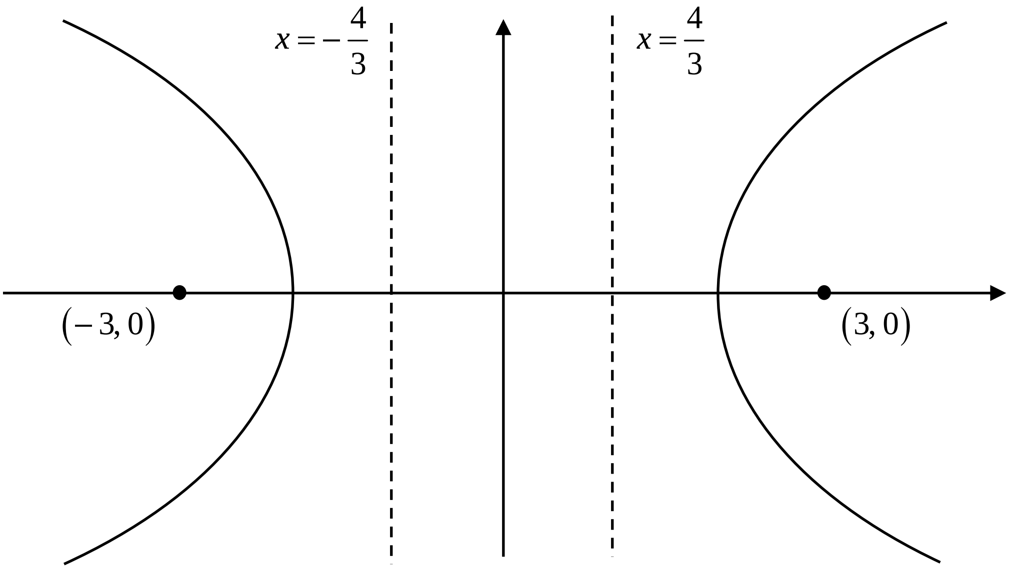


DIAGRAM NOT TO SCALE

The equation of this hyperbola is:

(A) $\frac{x^2}{4} - \frac{y^2}{5} = 1$

(B) $\frac{x^2}{5} - \frac{y^2}{4} = 1$

(C) $\frac{x^2}{16} - \frac{y^2}{25} = 1$

(D) $\frac{x^2}{2} - \frac{y^2}{5} = 1$

4. The polynomial $P(x) = x^4 + mx^2 + nx + 28$ has a double root at $x = 2$.

What are the values of m and n?

(A) $m = -11$ and $n = -12$

(B) $m = -5$ and $n = -12$

(C) $m = -11$ and $n = 12$

(D) $m = -5$ and $n = 12$

Section I continued

5. Diagram A shows the complex number z represented in the Argand plane.

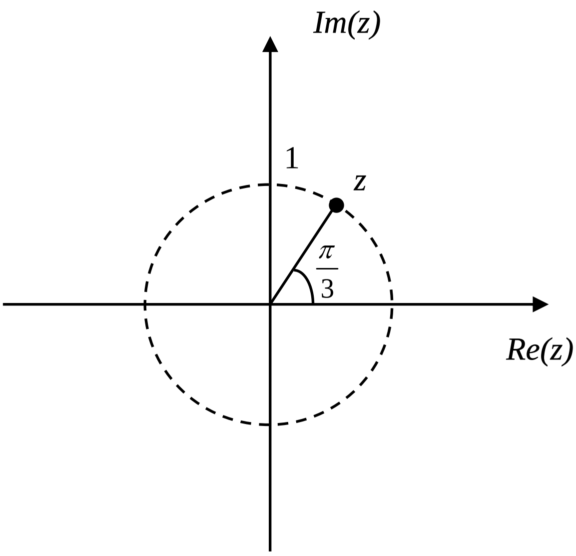


DIAGRAM A

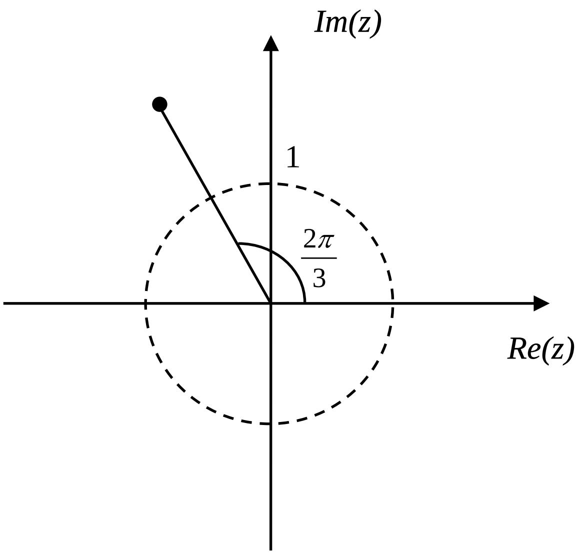


DIAGRAM B

Diagram B shows:

(A) z^2

(B) $2iz$

(C) $-2z$

(D) $2z^2$

6. Let α, β, γ be the roots of $2x^3 - 5x^2 + 2x - 1 = 0$.
The equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ is best given by:

(A) $x^3 + 2x^2 + 25x - 2 = 0$

(B) $x^3 - 2x^2 + 5x - 2 = 0$

(C) $x^3 - 2x^2 + 25x - 8 = 0$

(D) $-2x^3 + 5x^2 - 2x + 1 = 0$

Section I continued

7. How many distinct horizontal tangents can be drawn on the graph of $x^2 + y^2 + xy - 5 = 0$?

(A) 0

(B) 1

(C) 2

(D) More than 2

8. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3 x\, dx$ is equal in value to:

(A) $\displaystyle\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^3 x\, dx$

(B) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\pi}{4} - \tan^3 x\, dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \tan x\left(\sec^2 x + 1\right) dx$

(D) $\displaystyle\int_0^{\frac{\pi}{4}} \left(\frac{1-\tan x}{1+\tan x}\right)^3 dx$

Section I continued

9. The value of $\int_{-2}^{-1} \sqrt{4-x^2}\, dx + \int_{1}^{2} \sqrt{4-x^2}\, dx$ is:

(A) $\frac{4\pi}{3} - \sqrt{3}$

(B) $\frac{2\pi}{3} - \sqrt{3}$

(C) $\frac{2\pi}{3} - 2\sqrt{3}$

(D) $\frac{8\pi}{3} - \sqrt{3}$

10. The statement $\int_{0}^{1} \frac{dx}{1+x^n} < \int_{0}^{1} \frac{dx}{1+x^{n+1}}$ is:

(A) Always true

(B) Always false

(C) Only true for positive n

(D) Only true for negative n

End of Section I

Section II

90 marks

Attempt Questions 11-16.

Allow about 2 hours 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Show relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) (i) Find the square root of the complex number $15-8i$. **2**

(ii) Hence or otherwise, solve $z^2+6z-6+8i=0$. **1**

(b) (i) Write the complex number $z=1+\sqrt{3}i$ in modulus-argument form. **1**

(ii) For what values of *n* is $\left(1+\sqrt{3}i\right)^n$ completely real?

Justify your answer with appropriate reasoning. **2**

(c) The points *A* and *B* represent the complex numbers $z_1=2-i$ and $z_2=8+i$ respectively.

