
Student Number

Barker College

Mathematics Extension 2

2015
TRIAL
HIGHER SCHOOL CERTIFICATE

AM Friday 31 July

Section I – Multiple Choice

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

(A) ○ (B) ○ (C) ○ (D) ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) ● (B) ~~●~~ (C) ○ (D) ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

correct

(A) ● (B) ~~●~~ (C) ○ (D) ○

Start Here

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

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Student Number

Barker College

125
1890-2015

2015
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Staff Involved:

- **BHC***
- **RMH***
- **KJL**
- **MRB**

AM Friday 31 July

Number of copies: 50

General Instructions
- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on page 2
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I **Pages 2 - 6**

10 marks
- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II **Pages 7 - 14**

90 marks
- Attempt Questions 11 - 16
- Allow about 2 hours 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I — Multiple Choice

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 - 10.

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1. The eccentricity of the ellipse with equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ is

(A) $\dfrac{9}{25}$

(B) $\dfrac{4}{5}$

(C) $\dfrac{3}{5}$

(D) $\dfrac{\sqrt{41}}{5}$

2. The equation of degree 10 whose roots are the reciprocals of the equation $x^{10} - 5x^3 + x - 4 = 0$ is

(A) $4x^{10} - x^9 + 5x^7 - 1 = 0$

(B) $1 - 5x^7 - x^9 - 4x^{10} = 0$

(C) $4x^{10} + x^9 + 5x^7 - 1 = 0$

(D) $4x^{10} - x^9 - 5x^7 + 1 = 0$

3. $P(x) = x^4 + 2x^3 + 9x^2 + 8x + 20$ has a zero $x = 2i - 1$.

What is the value of $P\left(\overline{2i-1}\right)$?

(A) $-10 + 12i$

(B) $-10 - 12i$

(C) $10 - 12i$

(D) 0

4. $\int \tan^4 x \; dx$ equals

(A) $\frac{1}{3}\tan^3 x + \tan x + x + c$

(B) $\frac{1}{3}\tan^3 x - \tan x + x + c$

(C) $\frac{1}{3}\tan^3 x - \tan x - x + c$

(D) $\frac{1}{3}\tan^3 x + \tan x - x + c$

5. If w is one of the complex roots of $z^3 = 1$, what is the value of

$(1 - w)(1 - w^2)(1 - w^4)(1 - w^8)$?

(A) 9

(B) 6

(C) 3

(D) 0

6. The vertices of a conic are (2, 0) and (-2, 0) and the foci are (3, 0) and (-3, 0). What is the equation of this conic?

(A) $\frac{x^2}{4} + \frac{y^2}{5} = 1$

(B) $\frac{x^2}{4} - \frac{y^2}{5} = 1$

(C) $\frac{x^2}{4} - \frac{y^2}{\sqrt{5}} = 1$

(D) $\frac{x^2}{4} + \frac{y^2}{\sqrt{5}} = 1$

7. $\int x\, e^{-2x}\, dx$ equals

(A) $\frac{x}{2} e^{-2x} - \frac{1}{2}\int e^{-2x}\, dx$

(B) $\frac{x}{2} e^{-2x} + \frac{1}{2}\int e^{-2x}\, dx$

(C) $-\frac{x}{2} e^{-2x} - \frac{1}{2}\int e^{-2x}\, dx$

(D) $-\frac{x}{2} e^{-2x} + \frac{1}{2}\int e^{-2x}\, dx$

8. The polynomial $ax^8 + bx^7 + 2$ is divisible by $(x+1)^2$.

What are the values of a and b?

(A) $a = -14$ and $b = -16$

(B) $a = 14$ and $b = -16$

(C) $a = 14$ and $b = 16$

(D) $a = -14$ and $b = 16$

9. A unit circle has its centre at the origin O.

The point z_1 moves on the circle and $z_2 = \dfrac{\sqrt{2} - 3i}{z_1}$.

The Cartesian equation of the locus of z_2 is:

(A) $x^2 - y^2 = 11$

(B) $x^2 + y^2 = 11$

(C) $x^2 + y^2 = \sqrt{11}$

(D) $x^2 - y^2 = \sqrt{11}$

10. All the solutions of the equation $z^4 = (z - 1)^4$ are

(A) $z = \frac{1}{2}, \quad z = \frac{i}{-1 + i}, \quad z = \frac{i}{1 + i}$

(B) $z = \frac{1}{2}, \quad z = \frac{-i}{1 + i}, \quad z = \frac{i}{-1 + i}$

(C) $z = \frac{1}{2}, \quad z = \frac{2i}{-1 + i}, \quad z = \frac{2i}{1 + i}$

(D) $z = \frac{1}{2}, \quad z = \frac{2i}{1 + i}, \quad z = \frac{-2i}{1 + i}$

End of Section I

Section II

90 marks
Attempt Questions 11 - 16.
Allow about 2 hours 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Show relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) (i) Find the square root of the complex number $-5+12i$. **2**
Express your answer in the form $a + ib$.

(ii) Hence, or otherwise, solve $2z^2 - (6+i)z + 5 = 0$ for z. **2**
Express your answer in the form $a + ib$.

(b) Prove that $|z_1 - z_2|^2 = z_1\overline{z_1} + z_2\overline{z_2} - 2\,\mathrm{Re}\left(z_1\overline{z_2}\right)$. **2**

(c) (i) Write down the five roots of $z^5 = -1$. **2**

(ii) Hence show that

$$z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$$ **3**

(d) (i) If $z = \cos\theta + i\sin\theta$, show that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$ **1**

(ii) Hence show that

$$\sin^5\theta = \frac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right]$$ **3**

End of Question 11

Question 12 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) Use the substitution $5x - 1 = u^2$ to show that

$$\int x\sqrt{5x-1}\ dx = \frac{2(5x-1)(15x+2)\sqrt{5x-1}}{375} + C \qquad \mathbf{3}$$

(b) Use an appropriate 't-result' substitution to evaluate

$$\int_0^{\frac{\pi}{4}} \frac{d\theta}{2+\sin 2\theta} \qquad \mathbf{3}$$

(c) (i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **1**

(ii) Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}\ dx}{\sqrt{\sin x} + \sqrt{\cos x}}$. **2**

(d) (i) Let $I_n = \displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^n x\ dx$ and show that $I_n = \dfrac{-\sqrt{3}}{n2^n} + \left(\dfrac{n-1}{n}\right) I_{n-2}$ **3**

(ii) Hence find $\displaystyle\int_0^{\frac{1}{2}} \frac{x^3\ dx}{\sqrt{1-x^2}}$ by letting $x = \cos\theta$. **3**

End of Question 12

Question 13 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) The equation $2x^3 - 5x^2 + 8x - 3 = 0$ has roots α, β and γ. **2**
Find α, β and γ, given that one of the roots is $\alpha = 1 - \sqrt{2}\,i$.

(b) (i) Prove that if the polynomial $P(x)$ has a root of multiplicity m at $x = k$ then $P'(x)$ has a root of multiplicity $(m - 1)$ at $x = k$. **2**

(ii) If $P(x) = 4x^3 + 15x^2 + 12x - 4$ has a double zero, find all the zeros and factorise $P(x)$ fully over the real numbers. **3**

(c) The equation $x^4 + 4x^3 - 3x^2 - 4x + 2 = 0$ has roots α, β, γ and δ. **4**
Find the equation with roots α^2, β^2, γ^2 and δ^2.

(d) One of the roots of the equation $x^3 + ax^2 + bx + c = 0$ is double the sum of the other two roots. Show that $4a^3 - 18ab + 27c = 0$. **4**

End of Question 13

Question 14 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a) (i) For the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$, find the equation of the asymptote with positive gradient. **1**

(ii) If $P(x_1, y_1)$ is a point on the hyperbola in the first quadrant, show that the equation of the tangent at P is $9xx_1 - 4yy_1 = 36$. **2**

(iii) If this tangent passes through (1, 0) find the coordinates of P. **1**

(iv) Let α be the angle between the tangent at P and the asymptote with positive gradient. Draw a diagram indicating the location of α and show that **3**

$\alpha = \tan^{-1}\left[\dfrac{24 - 13\sqrt{3}}{23}\right]$.

(b) (i) Show that the equation **2**

$4x^2 - y^2 - 24hx + 2hy - 4a^2 + 35h^2 = 0$, where h and a are positive constants, represents a hyperbola.

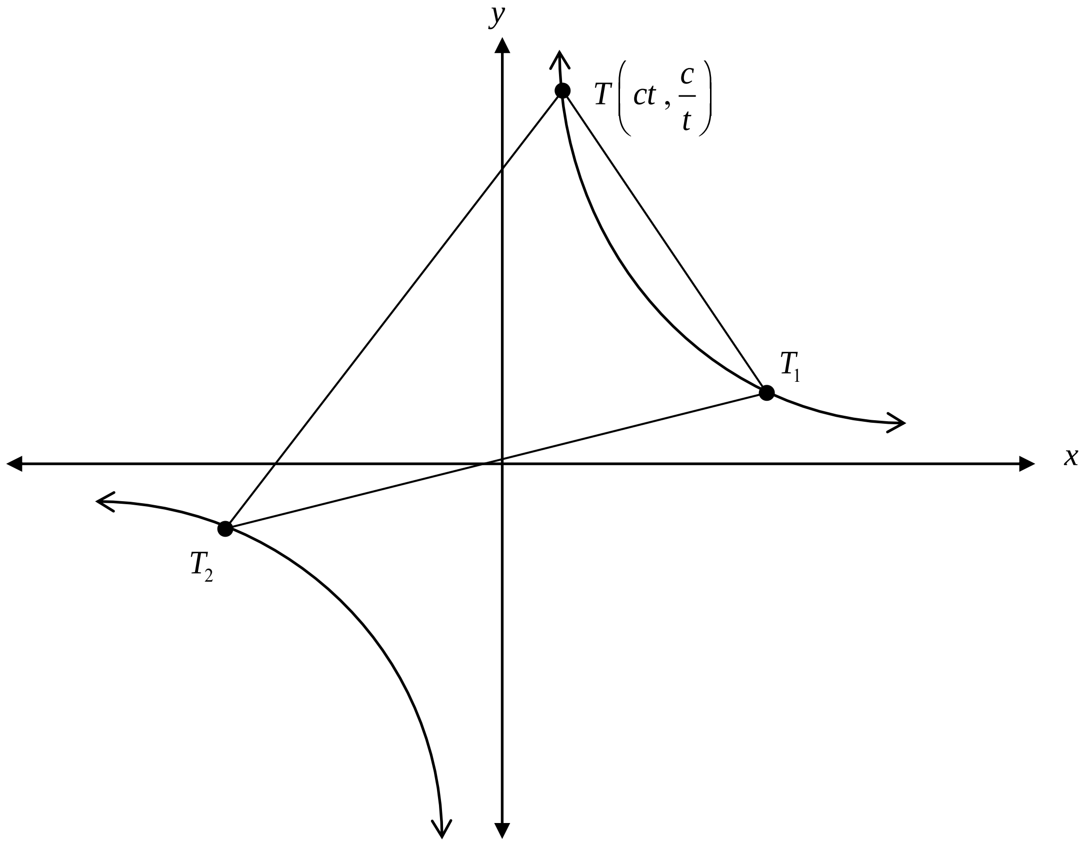
(ii) If the tangent to this hyperbola at the point (p, q) is perpendicular to the straight line $y = (e^2 - 1)x$, where e is the eccentricity of the hyperbola, show that $16p + q = 49h$. **3**

Question 14 continues on page 11

Question 14 (continued)

(c) The diagram below shows two points, T_1 and T_2, with parameters t_1 and t_2 respectively, on the rectangular hyperbola $xy = c^2$.

T is a third point with parameter t on the hyperbola such that $\angle T_1 T T_2$ is a right angle.



Show that the gradient of $T_1 T$ is $\dfrac{-1}{t_1 t}$ and deduce that, since $\angle T_1 T T_2$ is a right-angle, **3**

$$t^2 = \frac{-1}{t_1 t_2}.$$

End of Question 14

Question 15 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

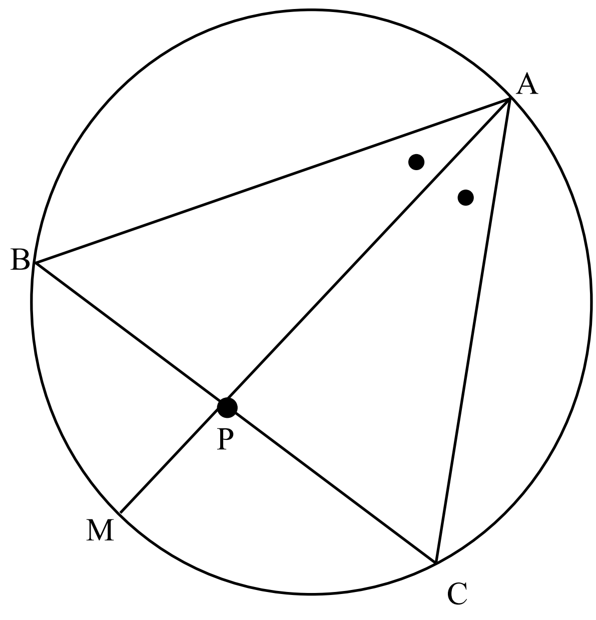
(a) Using the method of cylindrical shells, find the volume obtained by revolving the region bounded by the parabola $y = x^2 + 2$, the x-axis, the y-axis and the line $x = 2$, about the line $x = 2$. **4**

(b) The base of a solid is the ellipse $4x^2 + 25y^2 = 100$. **3**

All cross-sections perpendicular to the x-axis are isosceles right triangles (with the smallest side lying in the base).

Find the volume of the solid.

(c) In the figure below, the bisector AP of $\angle BAC$ is extended to meet the circle in M.