Find all possible complex numbers z_3, represented by *C* such that ΔABC is **isosceles** and **right-angled** at *C*. **3**

(d) On the Argand diagram, sketch the locus of *z* defined by $\arg\left(\dfrac{z-(3+3i)}{z-2}\right)=\dfrac{\pi}{3}$. **2**

(e) (i) Sketch $y=x\,(2^x)$, showing all key features. **2**

(ii) For what values of *k* does $x\,(2^x)-kx=0$ have exactly 2 real roots? **2**

End of Question 11

Question 12 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) (i) Find real numbers A, B and C, such that $\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+4}=\dfrac{1}{(x-2)(x^2+4)}$ **2**

(ii) Hence, or otherwise, find $\displaystyle\int \frac{1}{(x-2)(x^2+4)}\,dx$ **2**

(b) Using the substitution $t=\tan\dfrac{\theta}{2}$ or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta+2}$ **3**

(c) By considering $y=x^2-4x+3$, draw on separate diagrams sketches of:

(i) $y^2=x^2-4x+3$ **2**

(ii) $y=\dfrac{1}{x^2}-\dfrac{4}{x}+3$ *Hint: You may wish to consider this graph as* $y=f\left(\dfrac{1}{x}\right)$ **3**

(d) Let $I_n=\int x^n e^x dx$.

(i) Show that $I_n=x^n e^x - nI_{n-1}$. **1**

(ii) Hence evaluate $I=\displaystyle\int_1^2 x^2 e^x dx$ **2**

End of Question 12

Question 13 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) The base of a solid is the region bounded by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Its volume consists of semicircular cross sections.

Mr Lee initially believed that the volume of the solid would be the same regardless of whether the semicircular cross-sections were parallel to the x-axis or the y-axis.

Mr Lee was **wrong**.

By showing appropriate calculations, find the ratio of the volumes of the two solids. **4**

(b) Let $f(x)=5$ and $g(x)=x+\dfrac{4}{x}$.

(i) On the same diagram, sketch $y=f(x)$ and $y=g(x)$.

Clearly label any points of intersection. **2**

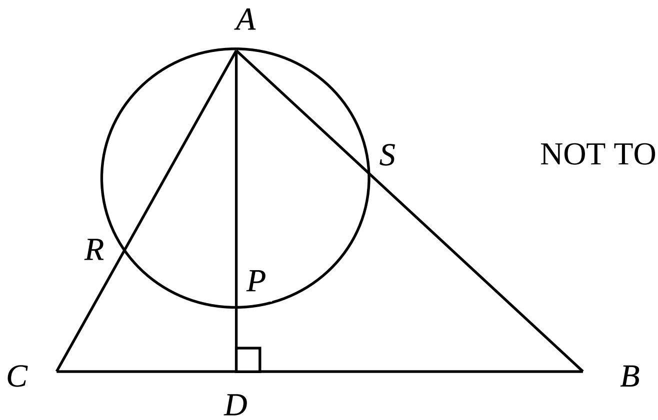
(ii) The region bounded by $y=f(x)$ and $y=g(x)$ is rotated about the line $x=-1$.

By using the method of cylindrical shells, find the volume of the solid generated. **3**

Question 13 continues on page 10

(c) In the triangle ABC, D is the foot of the altitude from A.

P is any point on AD. The circle drawn with diameter AP cuts AC at R and AB at S.



Prove that $BCRS$ is a cyclic quadrilateral. **3**

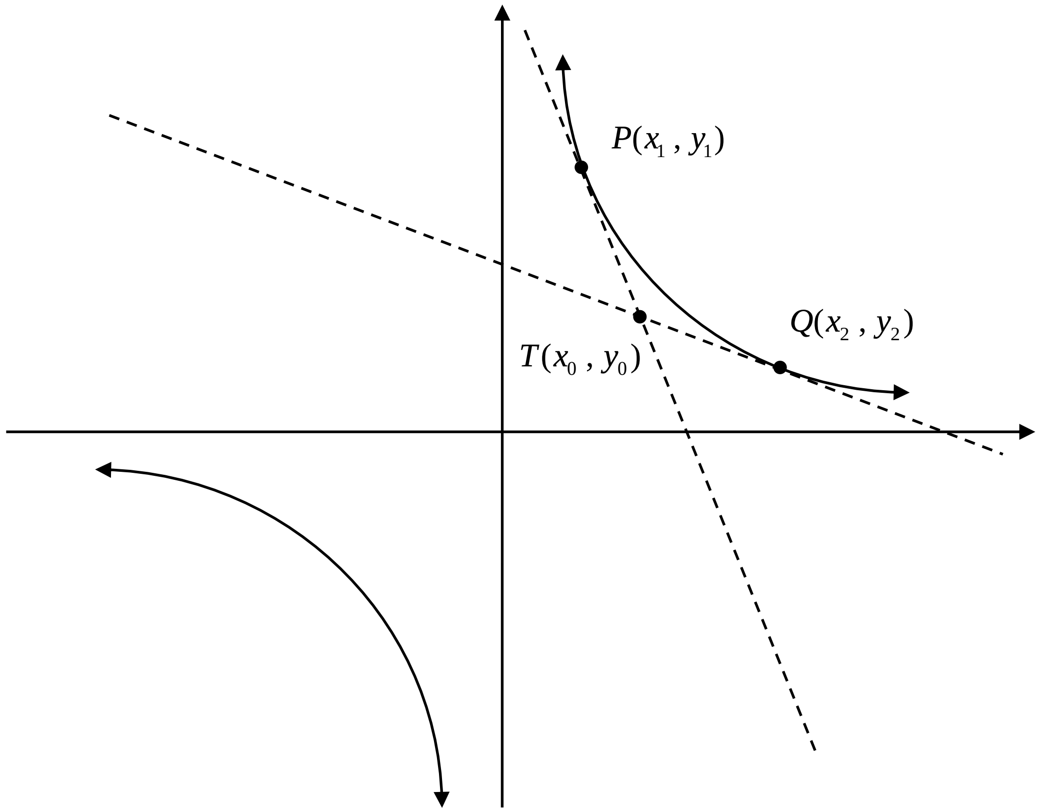
(d) The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has asymptotes given by $y = \pm \dfrac{bx}{a}$.

Show that the product of the lengths of the perpendiculars from any point $P\,(a\sec\theta, b\tan\theta)$ on the hyperbola to its asymptotes is equal to $\dfrac{b^2}{e^2}$. **3**

End of Question 13

Question 14 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) The tangents at $P(x_1\,,\,y_1)$ and $Q(x_2\,,\,y_2)$ on the hyperbola $xy = 1$ intersect at $T(x_0\,,\,y_0)$.



(i) Show that the tangent at $P(x_1\,,\,y_1)$ has the equation $xy_1 + yx_1 = 2$ **2**

(ii) Show that the chord of contact from T has the equation $xy_0 + yx_0 = 2$ **2**

(iii) Show that x_1 and x_2 are the roots of the equation $y_0x^2 - 2x + x_0 = 0$ **2**

(iv) Hence, or otherwise, show that the midpoint R of PQ has coordinates $\left(\dfrac{1}{y_0}\,,\,\dfrac{1}{x_0}\right)$. **2**

(v) Hence, or otherwise, show that as T moves on the hyperbola $xy = c^2$, $0 < c < 1$, R moves on the hyperbola $xy = \dfrac{1}{c^2}$. **1**

Question 14 continues on page 12

Question 14 continued

(b) A particle is travelling in a straight line with speed v m/s.