(i) Prove that $\triangle ABM \parallel\!| \triangle APC$. **2**

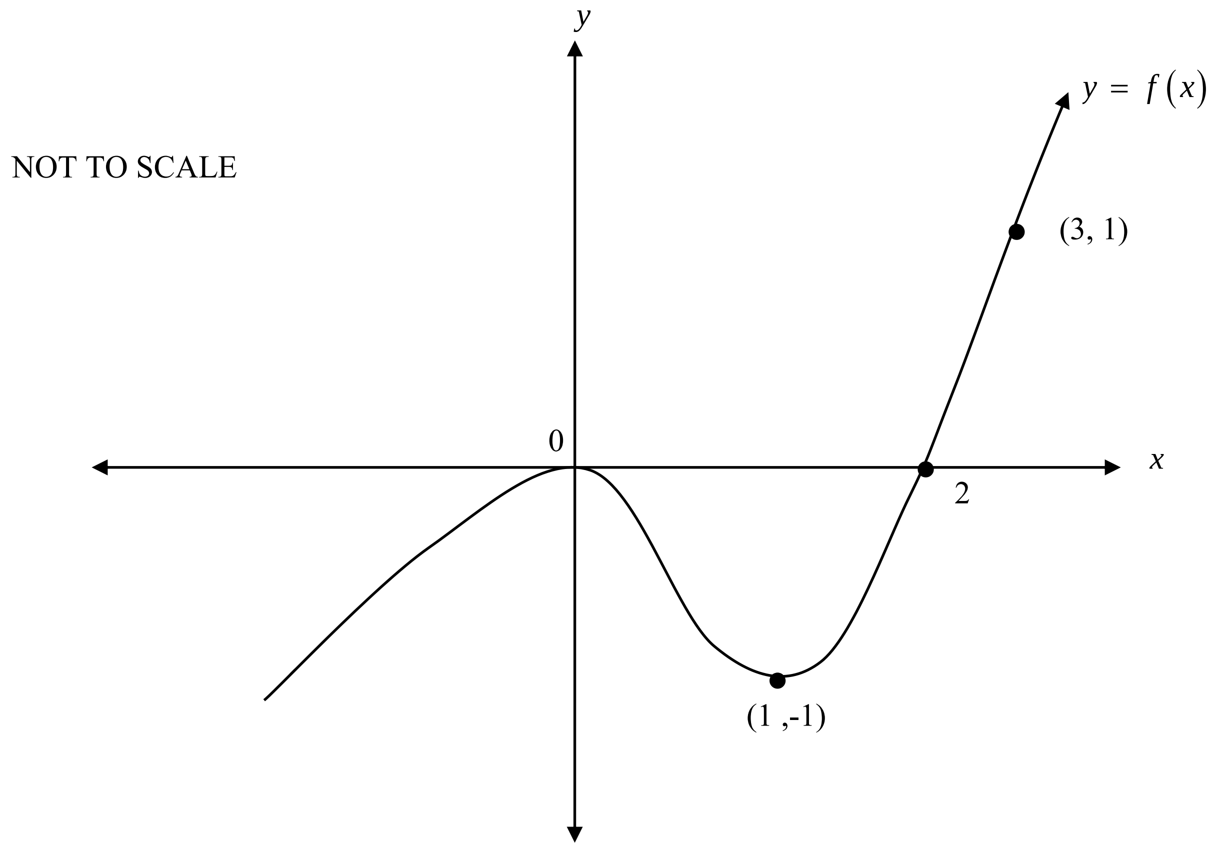
(ii) Prove that $BP \times PC = PM \times PA$. **2**

(d) Show by mathematical induction that $35^n + 3 \times 7^n + 2 \times 5^n + 6$ is divisible by 12 for all integers $n \geq 0$. **4**

End of Question 15

Question 16 (15 marks) **[Use a SEPARATE writing booklet]** **Marks**

(a)



The diagram shows the graph of $y = f(x)$.

Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(|x|)|$ **2**

(ii) $y = \ln(f(x))$ **2**

Question 16 continues on page 14

Question 16 (continued)

(b) A spherical planet of mass m_1 and radius R has a rocket launched from its surface with an initial speed of V. The mass of the rocket is m_2 and the distance between the rocket and the centre of the planet is x. The gravitational force, F, acting on the rocket is given by

$$F = \frac{G\, m_1\, m_2}{x^2}$$

where G is the constant of gravitation on that planet.
Assume that there are no other forces acting on the rocket.

(i) Write down an expression for the acceleration of the rocket in terms of x, taking the positive direction as away from the surface of the planet. **1**

(ii) Find an expression for the velocity v of the rocket in terms of x. **3**

(c) The rise and fall of a tide approximates simple harmonic motion. **4**
In a harbour, low tide is at 7am and high tide is at 1:40pm.
The corresponding depths are 20m and 40m.
Find the first time after 7am that a ship which requires $\left(5\sqrt{3} + 30\right)$ metres of water is able to enter the harbour.

(d) When a polynomial $P(x)$ is divided by $x^2 - a^2$, where $a \neq 0$, the remainder is of the form $px + q$ where **3**

$$p = \frac{1}{2a}\left[P(a) - P(-a)\right] \text{ and } q = \frac{1}{2}\left[P(a) + P(-a)\right].$$

(You do not have to prove these results.)

Find the remainder when $P(x) = x^n - a^n$, for n a positive integer, is divided by $x^2 - a^2$.

End of Paper

Extension 2 Mathematics 31.7.15 Solutions

Multiple Choice

Q1. $\frac{x^2}{25}+\frac{y^2}{16}=1$

$b^2=a^2(1-e^2)$

$16=25(1-e^2)$

$\frac{16}{25}=1-e^2$

$e^2=\frac{9}{25}$

$e=\frac{3}{5}$ (C)

Q2. The required equation is

$\left(\frac{1}{x}\right)^{10}-5\left(\frac{1}{x}\right)^3+\frac{1}{x}-4=0$

$1-5x^7+x^9-4x^{10}=0$

$4x^{10}-x^9+5x^7-1=0$ (A)

Q3. P(x) has real coefficients so complex roots occur in conjugate pairs. (D)

Q4. $\int \tan^4 x\,dx$

$=\int (\sec^2 x-1).\tan^2 x\,dx$

$=\int \tan^2 x.\sec^2 x-\tan^2 x\,dx$

$=\frac{1}{3}\tan^3 x-\int(\sec^2 x-1)\,dx$

$=\frac{1}{3}\tan^3 x-\tan x+x+c$ (B)

Q5. $(1-w)(1-w^2)(1-w^4)(1-w^8)$

$=(1-w)(1-w^2)(1-w)(1-w^2)$

$=[(1-w)(1-w^2)]^2$

$=[1-w-w^2+w^3]^2$

$=[2-w-w^2]^2$

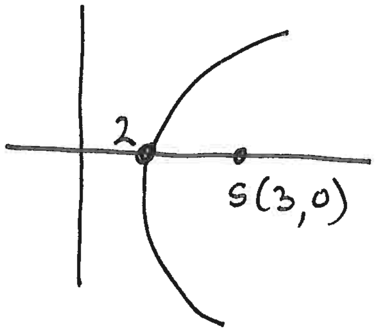
but $1+w+w^2=0$ so

$-w-w^2=1$

$=[2+1]^2$

$=9$ (A)

Q6.



It is a hyperbola.

Hence $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

And when $y=0$, $x=2$

$\Rightarrow a = 2$

Also $ae = 3$

$2e = 3$

$e = 3/2$

$b^2 = a^2(e^2-1)$

$b^2 = 4\left(\frac{9}{4} - 1\right)$

$b^2 = 5$

Hence $\frac{x^2}{4} - \frac{y^2}{5} = 1$ (B)

Q7. $\int x e^{-2x}\,dx$

$= \frac{-1}{2} e^{-2x} . x - \int -\frac{1}{2} e^{-2x} . (1)\,dx$

$= -\frac{x}{2} . e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$

$= -\frac{x}{2} . e^{-2x} + \frac{1}{2}\left[-\frac{1}{2} e^{-2x}\right]$

$= -\frac{x}{2} e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$ (B)

Q8. $P(x) = ax^8 + bx^7 + 2$

$P'(x) = 8ax^7 + 7bx^6$

$P(-1) = 0 \Rightarrow a - b + 2 = 0$

$P'(-1) = 0 \Rightarrow -8a + 7b = 0$

$8a - 8b = -16$

$\therefore b = 16$

$\therefore a = 14$ (C)

Q9. $|z_1| = 1$

$z_1 \times z_2 = \sqrt{2} - 3i$

$|z_1 z_2| = |\sqrt{2} - 3i| = \sqrt{11}$

$|z_1| \times |z_2| = 11 \quad \therefore |z_2| = \sqrt{11}$

Let $z_2 = x + iy$

then $\sqrt{x^2+y^2} = \sqrt{11}$

ie, $x^2 + y^2 = 11$ (B)

Q10 //

$$z^4 = (z-1)^4$$

$$\left(\frac{z}{z-1}\right)^4 = 1$$

Hence $\frac{z}{z-1} = 1, -1, i, -i$

If $\frac{z}{z-1} = 1$ then $z = z - 1$ (no solutions)

If $\frac{z}{z-1} = -1$ then $z = 1 - z \Rightarrow z = \frac{1}{2}$

If $\frac{z}{z-1} = i$ then $z = iz - i$

$$z(1-i) = -i$$

$$z = \frac{-i}{1-i} = \frac{i}{-1+i}$$

If $\frac{z}{z-1} = -i$ then $z = -iz + i$

$$z(1+i) = i$$

$$z = \frac{i}{1+i}$$

Hence A

Q11. (a)

(i) Let $(x+iy)^2 = -5+12i$

then $x^2 - y^2 = -5$ and $2xyi = 12i$

$$y = \frac{6}{x}$$

Hence $x^2 - \left(\frac{6}{x}\right)^2 = -5$

$$x^4 + 5x^2 - 36 = 0$$

$$(x^2+9)(x^2-4) = 0$$

$$x = \pm 2$$

When $x = 2$, $y = 3$

When $x = -2$, $y = -3$

Hence the square roots of $-5+12i$ are

$2+3i$ and $-2-3i$

(ii) $2z^2 - (6+i)z + 5 = 0$

$$z = \frac{6+i \pm \sqrt{(6+i)^2 - 4\times 2\times 5}}{4}$$

$$z = \frac{6+i \pm \sqrt{-5+12i}}{4}$$

$$z = \frac{6+i \pm (2+3i)}{4}$$

$z = 2+i$ and $1 - \frac{1}{2}i$

(b) RTP that

$$|z_1 - z_2|^2 = z_1\overline{z_1} + z_2\overline{z_2} - 2\,\mathrm{Re}(z_1\overline{z_2})$$

Let $z_1 = a + bi$ and $z_2 = c + di$

$$LHS = |z_1 - z_2|^2$$

$$= |(a+bi) - (c+di)|^2$$

$$= |(a-c) + i(b-d)|^2$$

$$= \left(\sqrt{(a-c)^2 + (b-d)^2}\right)^2$$

$$= (a-c)^2 + (b-d)^2$$

$$RHS = z_1\overline{z_1} + z_2\overline{z_2} - 2\,\mathrm{Re}(z_1\overline{z_2})$$

$$= (a+bi)(a-bi) + (c+di)(c-di) - 2\,\mathrm{Re}\left[(a+bi)(c-di)\right]$$

$$= a^2 + b^2 + c^2 + d^2 - 2\,\mathrm{Re}\left[ac + bd + bci - adi\right]$$

$$= a^2 + b^2 + c^2 + d^2 - 2ac - 2bd$$

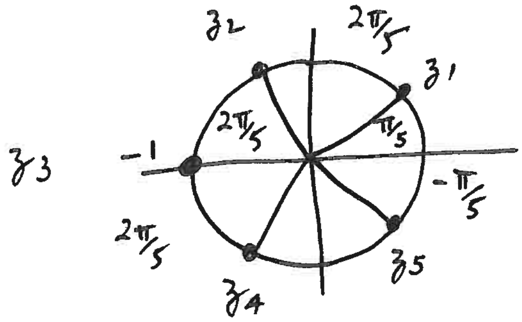
$$= (a-c)^2 + (b-d)^2$$

Hence $LHS = RHS$ as required.

(c) (i) $z^5 = -1$

$z = -1$ is one solution

The other roots are equally spaced around a unit circle.



The five roots are

$$-1,\ cis\left(\frac{\pi}{5}\right),\ cis\left(\frac{-\pi}{5}\right),\ cis\left(\frac{3\pi}{5}\right),\ cis\left(\frac{-3\pi}{5}\right).$$

(ii) Now

$$z^5+1 = (z+1)\left(z - cis\left(\frac{\pi}{5}\right)\right)\left(z - cis\left(\frac{-\pi}{5}\right)\right) \times \left(z - cis\left(\frac{3\pi}{5}\right)\right)\left(z - cis\left(\frac{-3\pi}{5}\right)\right)$$

$$= (z+1)\left(z - \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right)\left(z - \left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)\right) \times \left(z - \left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)\right)\left(z - \left(\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}\right)\right)$$

$$= (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + \cos^2\frac{\pi}{5} + \sin^2\frac{\pi}{5}\right) \times \left(z^2 - 2z\cos\frac{3\pi}{5} + \cos^2\frac{3\pi}{5} + \sin^2\frac{3\pi}{5}\right)$$

$$= (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$$

as required.

(d) (i) Let $z = \cos\theta + i\sin\theta$

then $z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$, by de Moivre's Theorem.

Also $z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$, by de Moivre.

$= \cos n\theta - i\sin n\theta$

Hence $z^n - \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$ as required.

(ii) RTP that $\sin^5\theta = \frac{1}{16}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

Now $z - \frac{1}{z} = 2i\sin\theta$

Hence $(2i\sin\theta)^5 = (z - \frac{1}{z})^5$

$32i\sin^5\theta = z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right)$
$+ 5z\left(\frac{1}{z^4}\right) - \left(\frac{1}{z^5}\right)$

$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$

$= 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$= 2i[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

Hence $\sin^5\theta = \frac{1}{16}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

as required.

Q12.

(a) $\int x\sqrt{5x-1}\ dx$

Let $u^2 = 5x-1 \Rightarrow x = \frac{u^2+1}{5}$

$2u\frac{du}{dx} = 5$

$\frac{2u}{5}du = dx$

$I = \int \frac{u^2+1}{5} \cdot \sqrt{u^2} \cdot \frac{2u}{5}\ du$

$= \frac{2}{25}\int u^2(u^2+1)\ du$

$= \frac{2}{25}\int u^4 + u^2\ du$

$= \frac{2}{25}\left[\frac{u^5}{5} + \frac{u^3}{3}\right] + C$

$= \frac{2u^3}{25}\left[\frac{3u^2+5}{15}\right] + C$

$= \frac{2}{25} \cdot (5x-1) \cdot \sqrt{5x-1} \cdot \left(\frac{3(5x-1)+5}{15}\right) + C$

$= \frac{2(5x-1)\sqrt{5x-1}(15x+2)}{375} + C$

$= \frac{2(5x-1)(15x+2)\sqrt{5x-1}}{375} + C$, as required.

(b) $\displaystyle\int_0^{\pi/4} \frac{d\theta}{2+\sin 2\theta}$

Let $t = \tan\theta$

$\frac{dt}{d\theta} = \sec^2\theta$

$\frac{dt}{d\theta} = 1+t^2$

$d\theta = \frac{dt}{1+t^2}$

when $\theta = \frac{\pi}{4}$, $t=1$

when $\theta = 0$, $t = 0$

$$I = \int_0^1 \frac{dt/1+t^2}{2+2t/1+t^2}$$

$$= \int_0^1 \frac{dt}{2(1+t^2)+2t}$$

$$= \int_0^1 \frac{dt}{2(t^2+t+1)}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{(\frac{\sqrt{3}}{2})^2+(t+\frac{1}{2})^2}$$

$$= \frac{1}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\sqrt{3}/2}\right)\right]_0^1$$

$$= \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} \times \frac{\pi}{3} - \frac{1}{\sqrt{3}} \times \frac{\pi}{6}$$

$$= \frac{1}{\sqrt{3}} \times \frac{\pi}{6}$$

$$= \frac{\pi}{6\sqrt{3}} = \frac{\sqrt{3}\pi}{18}$$

(c) (i) RTP that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Let $I = \int_0^a f(a-x)\,dx$

and let $u = a - x$

then $du = -dx$

so $dx = -du$

when $x = a$, $u = 0$

when $x = 0$, $u = a$

Hence $I = \int_a^0 f(u).(-du)$

$= -\int_a^0 f(u)\,du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

because u is a dummy variable.