Its acceleration is given by the equation $a = -kv$.

Initially it is at the origin travelling at a speed of U m/s.

(i) Prove that $a = v\dfrac{dv}{dx}$ **1**

(ii) Find an expression for the particle's velocity v in terms of its displacement x. **2**

(iii) Find an expression for the particle's velocity v in terms of time t. **2**

(iv) Hence, or otherwise, find its limiting displacement as $t \to \infty$ **1**

End of Question 14

Question 15 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) A cubic equation $x^3 + bx^2 + cx + d = 0$ has roots α, β, γ.

α, β, γ are in geometric progression and the middle root $\beta = -6$.

It is also known that $\alpha\beta + \beta\gamma + \alpha\gamma = 156$.

Find the equation of this cubic expressed in the form $x^3 + bx^2 + cx + d = 0$. **3**

(b) It is known that $x^3 - 6x^2 + 11x + a - 4 = 0$ has three distinct integer solutions.

Find the value of *a* and the three integer solutions. **3**

(c) (i) Find all the fifth roots of 1 and plot them on the unit circle. **1**

(ii) Let ω represent a non-real fifth root of 1, such that $0 < \arg\omega < \frac{\pi}{2}$.

Show that the five roots of 1 can be represented as $1, \omega, \omega^2, \omega^3, \omega^4$ **1**

(iii) Show that $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) = 5$ **2**

(iv) Show that $(1-\omega)(1-\omega^4) = 2 - 2\cos\frac{2\pi}{5}$ **2**

(v) Hence, or otherwise, find the exact value of $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}$ **3**

End of Question 15

Question 16 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) When a certain polynomial $P(x)$ is divided by $(x-2)$ its remainder is -5.

When $P(x)$ is divided by $(x-1)$ its remainder is -2.

Find the remainder when $P(x)$ is divided by $(x-1)(x-2)$. **2**

(b) Let $f(x)=\dfrac{\sin x+1}{3\sin x+2}$.

(i) Prove that $f(x)$ is a decreasing function for $0<x<\dfrac{\pi}{2}$. **1**

(ii) Hence, or otherwise, prove that $\dfrac{(\sqrt{2}-1)\,\pi}{12}<\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{\sin x+1}{3\sin x+2}\,dx<\frac{\pi}{28}$ **3**

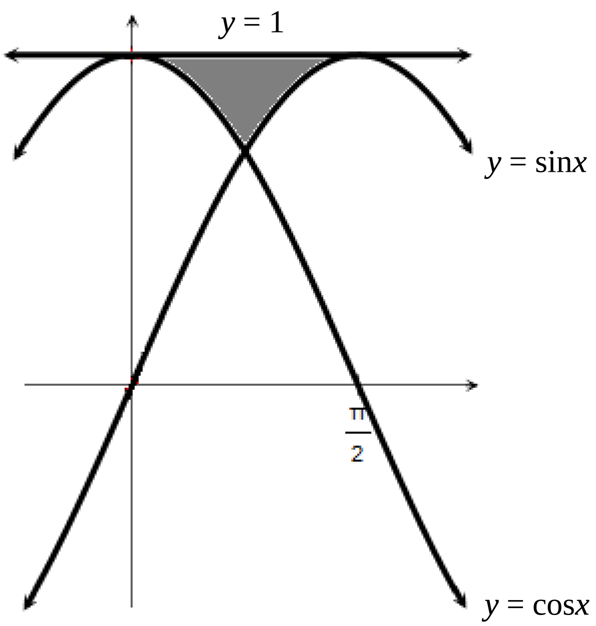
(c) (i) By induction, show that for each positive integer *n* there are unique positive integers p_n and q_n such that $\left(1+\sqrt{2}\right)^n=p_n+q_n\sqrt{2}$ **2**

(ii) Hence also show that $p_n{}^2-2q_n{}^2=\left(-1\right)^n$ **2**

Question 16 continues on page 15

(d) The shaded area in the diagram is bounded by the curves $y = \sin x$, $y = \cos x$ and $y = 1$.

This area is rotated around the y-axis.



(i) By differentiation or otherwise, show that $\sin^{-1} y + \cos^{-1} y = \dfrac{\pi}{2}$ **1**

(ii) By using the method of slicing discs, find the volume of the solid generated. **4**

End of Question 16
End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1 \, ; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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(1) Since $b > a$,

$a^2 = b^2(1-e^2)$

$\therefore 4 = 9(1-e^2)$

$\therefore 1-e^2 = \frac{4}{9}$

$e^2 = \frac{5}{9} \quad \therefore e = \frac{\sqrt{5}}{3} \quad \therefore C$

(2) D

(3) $ae = 3$

$\frac{a}{e} = \frac{4}{3} \rightarrow 3a = 4e$

$e = \frac{3a}{4}$

$\therefore a \times \frac{3a}{4} = 3$

$a^2 = 4$

$a = 2 \rightarrow e = 3/2$

$b^2 = a^2(e^2 - 1)$

$\therefore b^2 = 4\left(\frac{9}{4} - 1\right)$

$b^2 = 4 \times \frac{5}{4} = 5$

$\therefore \frac{x^2}{4} - \frac{y^2}{5} = 1 \quad \therefore A$

(4) $P(2) = P'(2) = 0$

$\therefore 16 + 4m + 2n + 28 = 0$

$4m + 2n = -44$

$2m + n = -22$

$P'(2) = 4(2)^3 + 2m(2) + n = 0$

$32 + 4m + n = 0$

$4m + n = -32$

$\therefore 2m = -10$

$m = -5$

$n = -12 \quad \therefore B$

(5) D since z^2 doubles the argument and the 2 doubles the modulus

(6) Let $y = \frac{1}{x} \quad \therefore x = \frac{1}{y}$

$\therefore \frac{2}{y^3} - \frac{5}{y^2} + \frac{2}{y} - 1 = 0$

$\therefore 2 - 5y + 2y^2 - y^3 = 0$

$\therefore x^3 - 2x^2 + 5x - 2 = 0 \quad \therefore B$

(7) $2x + 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-(2x+y)}{2y+x} = 0$ for horizontal tangents

$\therefore 2x = -y \quad (x \neq 0)$

$\therefore x^2 + 4x^2 - 2x^2 = 5$

$3x^2 = 5$

$x = \pm\sqrt{\frac{5}{3}}$

$\therefore$ At points $\left(\sqrt{\frac{5}{3}}, -2\sqrt{\frac{5}{3}}\right)$ & $\left(-\sqrt{\frac{5}{3}}, 2\sqrt{\frac{5}{3}}\right)$, two distinct horizontal tangents $\therefore C$

(8) Not A since A is 0 due to odd fn

Not C since $\tan^3 x = \tan x \tan^2 x = \tan x(\sec^2 x - 1)$

Can't see how B works...