(ii) $$\int_0^{\pi/2} \frac{\sqrt{\sin x}\ dx}{\sqrt{\sin x} + \sqrt{\cos x}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}\ dx}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}\ dx}{\sqrt{\cos x} + \sqrt{\sin x}}$$

Hence $$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}\ dx}{\sqrt{\sin x} + \sqrt{\cos x}}$$

$$= \int_0^{\pi/2} 1\ dx$$

$$= \frac{\pi}{2}$$

Hence $$I = \frac{\pi}{4}$$

(d) (i) $I_n = \int_{\pi/3}^{\pi/2} \cos^n x \, dx$

$$= \int_{\pi/3}^{\pi/2} \cos x . \cos^{n-1} x \, dx$$

$$= \left[\sin x . \cos^{n-1} x \right]_{\pi/3}^{\pi/2}$$

$$- \int_{\pi/3}^{\pi/2} \sin x . (n-1) \cos^{n-2} x . (-\sin x) \, dx$$

$$= - \sin \frac{\pi}{3} \times \left(\cos \frac{\pi}{3}\right)^{n-1} + \int_{\pi/3}^{\pi/2} \sin^2 x . (n-1) \cos^{n-2} x \, dx$$

$$= - \frac{\sqrt{3}}{2} . \left(\frac{1}{2}\right)^{n-1} + (n-1) \int_{\pi/3}^{\pi/2} (1 - \cos^2 x) . \cos^{n-2} x \, dx$$

$$= \frac{-\sqrt{3}}{2^n} + (n-1) \int_{\pi/3}^{\pi/2} \cos^{n-2} x \, dx - (n-1) \int_{\pi/3}^{\pi/2} \cos^n x \, dx$$

$$= \frac{-\sqrt{3}}{2^n} + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore \quad (n-1) I_n + I_n = \frac{-\sqrt{3}}{2^n} + (n-1) I_{n-2}$$

$$n I_n = - \frac{\sqrt{3}}{2^n} + (n-1) I_{n-2}$$

$$\therefore \quad I_n = \frac{-\sqrt{3}}{n 2^n} + \left(\frac{n-1}{n}\right) I_{n-2}$$

(ii) $\displaystyle\int_0^{1/2} \frac{x^3\,dx}{\sqrt{1-x^2}}$

Let $x = \cos\theta$ when $x = \frac{1}{2}$, $\theta = \frac{\pi}{3}$

$dx = -\sin\theta\,d\theta$ when $x = 0$, $\theta = \frac{\pi}{2}$

Hence $$I = \int_{\pi/2}^{\pi/3} \frac{\cos^3\theta.(-\sin\theta\,d\theta)}{\sqrt{1-\cos^2\theta}}$$

$$= \int_{\pi/3}^{\pi/2} \cos^3\theta \quad d\theta \quad = I_3$$

And $I_3 = \dfrac{-\sqrt{3}}{3\times 2^3} + \dfrac{2}{3} I_1$ (from part (i))

And $$I_1 = \int_{\pi/3}^{\pi/2} \cos x \; dx$$

$$= \left[\sin x\right]_{\pi/3}^{\pi/2}$$

$$= 1 - \frac{\sqrt{3}}{2}$$

Hence $$I_3 = \frac{-\sqrt{3}}{24} + \frac{2}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{-\sqrt{3}}{24} + \frac{16}{24} - \frac{8\sqrt{3}}{24} = \frac{16 - 9\sqrt{3}}{24}$$

QB.// (a)

$2x^3 - 5x^2 + 8x - 3 = 0$ has $\alpha = 1 - \sqrt{2}i$

Hence $\beta = 1 + \sqrt{2}i$ since complex roots occur in conjugate pairs when all coefficients are real.

But $\Sigma(\alpha) = -\frac{b}{a} = \frac{5}{2}$

Hence $(1 - \sqrt{2}i) + (1 + \sqrt{2}i) + \gamma = \frac{5}{2}$

$\gamma = \frac{1}{2}$

(b) (i) Let $P(x) = (x-k)^m . Q(x)$

then $P'(x) = (x-k)^m \times Q'(x) + Q(x) \times m(x-k)^{m-1}$

$= (x-k)^{m-1}\left[(x-k).Q'(x) + mQ(x)\right]$

$\therefore \quad P'(k) = P(k) = 0$

$\therefore$ $P'(x)$ has a root of multiplicity $(m-1)$

(b) (ii) $P(x) = 4x^3 + 15x^2 + 12x - 4$

$$P'(x) = 12x^2 + 30x + 12$$
$$= 6(2x^2 + 5x + 2)$$
$$= 6(2x+1)(x+2)$$

$\therefore P'(x) = 0$ when $x = -\frac{1}{2}$ and $x = -2$

But $P(-\frac{1}{2}) = -\frac{4}{8} + \frac{15}{4} - 6 - 4 \neq 0$

and $P(-2) = -4 \times 8 + 60 - 24 - 4$
$$= -32 + 60 - 28$$
$$= 0$$

Hence $x = -2$ is the double zero.

Hence $P(x) = (x+2)^2(ax+b)$

but $a = 4$ and $b = -1$ by inspection.

Hence $P(x) = (x+2)^2(4x-1)$

and the zeros are $-2, -2, \frac{1}{4}$.

(c) The Required equation is

$$(\sqrt{x})^4 + 4(\sqrt{x})^3 - 3(\sqrt{x})^2 - 4(\sqrt{x}) + 2 = 0$$

$$x^2 + 4x\sqrt{x} - 3x - 4\sqrt{x} + 2 = 0$$

$$x^2 - 3x + 2 = 4\sqrt{x}(1-x)$$

$$(x^2-3x+2)(x^2-3x+2) = 16x(1-x)^2$$

$$x^4 - 3x^3 + 2x^2 - 3x^3 + 9x^2 - 6x + 2x^2 - 6x + 4 = 16x - 32x^2 + 16x^3$$

$$x^4 - 6x^3 + 13x^2 - 12x + 4 = 16x - 32x^2 + 16x^3$$

ie,

$$x^4 - 22x^3 + 45x^2 - 28x + 4 = 0$$

(d) Let the roots of $x^3 + ax^2 + bx + c = 0$ be α, β and γ where $\alpha = 2(\beta + \gamma)$.

Now the sum of the roots is

$$\alpha + \beta + \gamma = -a$$

ie,

$$\alpha + \frac{\alpha}{2} = -a$$

$$\frac{3\alpha}{2} = -a$$

$$\alpha = \frac{-2a}{3}$$

And the sum of the product of the roots 2 at a time is

$$\alpha\beta + \beta\gamma + \alpha\gamma = b$$

$$2(\beta+\gamma).\beta + \beta\gamma + 2(\beta+\gamma).\gamma = b$$

$$(\beta+\gamma)(2\beta+2\gamma) + \beta\gamma = b$$

$$2(\beta+\gamma)(\beta+\gamma) + \beta\gamma = b$$

$$2\left(\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}\right) + \beta\gamma = b$$

$$\beta\gamma = b - \frac{\alpha^2}{2}$$

$$\beta\gamma = b - \frac{\left(\frac{-2a}{3}\right)^2}{2}$$

$$\beta\gamma = b - \frac{4a^2}{18} \quad \text{---} \quad (1)$$

And the product of the roots 3 at a time is

$$\alpha\beta\gamma = -c$$

$$\beta\gamma = -c/\alpha = -c/_{-2a/3} = \frac{3c}{2a} \quad \text{---} \quad (2)$$

Equating (1) and (2) gives

$$\frac{3c}{2a} = b - \frac{4a^2}{18}$$

$$3c = 2ab - \frac{8a^3}{18}$$

$$54c = 36ab - 8a^3$$

ie,

$$4a^3 - 18ab + 27c = 0$$

Q14.

(a) (i) $y = \frac{3}{2}x$

(ii) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

$$\frac{2x}{4} - \frac{2y}{9} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{18x}{8y} = \frac{9x}{4y}$$

Hence the gradient of the tangent at (x_1, y_1) is $\frac{9x_1}{4y_1}$

Hence the equation of the tangent is

$$y - y_1 = \frac{9x_1}{4y_1}(x - x_1)$$

$$4yy_1 - 4(y_1)^2 = 9xx_1 - 9(x_1)^2$$

$$9(x_1)^2 - 4(y_1)^2 = 9xx_1 - 4yy_1$$

But from $\frac{x^2}{4} - \frac{y^2}{9} = 1$ $\quad 9x^2 - 4y^2 = 36$

So $9(x_1)^2 - 4(y_1)^2 = 36$

Hence the equation of the tangent is

$$9xx_1 - 4yy_1 = 36$$

(iii) $9xx_1 - 4yy_1 = 36$

passes through $(1, 0)$ so

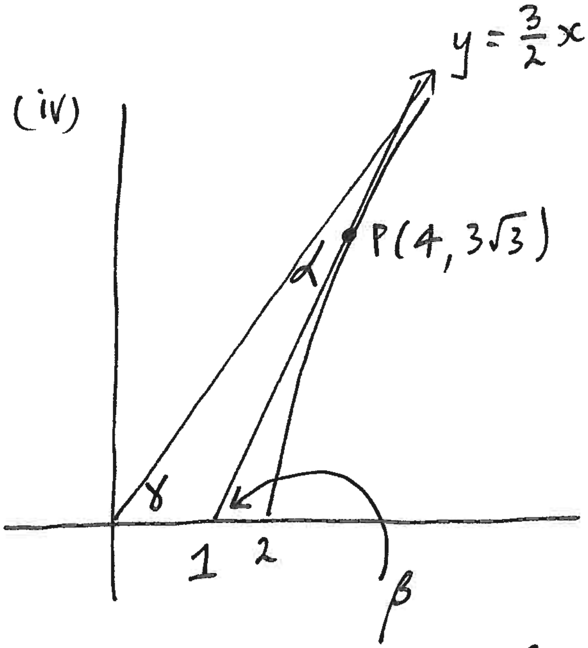
$$9x_1 = 36$$
$$x_1 = 4$$

And $9(x_1)^2 - 4(y_1)^2 = 36$

$$9 \times 4^2 - 4(y_1)^2 = 36$$
$$108 = 4(y_1)^2$$
$$y_1 = \sqrt{27}$$
$$y_1 = 3\sqrt{3}$$

Hence $(x_1, y_1) = (4, 3\sqrt{3})$

(iv)



$\beta = \alpha + \gamma$ (exterior angle)

$$\alpha = \beta - \gamma$$

$$\tan\alpha = \tan(\beta - \gamma)$$
$$= \frac{\tan\beta - \tan\gamma}{1 + \tan\beta . \tan\gamma}$$

But $\tan\beta = \frac{3\sqrt{3}}{4-1} = \sqrt{3}$ and $\tan\gamma = \frac{3}{2}$

Hence $\tan\alpha = \dfrac{\sqrt{3} - \frac{3}{2}}{1 + \sqrt{3} \times \frac{3}{2}} = \dfrac{2\sqrt{3} - 3}{2 + 3\sqrt{3}} \times \dfrac{2 - 3\sqrt{3}}{2 - 3\sqrt{3}} = \dfrac{-24 + 13\sqrt{3}}{-23}$

$$\therefore \alpha = \tan^{-1}\left[\frac{24 - 13\sqrt{3}}{23}\right]$$

(b) (i)

$$4x^2 - y^2 - 24hx + 2hy - 4a^2 + 35h^2 = 0$$

$$4x^2 - 24hx - y^2 + 2hy = 4a^2 - 35h^2$$

$$(2x - 6h)^2 - (y-h)^2 - 36h^2 + h^2 = 4a^2 - 35h^2$$

$$(2x-6h)^2 - (y-h)^2 = 4a^2$$

$$4(x-3h)^2 - (y-h)^2 = 4a^2$$

$$\frac{(x-3h)^2}{a^2} - \frac{(y-h)^2}{4a^2} = 1 \quad \text{———} \quad \alpha$$

α is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hence the equation Represents a hyperbola.

(ii) Differentiating α we get

$$\frac{2(x-3h)}{a^2} - \frac{2(y-h)}{4a^2} \times \frac{dy}{dx} = 0$$

So $$\frac{dy}{dx} = \frac{2 \times 4a^2 (x-3h)}{2 \times a^2 (y-h)}$$

$$\frac{dy}{dx} = \frac{4(x-3h)}{(y-h)}$$

Hence the gradient of the tangent at (p, q) is

$$\frac{4(p-3h)}{q-h}.$$

Now the gradient of the line perpendicular to $y = (e^2-1)x$ is $\dfrac{-1}{e^2-1}$

Hence $$\frac{-1}{e^2-1} = \frac{4(p-3h)}{q-h} \quad \cdots\cdots \text{①}$$

But $b^2 = a^2(e^2-1)$

so $e^2-1 = \dfrac{b^2}{a^2}$

But $b^2 = 4a^2$ so $e^2-1=4$.

Substituting into ① we get

$$\frac{-1}{4} = \frac{4(p-3h)}{q-h}$$

$$h-q = 16p-48h$$

$$\therefore \quad 16p+q = 49h,$$ as required.

(c) Now $T_1 = \left(ct_1, \frac{c}{t_1}\right)$

and $T = \left(ct, \frac{c}{t}\right)$

Hence m of T_1T is $\dfrac{\frac{c}{t} - \frac{c}{t_1}}{ct - ct_1} = \dfrac{\frac{1}{t} - \frac{1}{t_1}}{t - t_1}$

$$= \frac{\frac{t_1 - t}{t_1 t}}{t - t_1} = \frac{\frac{-(t - t_1)}{t_1 t}}{t - t_1} = \frac{-1}{t_1 t}$$

Similarly m of T_2T is $\dfrac{-1}{t_2 t}$.

Since $\angle T_1 T T_2$ is $90°$ then

$$(m \text{ of } T_1T) \times (m \text{ of } T_2T) = -1$$

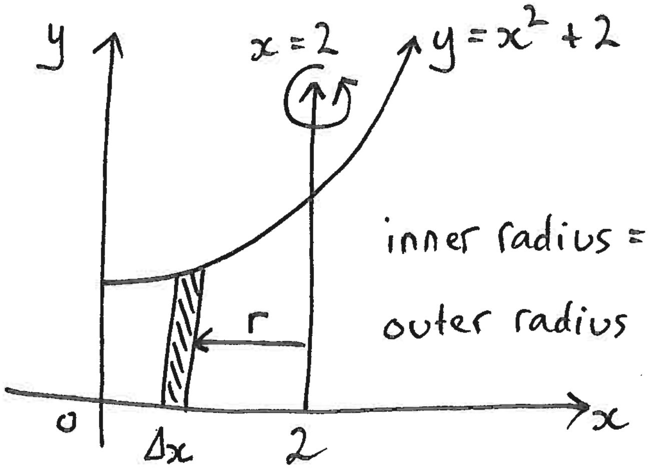
$$\frac{-1}{t_1 t} \times \frac{-1}{t_2 t} = -1$$

$$\therefore \frac{1}{t_1 t_2 t^2} = -1$$

$$\therefore \frac{1}{t_1 t_2} = -t^2$$

$$\therefore t^2 = \frac{-1}{t_1 t_2}$$, as required.