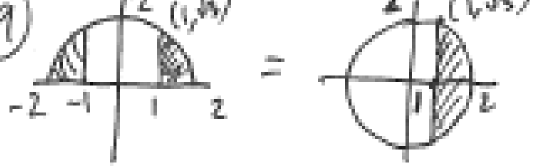
Maybe... $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ since $\tan\frac{\pi}{4} = 1$

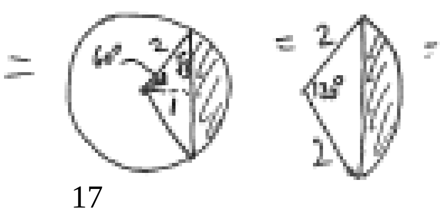
$\therefore \int_0^{\frac{\pi}{4}} (\tan x)^3\,dx = \int_0^{\frac{\pi}{4}} \left[\tan\left(\frac{\pi}{4} - x\right)\right]^3 dx$

$= \int_0^{\frac{\pi}{4}} \left[\frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right]^3 dx$

$= \int_0^{\frac{\pi}{4}} \left[\frac{1 - \tan x}{1 + \tan x}\right]^3 dx \quad \therefore D$

(9)





$= \frac{1}{2} 2^2 \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

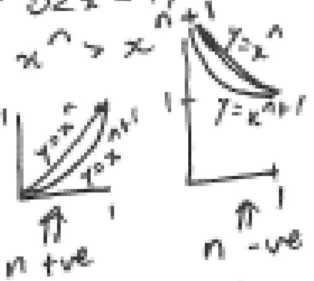
$= 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= \frac{4\pi}{3} - \sqrt{3} \quad \therefore A$

(10) For $x = 0$ & $x = 1$, $x^n = x^{n+1}$

For $0 < x < 1$, $x^n > x^{n+1}$

eg



$\therefore \int_0^1 x^n\,dx > \int_0^1 x^{n+1}\,dx$

$\therefore \int_0^1 1 + x^n\,dx > \int_0^1 1 + x^{n+1}\,dx$

$\therefore \int_0^1 \frac{1}{1+x^n}\,dx < \int_0^1 \frac{1}{1+x^{n+1}}\,dx$

$\therefore$ A, since always true

(11) (a) (i) $(a+bi)^2 = 15-8i$

$\therefore a^2-b^2=15$

$2ab=-8 \rightarrow b=-\frac{4}{a}$

$\therefore a^2-\frac{16}{a^2}=15$

$a^4-15a^2-16=0$

$(a^2-16)(a^2+1)=0$

$\therefore a=\pm 4$, since a is real

$\therefore b=\mp 1$

$\therefore \sqrt{15-8i} = \pm(4-i)$

(ii) $z=\dfrac{-6\pm\sqrt{36-4(-6+8i)}}{2}$

$=\dfrac{-6\pm\sqrt{36+24-32i}}{2}$

$=\dfrac{-6\pm\sqrt{60-32i}}{2}$

$=\dfrac{-6\pm 2\sqrt{15-8i}}{2}$

$=-3\pm\sqrt{15-8i}$

$=-3\pm(4-i)$

$z = 1-i$ or $-7+i$

(b) (i)

$2\,\text{cis}\,\frac{\pi}{3}$

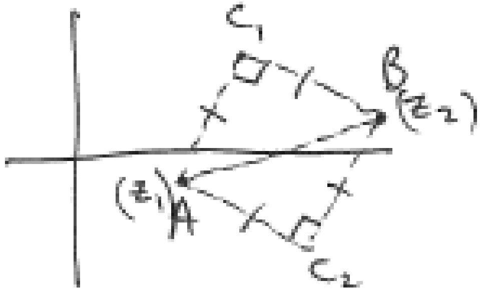
(ii) $2^n \text{cis}\,\frac{n\pi}{3}$ real?

when $\frac{n\pi}{3}=\pm k\pi$ (k integer)

$\therefore n=\pm 3k$

$\therefore n$ is a multiple of 3

(c)



AC_1BC_2 is a square

Midpt $AB=(5,0)$ which is up 1, across 3 from A

$m_{AB}=\frac{2}{6}=\frac{1}{3}$ $\therefore m_{C_1C_2}=-3$

$\therefore$ From $(5,0)$ go right 1, down 3 for C_2

& go left 1, up 3 for C_1

This is because diagonals bisect each other at right angles and are the same length

$\therefore C_2(6,-3)$ & $C_1(4,3)$

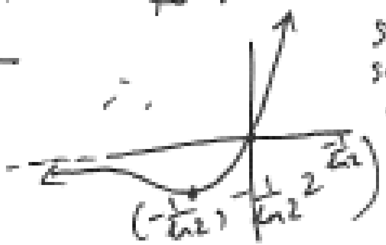
$\therefore z_3 = 6-3i$ & $4+3i$

(d)

(e)(i)

$y=2^x$ is more powerful for $x\to-\infty$.

$\therefore$



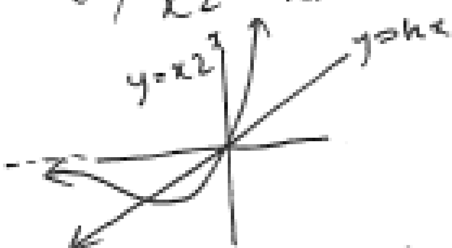
shaped similar to $y=xe^x$

$\frac{dy}{dx}=1(2^x)+(2^x\ln 2)x$

$=2^x(1+x\ln 2)=0$ for stat pts

$\therefore x\ln 2=-1$ $\therefore x=\frac{-1}{\ln 2}$

(ii) $x2^x=kx$



we need $k>0$, except where $y=kx$ is tangent to $y=x2^x$ at the origin.

Gradient of $y=x2^x$ at $x=0$ is $2^0(1+0)=1$

$\therefore k>0$ except $k=1$

(12) (a) (i) $A(x^2+4)+(x-2)(Bx+C)\equiv 1$

$x=2: \ 8A=1 \ \therefore A=\frac{1}{8}$

$x=0: \ 4A-2C=1 \ \therefore \frac{1}{2}-2C=1$

$\therefore 2C=-\frac{1}{2}$

$C=-\frac{1}{4}$

Coeff x^2: $A+B=0 \ \therefore B=-\frac{1}{8}$

(ii) $\int \frac{\frac{1}{8}}{x-2}+\frac{-\frac{1}{8}x-\frac{1}{4}}{x^2+4}\,dx$

$=\frac{1}{8}\int \frac{1}{x-2}-\frac{x+2}{x^2+4}\,dx$

$=\frac{1}{8}\int \frac{1}{x-2}-\frac{x}{x^2+4}-\frac{2}{x^2+4}\,dx$

$=\frac{1}{8}\int\frac{1}{x-2}\,dx-\frac{1}{16}\int\frac{2x\,dx}{x^2+4}-\frac{1}{4}\int\frac{1}{x^2+4}\,dx$

$=\frac{1}{8}\ln|x-2|-\frac{1}{16}\ln|x^2+4|-\frac{1}{4}\times\frac{1}{2}\tan^{-1}\frac{x}{2}+c$

$=\frac{1}{8}\ln|x-2|-\frac{1}{16}\ln|x^2+4|-\frac{1}{8}\tan^{-1}\frac{x}{2}+c$

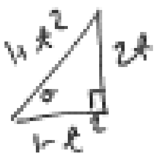
(b) $\frac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{\theta}{2}=\frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)=\frac{1}{2}\left(1+t^2\right)$