Q15. (a)



inner radius $= 2 - x$

outer radius $= r + \Delta x$

Rotation of the shaded strip about the line $x = 2$ produces a cylindrical shell of volume ΔV where

$$\Delta V = \pi\left[(r+\Delta x)^2 - r^2\right] \times y$$

$$= \pi\left[(2R + \Delta x)\Delta x\right] \times (x^2+2)$$

$$= \pi\left[2R\,\Delta x\right](x^2+2) \quad \text{because } (\Delta x)^2 \text{ is very small}$$

$$= \pi\left[2(2-x)(x^2+2)\Delta x\right]$$

$$\therefore V = \lim_{\Delta x \to 0} \Delta V$$

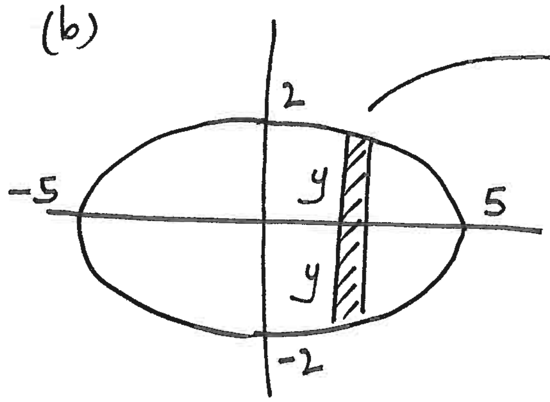
$$= 2\pi \int_0^2 (2-x)(x^2+2)\,dx$$

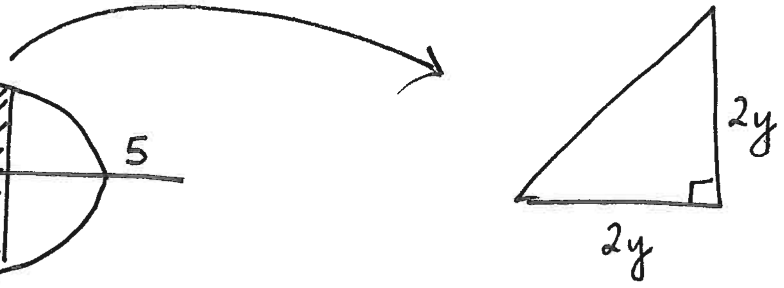
$$= 2\pi \int_0^2 (2x^2 + 4 - x^3 - 2x)\,dx$$

$$= 2\pi \left[\frac{2x^3}{3} + 4x - \frac{x^4}{4} - x^2\right]_0^2$$

$$= 2\pi\left(\frac{16}{3} + 8 - 4 - 4\right) = \frac{32\pi}{3} \text{ units}^3$$

(b)





$$4x^2 + 25y^2 = 100$$

The volume of a slice is ΔV.

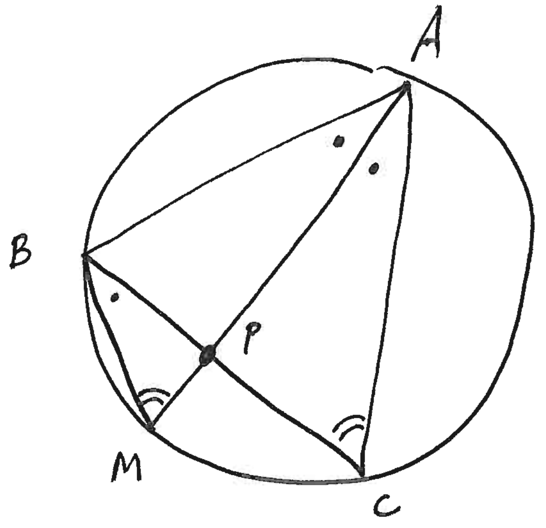
$$\Delta V = \frac{1}{2} \times 2y \times 2y \times \Delta x$$
$$= 2y^2 \, \Delta x$$

Now $y^2 = \dfrac{100 - 4x^2}{25}$ $\quad \therefore \; \Delta V = 2\left(\dfrac{100 - 4x^2}{25}\right)\Delta x$

The required volume is V where

$$V = \lim_{\Delta x \to 0} \Delta V$$
$$= \int_{-5}^{5} \frac{2}{25}\left(100 - 4x^2\right) dx$$
$$= \int_{-5}^{5} \frac{8}{25}\left(25 - x^2\right) dx$$
$$= \frac{16}{25}\int_{0}^{5}\left(25 - x^2\right) dx$$
$$= \frac{16}{25}\left[25x - \frac{x^3}{3}\right]_0^5 = \frac{160}{3} \text{ units}^3$$

(c)



(i) Join B to M.

In ΔABM and ΔPCA,

$\angle BAP = \angle PAC$ (given)

$\angle BMA = \angle ACP$

(equal angles at the circumference standing on the same arc AB)

Hence $\Delta ABM \parallel\!| \Delta APC$ (equiangular)

(ii) Now $\angle MBC = \angle CAP$

(equal angles standing on arc MC)

$\therefore \Delta BMP \parallel\!| \Delta APC$ (equiangular)

$\therefore \dfrac{BP}{PA} = \dfrac{PM}{PC}$ (ratios of corresponding sides of similar triangles)

$\therefore BP \times PC = PM \times PA$, as required.

(d) STEP 1: when $n=0$,

$35^n + 3\times 7^n + 2\times 5^n + 6$ is equal to

$1 + 3\times 1 + 2\times 1 + 6 = 12$, which is divisible by 12.

Hence the statement is true when $n=0$.

STEP 2: We will assume that

$35^k + 3\times 7^k + 2\times 5^k + 6 = 12M$ (M an integer).

We will now try to prove that

$35^{k+1} + 3\times 7^{k+1} + 2\times 5^{k+1} + 6$ is divisible by 12.

Now $35^{k+1} + 3\times 7^{k+1} + 2\times 5^{k+1} + 6$

$= 35.35^k + 3.7^{k+1} + 2.5^{k+1} + 6$

but $35^k = 12M - 6 - 2.5^k - 3.7^k$ (by the assumption)

$= 35(12M - 6 - 2.5^k - 3.7^k) + 3.7^{k+1} + 2.5^{k+1} + 6$

$= 420M - 210 - 70.5^k - 105.7^k + 3.7^{k+1} + 2.5^{k+1} + 6$

$= 420M - 210 - 70.5^k - 105.7^k + 21.7^k + 10.5^k + 6$

$= 420M - 210 - 60.5^k - 84.7^k + 6$

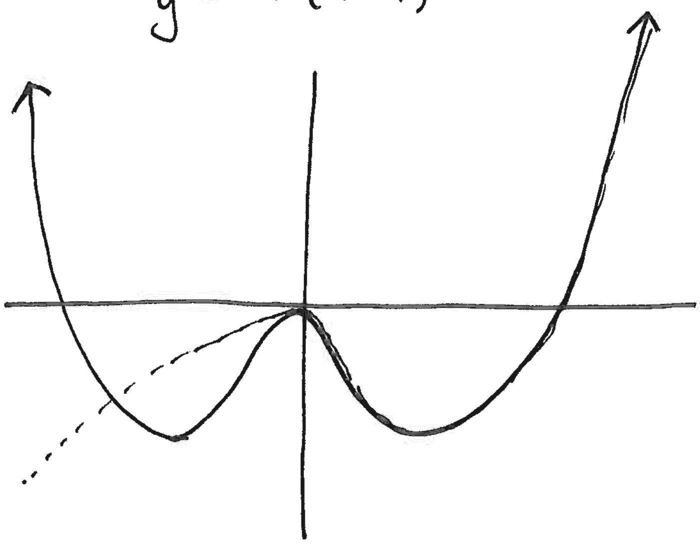
$= 420M - 204 - 60.5^k - 84.7^k$

$= 12[35M - 17 - 5^{k+1} - 7^{k+1}]$ which is divisible by 12.