$\therefore 2\,dt=(1+t^2)\,d\theta$

$\therefore d\theta=\frac{2\,dt}{1+t^2}$

$\theta=0: \ t=0$

$\theta=\frac{\pi}{2}: \ t=\tan\frac{\pi}{4}=1$



$\therefore \int_0^1 \frac{2\,dt}{1+t^2}\times\frac{1}{\frac{2t}{1+t^2}+2}$

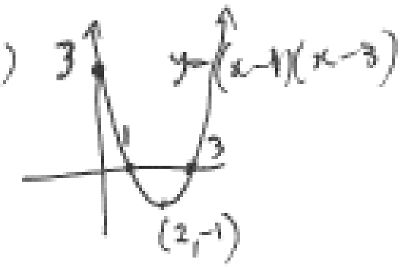
$=\int_0^1\frac{2\,dt}{2t+2+2t^2}=\int_0^1\frac{dt}{t^2+t+1}$

$=\int_0^1\frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}=\left[\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{(t+\frac{1}{2})}{\frac{\sqrt{3}}{2}}\right]_0^1$

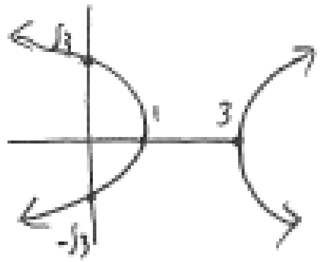
$=\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1=\frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{3}{\sqrt{3}}-\tan^{-1}\frac{1}{\sqrt{3}}\right)$

$=\frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3}-\tan^{-1}\frac{1}{\sqrt{3}}\right)=\frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}\times\frac{\pi}{6}=\frac{\pi}{3\sqrt{3}}$

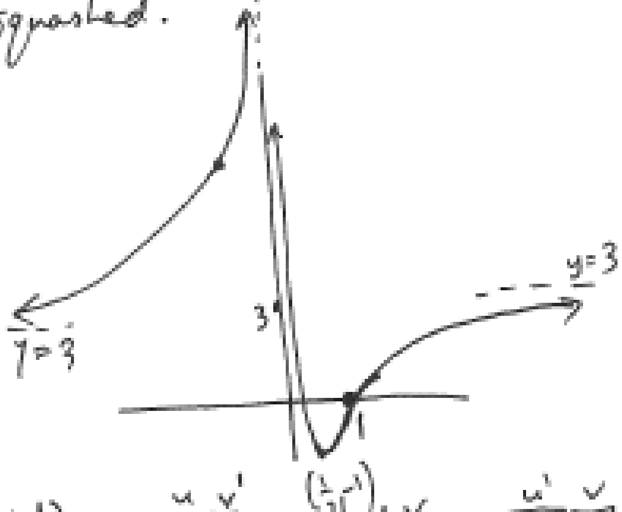
(c)



(i) $y=\pm\sqrt{x^2-4x+3}$



(ii) $f(x)$ and $f\left(\frac{1}{x}\right)$ are the same for $x=\pm 1$, and are reflections of each other in the lines $x=\pm 1$, albeit the reflections are stretched and squashed.



(d) (i) $I_n=\int \overset{u}{x^n}\,\overset{v'}{e^x}\,dx=\overset{u}{x^n}\,\overset{v}{e^x}-\int \overset{u'}{nx^{n-1}}\,\overset{v}{e^x}\,dx$

$I_n=x^ne^x-n\int x^{n-1}e^x\,dx$

$I_n=x^ne^x-nI_{n-1}$

(ii) $I_2=x^2e^x-2I_1$

$I_1=xe^x-I_0$

$I_0=e^x$

$\therefore I_1=xe^x-e^x$

$\therefore I_2=x^2e^x-2xe^x+2e^x=e^x(x^2-2x+2)$

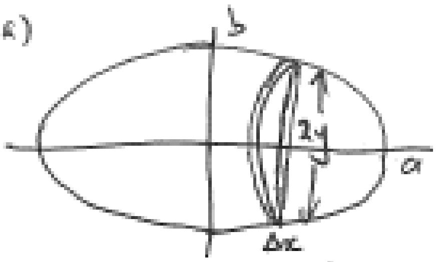
$\therefore I=\left[e^x(x^2-2x+2)\right]_1^2$

$=e^2(4-4+2)-e(1-2+2)$

$=2e^2-e$

$=e(2e-1)$

(13) (a)



V_1: Cross-section parallel to y-axis

Radius is y

$\therefore \Delta V_1 = \frac{\pi y^2}{2}\Delta x$

$V_1 = 2\int_0^a \frac{\pi y^2}{2}\,dx$

$= \pi\int_0^a y^2\,dx$

Now $\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

$\therefore y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

$= \pi\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$

$= \pi b^2\int_0^a 1-\frac{x^2}{a^2}\,dx$

$= \pi b^2\left[x-\frac{x^3}{3a^2}\right]_0^a$

$= \pi b^2\left[a-\frac{a}{3}\right] = \frac{2\pi ab^2}{3}$

V_2: Cross-section parallel to x-axis

Radius is x

$\therefore \Delta V_2 = \frac{\pi x^2}{2}\Delta y$

$V_2 = 2\int_0^b \frac{\pi x^2}{2}\,dy$

Now $\frac{x^2}{a^2} = 1-\frac{y^2}{b^2}$

$\therefore x^2 = a^2\left(1-\frac{y^2}{b^2}\right)$

$= \pi\int_0^b x^2\,dy$

$= \pi a^2\int_0^b 1-\frac{y^2}{b^2}\,dy$

$= \pi a^2\left[y-\frac{y^3}{3b^2}\right]_0^b$

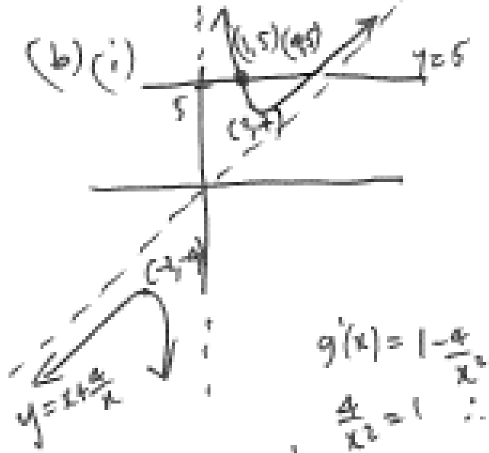
$= \pi a^2\left[b-\frac{b}{3}\right] = \frac{2\pi a^2 b}{3}$

$\therefore \frac{V_1}{V_2} = \frac{\frac{2\pi ab^2}{3}}{\frac{2\pi a^2 b}{3}} = \frac{6\pi ab^2}{6\pi a^2 b} = \frac{b}{a}$

$\therefore$ Ratio $V_1 : V_2 = b : a$

or Ratio $V_2 : V_1 = a : b$

(b) (i)



$5 = x + \frac{4}{x}$

$x^2 - 5x + 4 = 0$

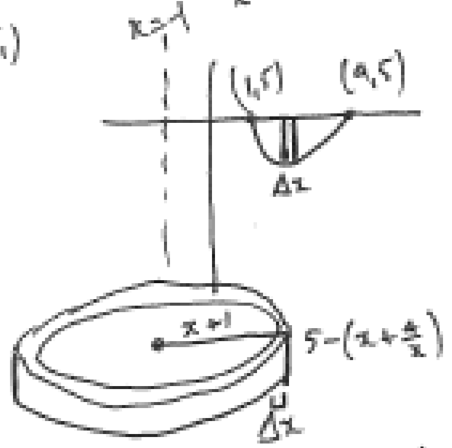
$(x-1)(x-4) = 0$

$x = 1$ or 4

$g'(x) = 1 - \frac{4}{x^2} = 0$ for stat pts

$\frac{4}{x^2} = 1 \quad \therefore x^2 = 4 \quad \therefore x = \pm 2$

(ii)



$\therefore \Delta V = 2\pi(x+1)\left(5-x-\frac{4}{x}\right)\Delta x$

$\therefore V = 2\pi\int_1^4 (x+1)\left(5-x-\frac{4}{x}\right)dx$

$V = 2\pi\int_1^4 5x - x^2 - 4 + 5 - x - \frac{4}{x}\,dx$

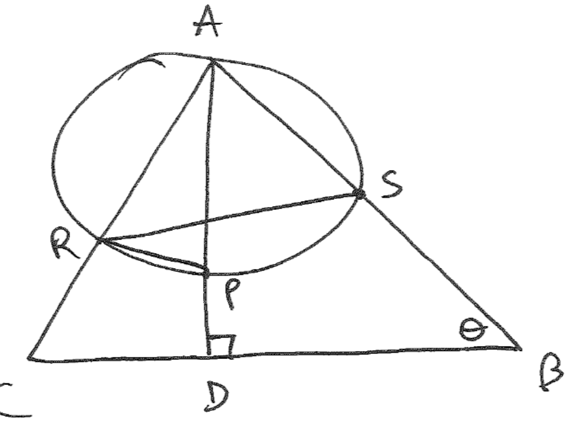
$= 2\pi\int_1^4 4x - x^2 + 1 - \frac{4}{x}\,dx$

$= 2\pi\left[2x^2 - \frac{x^3}{3} + x - 4\ln x\right]_1^4$

$= 2\pi\left\{32 - \frac{64}{3} + 4 - 4\ln 4 - 2 + \frac{1}{3} - 1 + 0\right\}$

$= 2\pi\{12 - 4\ln 4\} = 8\pi(3 - \ln 4)\,u^3$

(c)



Join RS and RP

let $\angle ABD = \theta$

$\therefore \angle DAB = 90 - \theta$ (angle sum ΔABD)

$\therefore \angle PRS = 90 - \theta$ (angles at circumference on same arc PS)

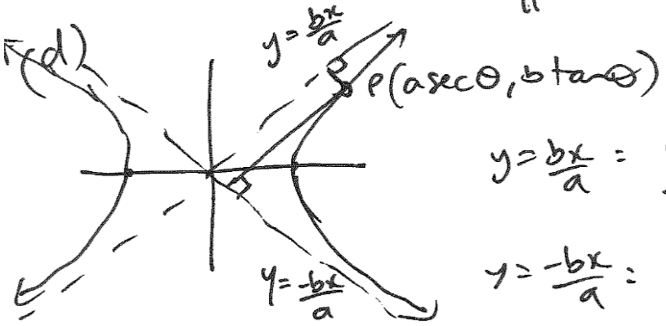
$\angle ARP = 90^\circ$ (angle in semicircle since AP is diameter)

$\therefore \angle ARS = 90 - (90 - \theta) = \theta$ (subtraction)

$\therefore \angle ARS = \angle ABD$

$\therefore$ BCRS cyclic quad (exterior angle = opposite interior angle)

(d)



$y = \frac{bx}{a}$: $ay = bx$ $\therefore bx - ay = 0$

$y = \frac{-bx}{a}$: $ay = -bx$, $bx + ay = 0$

$\therefore$ Product =

$$\frac{|ba\sec\theta - ab\tan\theta|}{\sqrt{b^2 + (-a)^2}} \times \frac{|ba\sec\theta + ab\tan\theta|}{\sqrt{b^2 + a^2}}$$

$$= \frac{\left|b^2a^2\sec^2\theta + \cancel{b^2a^2\sec\theta\tan\theta} - \cancel{b^2a^2\tan\theta\sec\theta} - b^2a^2\tan^2\theta\right|}{\sqrt{b^2+a^2}\sqrt{b^2+a^2}}$$

$$= \frac{|b^2a^2(\sec^2\theta - \tan^2\theta)|}{b^2+a^2} = \frac{|b^2a^2(1 + \tan^2\theta - \tan^2\theta)|}{b^2+a^2}$$

$$= \frac{b^2a^2}{b^2+a^2} \quad \text{since } a, b > 0$$

Now $b^2 = a^2(e^2 - 1)$ $\therefore e^2 - 1 = \frac{b^2}{a^2}$ $\therefore e^2 = \frac{b^2}{a^2} + 1 = \frac{b^2+a^2}{a^2}$

$\therefore \frac{1}{e^2} = \frac{a^2}{b^2+a^2}$

$\therefore \frac{b^2a^2}{b^2+a^2} = b^2 \times \frac{1}{e^2} = \frac{b^2}{e^2}$

14 (a) (i) $y = \frac{1}{x}$ $\therefore \frac{dy}{dx} = -\frac{1}{x^2}$

At $x = x_1$, $m = -\frac{1}{x_1^2}$

$\therefore y - y_1 = -\frac{1}{x_1^2}(x - x_1)$

$\therefore x_1^2 y - x_1^2 y_1 = -x + x_1$

But $x_1 y_1 = 1$ $\therefore x_1^2 y - x_1 = -x + x_1$

$\therefore x_1^2 y + x = 2x_1$ $(\div x_1)$

$\therefore x_1 y + \frac{x}{x_1} = 2$

Now $\frac{1}{x_1} = y_1$

$\therefore x_1 y + x y_1 = 2$

ie $xy_1 + yx_1 = 2$

(ii) Tangent at P passes through T

$\therefore$ FACT: $x_0 y_1 + y_0 x_1 = 2$

Tangent at Q has eqn $xy_2 + yx_2 = 2$

It passes through T

$\therefore$ FACT: $x_0 y_2 + y_0 x_2 = 2$

Consider the <u>line</u> $xy_0 + yx_0 = 2$.

Does it pass through $P(x_1, y_1)$?

ie does $x_1 y_0 + y_1 x_0 = 2$? YES!

Does it pass through $Q(x_2, y_2)$?

ie does $x_2 y_0 + y_2 x_0 = 2$? YES!

$\therefore$ Since $xy_0 + yx_0 = 2$ is a <u>line</u> that passes through P and Q, the equation of PQ <u>is</u> $xy_0 + yx_0 = 2$

ie chord of contact from T is $xy_0 + yx_0 = 2$

(iii) x_1 & x_2 are x-coordinates of simultaneous eqns involving $xy_0 + yx_0 = 2$ & $y = \frac{1}{x}$

∴ $xy_0 + \frac{x_0}{x} = 2$

∴ $x^2y_0 + x_0 = 2x$

∴ $y_0x^2 - 2x + x_0 = 0$

(iv) $\frac{x_1+x_2}{2} = \frac{\frac{2}{y_0}}{2} = \frac{1}{y_0}$

∴ x coordinate of midpt PQ is $\frac{1}{y_0}$

When $x = \frac{1}{y_0}$, $\frac{1}{y_0}y_0 + yx_0 = 2$ ∴ $1 + yx_0 = 2$

∴ $yx_0 = 1$

∴ $y = \frac{1}{x_0}$

∴ y coordinate of midpt PQ is $\frac{1}{x_0}$

i.e. $R\left(\frac{1}{y_0}, \frac{1}{x_0}\right)$

(v) If $T(x_0, y_0)$ lies on $xy = c^2$ then

$x_0y_0 = c^2$

Does R lie on $xy = \frac{1}{c^2}$?