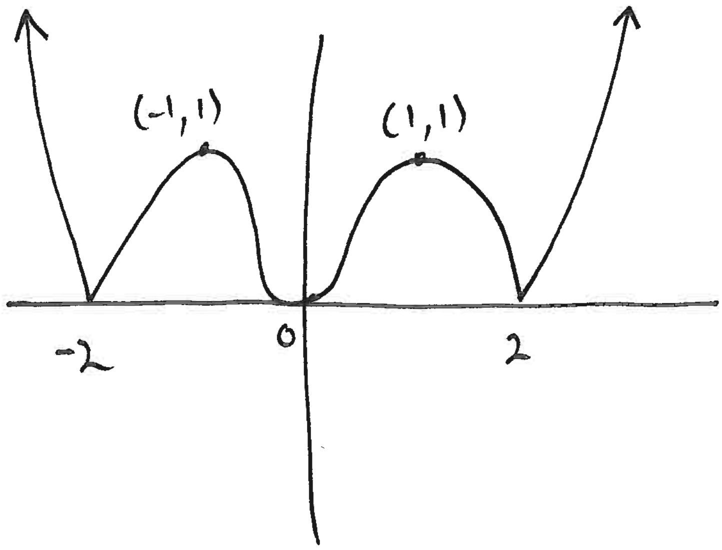
Hence the statement is true by mathematical induction.

Q16. (a)

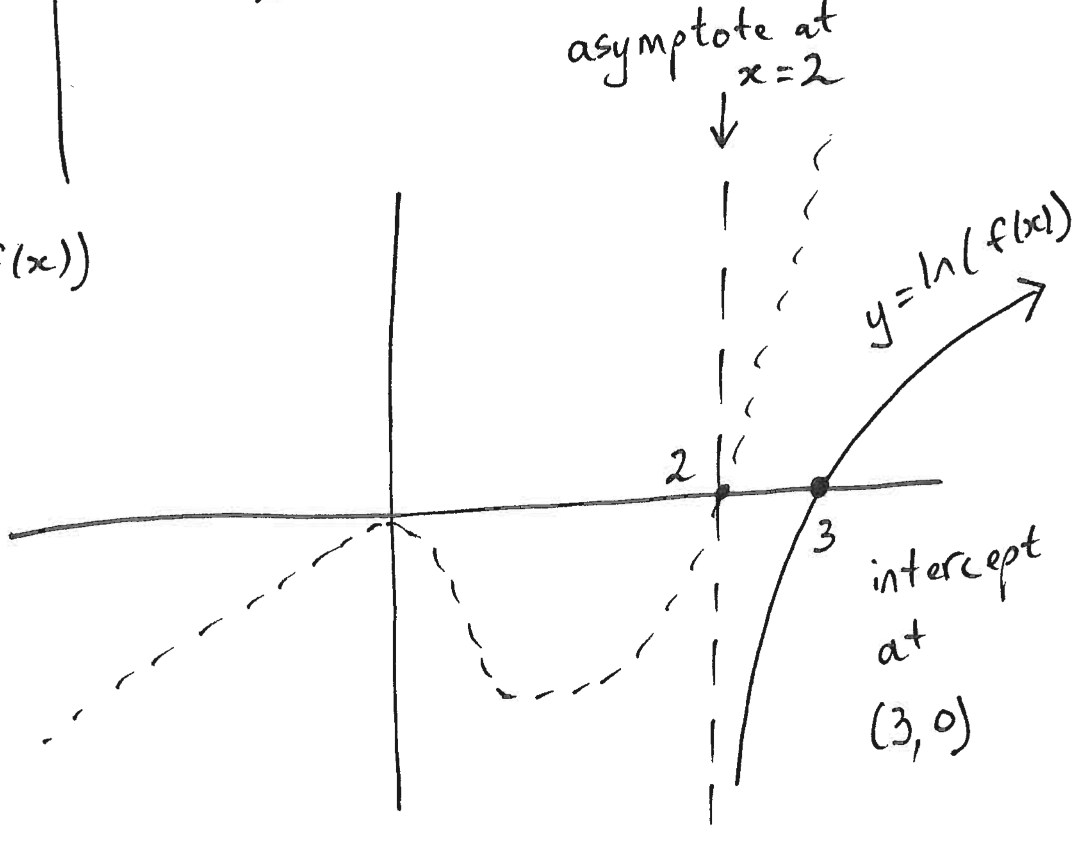
(i) $y = f(|x|)$



$y = |f(|x|)|$



(ii) $y = \ln(f(x))$



(b)

(i) $F = \dfrac{G m_1 m_2}{x^2}$

But $F = m_2 \times$ acceleration

$\therefore \quad m_2 \ddot{x} = \dfrac{-G m_1 m_2}{x^2}$

Hence $\ddot{x} = \dfrac{-G m_1}{x^2}$

(ii) $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = -\dfrac{G m_1}{x^2}$

$$\frac{1}{2}v^2 = -G m_1 \int \frac{dx}{x^2}$$

$$\frac{1}{2}v^2 = -G m_1 \left(\frac{-1}{x}\right) + c$$

But when $x = R$, $v = V$ so

$$\frac{1}{2}V^2 = \frac{G m_1}{R} + c$$

$$\therefore \quad c = \frac{1}{2}V^2 - \frac{G m_1}{R}$$

$$\therefore \quad \frac{1}{2}v^2 = \frac{G m_1}{x} + \frac{1}{2}V^2 - \frac{G m_1}{R}$$

$$v^2 = \frac{2 G m_1}{x} + V^2 - \frac{2 G m_1}{R}$$

$$\therefore \quad v = \sqrt{V^2 + \frac{2 G m_1}{x} - \frac{2 G m_1}{R}}$$

(c) The time between low and high tide is 6 h 40 min.

So the period of tide $= 13$ h 20 mins

$= \frac{40}{3}$ hours

But $T = \frac{2\pi}{n} = \frac{40}{3} \Rightarrow n = \frac{3\pi}{20}$

The centre of motion is at 30 m and the amplitude is 10 m.

Now $x = b + a\cos(nt + \alpha)$

$= 30 + 10\cos(nt + \alpha)$

Let $t = 0$ when $x = 20$ (ie, at 7am) then

$20 = 30 + 10\cos\alpha$

$\alpha = -\pi$

Hence $x = 30 + 10\cos\left(\frac{3\pi}{20}t - \pi\right)$

Solve $5\sqrt{3} + 30 = 30 + 10\cos\left(\frac{3\pi}{20}t - \pi\right)$

$\frac{\sqrt{3}}{2} = \cos\left(\frac{3\pi t}{20} - \pi\right)$

$-\frac{\pi}{6} = \frac{3\pi t}{20} - \pi$

$t = \frac{50}{9}$ hours

$\therefore$ Ship can enter at 12:33 pm

(d) Now $P(x) = x^n - a^n$

and $x^n - a^n = (x^2 - a^2).Q(x) + px + q$.

when n is even, $P(a) = 0$

$P(-a) = 0$

Hence $p = 0$ and $q = 0$.

Hence there is no remainder when n is even.

when n is odd, $P(a) = 0$

$P(-a) = -a^n - a^n = -2a^n$

$\therefore px + q = \frac{1}{2a}[0 - (-2a^n)]x + \frac{1}{2}[0 - 2a^n]$

$= a^{n-1}x - a^n$

when n is odd, the remainder is $a^{n-1}x - a^n$.