Subst: Does $\frac{1}{y_0}\times\frac{1}{x_0} = \frac{1}{c^2}$?

Does $\frac{1}{x_0y_0} = \frac{1}{c^2}$?

Does $\frac{1}{c^2} = \frac{1}{c^2}$? Yes!

∴ R moves on hyperbola $xy = \frac{1}{c^2}$

(b)(i) $a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot v = v\frac{dv}{dx}$

(ii) $-kv = v\frac{dv}{dx}$ ∴ $\frac{dv}{dx} = -k$

∴ $v = -kx + c$. when $x = 0, v = U$

∴ $U = c$ ∴ $v = U - kx$

(iii) $\frac{dt}{dv} = -\frac{1}{kv}$

∴ $t = -\frac{1}{k}\ln V + c$

When $t = 0, V = U$

∴ $0 = \frac{-\ln U}{k} + c$ ∴ $c = \frac{\ln U}{k}$

∴ $t = \frac{1}{k}\ln\left(\frac{U}{V}\right)$

∴ $e^{kt} = \frac{U}{V}$ ∴ $V = Ue^{-kt}$

(iv) As $t \to \infty$, $V \to 0$ (assuming $k > 0$)

As $V \to 0$, $U - kx \to 0$

ie $U \to kx$

ie $x \to \frac{U}{k}$

∴ As $t \to \infty$, $x \to \frac{U}{k}$

15 (a) Roots: $\frac{-6}{r}, -6, -6r$

Now $\frac{36}{r} + 36 + 36r = 156$

∴ $36 + 36r + 36r^2 = 156r$

$9 + 9r + 9r^2 = 39r$

∴ $9r^2 - 30r + 9 = 0$

$3r^2 - 10r + 3 = 0$

$(3r-1)(r-3) = 0$

∴ $r = \frac{1}{3}$ or 3

∴ Roots: $-18, -6, -2$

Sum $= -26$. Product $= -216$

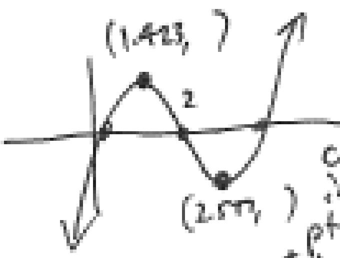
∴ $x^3 + 26x^2 + 156x + 216 = 0$

(b) If $y = x^3 - 6x^2 + 11x + a - 4$,

$\frac{dy}{dx} = 3x^2 - 12x + 11 = 0$ stat pts

∴ $x = \frac{12 \pm \sqrt{144 - 132}}{6} = \frac{12 \pm \sqrt{12}}{6}$

$= 2 \pm 0.577\ldots$



Since positive cubic with 3 distinct integer solns, 2 stat pts must be above & below x-axis. Thus $x = 2$ must be integer soln.

$\therefore P(2) = 0$

$\therefore 8 - 24 + 22 + a - 4 = 0$

$\therefore a = -2$

$\therefore x^3 - 6x^2 + 11x - 6 = 0$

$\therefore (x-2)(x^2 - 4x + 3) = 0$

$\therefore (x-2)(x-1)(x-3) = 0$

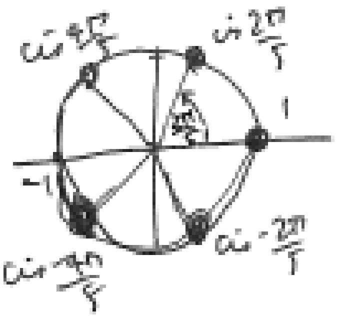
$\therefore a = -2$ & integer solns: 1, 2, 3

(c) (i) $z^5 = 1$

$\therefore z^5 = \operatorname{cis}(0 + 2k\pi)$, k integer

$\therefore z = \operatorname{cis} \frac{2k\pi}{5}$

$\therefore z = 1, \operatorname{cis}\frac{2\pi}{5}, \operatorname{cis}\frac{4\pi}{5}, \operatorname{cis}\frac{-2\pi}{5}, \operatorname{cis}\frac{-4\pi}{5}$



(ii) $\omega = \operatorname{cis}\frac{2\pi}{5}$

$\therefore \omega^2 = \operatorname{cis}\frac{4\pi}{5}$

$\omega^3 = \operatorname{cis}\frac{6\pi}{5} = \operatorname{cis}\left[-\left(2\pi - \frac{6\pi}{5}\right)\right] = \operatorname{cis}\left(-\frac{4\pi}{5}\right)$

$\omega^4 = \operatorname{cis}\frac{8\pi}{5} = \operatorname{cis}\left[-\left(2\pi - \frac{8\pi}{5}\right)\right]$

$= \operatorname{cis}\left(-\frac{2\pi}{5}\right)$

$\therefore 5^{th}$ roots of $1 = 1, \omega, \omega^2, \omega^3, \omega^4$

(iii) $z^5 = 1 \quad \therefore z^5 - 1 = 0$

$\therefore (z-1)(z^4 + z^3 + z^2 + z + 1) = 0$

But also

$(z-1)(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4) = 0.$

since the 5^{th} roots of 1 are defined in (ii)

$\therefore (z-1)(z^4+z^3+z^2+z+1) = (z-1)(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$

$\therefore z^4+z^3+z^2+z+1 = (z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$

Let $z=1 \quad \therefore \quad 5 = (1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$

(iv) LHS $= 1 - \omega - \omega^4 + \omega^5$

$= 1 - (\omega + \omega^4) + 1$

$= 2 - (\omega + \omega^4)$

Now $\omega + \omega^4 = \operatorname{cis}\frac{2\pi}{5} + \operatorname{cis}\frac{-2\pi}{5}$

$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5}$

$= 2\cos\frac{2\pi}{5}$

$\therefore$ LHS $= 2 - 2\cos\frac{2\pi}{5}$

(v) Similarly, $(1-\omega^2)(1-\omega^3) = 2 - (\omega^2 + \omega^3)$

$= 2 - 2\cos\frac{4\pi}{5}$

$\therefore \left(2 - 2\cos\frac{2\pi}{5}\right)\left(2 - 2\cos\frac{4\pi}{5}\right) = 5$

$\therefore \left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = \frac{5}{4}$

Now $\cos 2\theta = 1 - 2\sin^2\theta$

$\therefore -\cos 2\theta = 2\sin^2\theta - 1$

$\therefore 1 - \cos 2\theta = 2\sin^2\theta$

$\therefore \left(2\sin^2\frac{\pi}{5}\right)\left(2\sin^2\frac{2\pi}{5}\right) = \frac{5}{4}$

$\therefore \sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} = \frac{5}{16}$

$\therefore \sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$ (take +ve since $\frac{\pi}{5}, \frac{2\pi}{5}$ acute)

(16) (a) $P(2) = -5$, $P(1) = -2$

$P(x) = Q(x)(x-1)(x-2) + (ax + b)$

since remainder is linear if divisor is quadratic

$\therefore P(2) = -5 = 0 + 2a + b \quad \therefore 2a + b = -5$

$P(1) = -2 = 0 + a + b \quad \therefore a + b = -2$

$\therefore a = -3$

$b = 1$

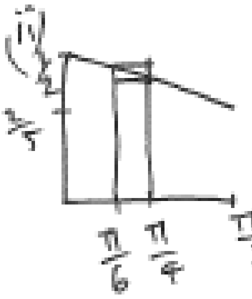
$\therefore$ Remainder is $-3x + 1$

(b) (i) $f'(x) = \frac{\cos x(3\sin x+2) - 3\cos x(\sin x+1)}{(3\sin x+2)^2}$

$= \frac{-\cos x}{(3\sin x+2)^2}$. For $0<x<\frac{\pi}{2}$, top -ve, bottom +ve

∴ $f'(x) < 0$ ∴ decreasing

(ii)



Area squeezed between 2 rectangles, width $\frac{\pi}{4}-\frac{\pi}{6} = \frac{\pi}{12}$

∴ $\frac{\pi}{12}f\left(\frac{\pi}{4}\right) < \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin x+1}{3\sin x+2}\,dx < \frac{\pi}{12}f\left(\frac{\pi}{6}\right)$

∴ $\frac{\pi}{12}\left(\frac{\frac{1}{\sqrt2}+1}{\frac{3}{\sqrt2}+2}\right) < \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin x+1}{3\sin x+2}\,dx < \frac{\pi}{12}\left(\frac{\frac{1}{2}+1}{\frac{3}{2}+2}\right)$

∴ $\frac{\pi}{12}\left(\frac{1+\sqrt2}{3+2\sqrt2}\right) <$ " $< \frac{\pi}{12}\left(\frac{1+2}{3+4}\right)$

∴ $\frac{\pi}{12}\frac{(1+\sqrt2)(3-2\sqrt2)}{9-8} <$ " $< \frac{3\pi}{84}$

∴ $\frac{(\sqrt2-1)\pi}{12} <$ " $< \frac{\pi}{28}$

(c) (i) Let $n=1$. LHS $= 1+\sqrt2$ ∴ $p_1 = 1$ & $q_1 = 1$ ∴ True for $n=1$

Let $n=k$ be an integer such that $(1+\sqrt2)^k = p_k + q_k\sqrt2$ where p_k & q_k are unique integers

Prove $n=k+1$ is an integer such that $(1+\sqrt2)^{k+1} = p_{k+1} + q_{k+1}\sqrt2$ where p_{k+1} & q_{k+1} are unique integers

Proof: LHS $= (1+\sqrt2)^k(1+\sqrt2) = (p_k + q_k\sqrt2)(1+\sqrt2)$
$= p_k + 2q_k + \sqrt2(p_k+q_k)$

Since p_k & q_k are unique integers, so are p_k+2q_k & p_k+q_k

∴ $= p_{k+1} + q_{k+1}\sqrt2$

∴ True by mathematical induction

(ii) Now prove $p_n - q_n\sqrt2 = (1-\sqrt2)^n$, with p_n & q_n same values found in (i)

For $n=1$, RHS $= 1-\sqrt2$, LHS $= p_1 - q_1\sqrt2$ ∴ $p_1 = 1$, $q_1 = 1$ which are same as in (i) ∴ True for $n=1$

Let $n=k$ be integer such that $(1-\sqrt2)^k = p_k - q_k\sqrt2$, p_k, q_k as in (i)

Prove $n=k+1$ is an integer such that $(1-\sqrt2)^{k+1} = p_{k+1} - q_{k+1}\sqrt2$, p_{k+1}, q_{k+1} as in (i)

Proof: LHS $= (1-\sqrt2)(p_k - q_k\sqrt2) = p_k + 2q_k - \sqrt2(p_k+q_k) = p_{k+1} - \sqrt2 q_{k+1}$ which are same as in (i)

∴ True by mathematical induction.

∴ LHS $= (p_n - \sqrt2 q_n)(p_n + \sqrt2 q_n)$
$= (1-\sqrt2)^n(1+\sqrt2)^n$
$= ((1-\sqrt2)(1+\sqrt2))^n = (1-2)^n$
$= (-1)^n =$ RHS

(d) (i) Let $x = \sin^{-1}y + \cos^{-1}y$

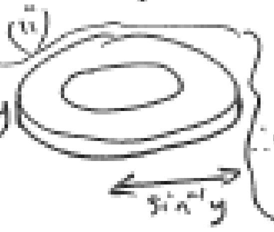
∴ $\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}} - \frac{1}{\sqrt{1-y^2}} = 0$

∴ $x = \sin^{-1}y + \cos^{-1}y$ always has the same value for $-1 \le y \le 1$

∴ Substitute $y = 1$ ∴ $x = \sin^{-1}1 + \cos^{-1}1 = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

∴ $\sin^{-1}y + \cos^{-1}y = \frac{\pi}{2}$

(ii)



∴ $\Delta V = \pi\left((\sin^{-1}y)^2 - (\cos^{-1}y)^2\right)dy$

∴ $V = \int_{\frac{1}{\sqrt2}}^{1} \pi(\sin^{-1}y - \cos^{-1}y)(\sin^{-1}y + \cos^{-1}y)\,dy$

$= \frac{\pi^2}{2}\int_{\frac{1}{\sqrt2}}^{1} \sin^{-1}y - \cos^{-1}y\,dy$

Now $\sin x = \cos x$ has soln $x = \frac{\pi}{4}$ & $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt2}$

$= \frac{\pi^2}{2}\int_{\frac{1}{\sqrt2}}^{1} \sin^{-1}y - \left(\frac{\pi}{2} - \sin^{-1}y\right)dy$

$= \frac{\pi^2}{2}\int_{\frac{1}{\sqrt2}}^{1} 2\sin^{-1}y - \frac{\pi}{2}\,dy$

$= \pi^2\int_{\frac{1}{\sqrt2}}^{1} \sin^{-1}y\,dy - \frac{\pi^3}{4}\int_{\frac{1}{\sqrt2}}^{1} 1\,dy$

$= \pi^2\left[\left[y\sin^{-1}y\right]_{\frac{1}{\sqrt2}}^{1} - \int_{\frac{1}{\sqrt2}}^{1}\frac{y\,dy}{\sqrt{1-y^2}}\right] - \frac{\pi^3}{4}\left[y\right]_{\frac{1}{\sqrt2}}^{1}$

$= \pi^2\left[\frac{\pi}{2} - \frac{1}{\sqrt2}\frac{\pi}{4}\right] + \frac{\pi^2}{2}\int_{\frac{1}{\sqrt2}}^{1}\frac{-2y\,dy}{\sqrt{1-y^2}} - \frac{\pi^3}{4}\left[1 - \frac{1}{\sqrt2}\right]$

$= \frac{\pi^3}{2} - \frac{\pi^3}{4\sqrt2} + \left[\frac{\pi^2}{2}\frac{\times 2}{1}\sqrt{1-y^2}\right]_{\frac{1}{\sqrt2}}^{1} - \frac{\pi^3}{4} + \frac{\pi^3}{4\sqrt2}$

$= \frac{\pi^3}{2} - \frac{\pi^3}{4\sqrt2} - \pi^2\sqrt{\frac{1}{2}} - \frac{\pi^3}{4} + \frac{\pi^3}{4\sqrt2}$

$= \frac{\pi^3}{4} - \frac{\pi^2}{\sqrt2}\ u^3$