Student Number

BAULKHAM HILLS HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

2008

MATHEMATICS
EXTENSION 2

GENERAL INSTRUCTIONS

- Reading time - 5 minutes
- Working time - 3 hours
- Write, using black or blue pen
- Start each question on a new page
- Write your student number at the top of each page
- Calculators may be used
- A table of standard integrals is provided
- ALL necessary working should be shown in every question

Marks

Question 1

a) Find $\int \frac{dx}{\sqrt{x^2 - 49}}$. **1**

b) Find $\int \sin 2x \cos^2 x dx$. **2**

c) Evaluate $\int_0^{\frac{\pi}{3}} \frac{dx}{1 + \cos x + \sin x}$ using the substitution $t = \tan\frac{x}{2}$. **4**

d) Show that $\int_1^2 (2 - x)\sqrt{(x-1)^3}\, dx = \frac{4}{35}$. **3**

e) i) Find real numbers A, B and C such that **2**

$$\frac{3}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}.$$

ii) Hence find $\int \frac{3dx}{x^3 - 1}$. **3**

Marks

Question 2 (Start on a new page)

a) If $z = 1 + i$ and $w = 1 - 3i$ find, in the form $x + iy$,

i) $\bar{z} - w$ **2**

ii) $\dfrac{z}{w}$ **2**

b) i) Express $-2 + 2\sqrt{3}\,i$ in modulus argument form **2**

ii) Hence, evaluate $\sqrt[4]{-2 + 2\sqrt{3}i}$ in the form $x + iy$ **2**

c) Prove $3\,|z - 1|^2 = |z + 1|^2$ if and only if $|z - 2|^2 = 3$ **2**

d) z is a complex number. **2**
Sketch the locus of z satisfied by

$$\arg\left(\frac{z-2}{z-i}\right) = \frac{\pi}{4}$$

e) If $z = r(\cos\theta + i\sin\theta)$ find r and the smallest value of θ **3**
which satisfies the equation $2z^2 = 9 + 3\sqrt{3}i$.

Marks

Question 3 (Start on a new page)

a) Sketch the following curves on separate axes, showing all intercepts and turning points

i) $y = x^3 - 9x$ **3**

ii) $y = \left(x^3 - 9x\right)^2$ **2**

iii) $y^2 = x^3 - 9x$ **2**

b) i) Find the stationary points and the asymptotes of the function **3**

$$y = \frac{(x+1)^4}{x^4 + 1}.$$

ii) Sketch the function labelling all essential features. **1**

iii) Use the graph to find the set of values of k for which **2**

$(x+1)^4 = k\left(x^4 + 1\right)$ has two distinct real roots.

c) A periodic function $f(x)$ has the following properties **2**

$$\left.\begin{array}{l} f(x) = 1 \\ f(x) = 3 - 2x^3 \end{array}\right\} \begin{array}{l} -1 < x < 0 \\ 0 \le x \le 1 \end{array}$$

$f(x)$ has a period of 2.

Sketch this function in the domain $-3 \le x \le 3$.

Marks

Question 4 (Start a new page)

For the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

a) Find the eccentricity. **1**

b) Find the coordinates of the foci S and S'. **1**

c) Find the equations of the directrices. **1**

d) Sketch the curve $\frac{x^2}{25}+\frac{y^2}{16}=1$. **1**

e) Show that the coordinates of any point P can be represented by $(5\cos\theta, 4\sin\theta)$. **2**

f) Show that PS + PS' is independent of the position of P. **2**

g) Show that the equation of the normal at the point P on the ellipse is **2**

$$5x\sin\theta - 4y\cos\theta - 9\sin\theta\cos\theta = 0.$$

g) If the normal meets the major axis at L and the minor axis at M, prove that **2**

$$\frac{PL}{PM}=\frac{16}{25}.$$

h) Show that the normal bisects $\angle SPS'$. **3**

Marks

Question 5 (Start a new page)

a) It is given that $3 - i$ is a root of $P(z) = z^3 + rz + 60$ where r is a real number

i) State why $3 + i$ is also a root of $P(z)$. **1**

ii) Factorise $P(z)$ over the real numbers. **2**

b) i) If $P(x)$ and $Q(x)$ are distinct polynomials which have a common factor $(x - a)$ show that $R(x) = P(x) - Q(x)$ will have the same common factor. **2**

ii) Hence, if $P(x) = 6x^3 + 7x^2 - x - 2$ and $Q(x) = 6x^3 - 5x^2 - 3x + 2$ find the two zeroes that $P(x)$ and $Q(x)$ have in common. **2**

c) For the polynomial $P(x) = 3x^3 + 4x^2 - 2x - 1$, find

i) $\alpha + \beta + \gamma$ and $\alpha\beta + \alpha\gamma + \beta\gamma$ **2**

ii) $\alpha^2 + \beta^2 + \gamma^2$ **1**

iii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

d) Solve $x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2 = 0$ if it has a root of multiplicity 4. **3**

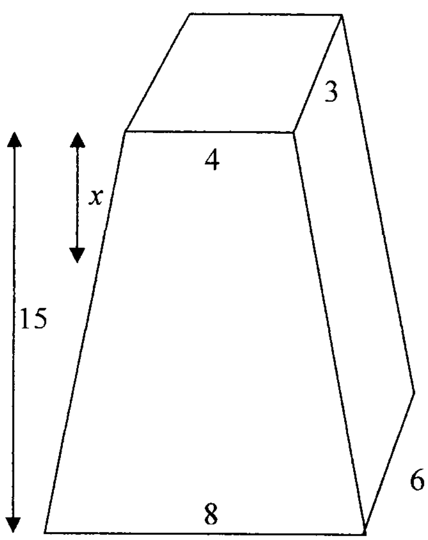
Marks

Question 6 (Start a new page)

a) A solid has, as its base, the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$. **4**

If each cross-section perpendicular to the major axis of the base is an equilateral triangle, show that the volume of the solid is $128\sqrt{3}$ cubic units.

b) A concrete beam of length 15m has plane sides.
Cross sections parallel to the ends are rectangular.
The beam measures 4m by 3m at one end and
8m by 6m at the other.



NOT TO SCALE

i) Find an expression for the area of a cross section at a distance x metres from the smaller end. **3**

ii) Hence find the volume of the beam. **2**

c) The region $(x-2R)^2+y^2\le R^2$ is rotated about the y axis forming a solid of revolution called a torus. **6**

By using the method of cylindrical shells, show that the volume of the torus is $4\pi^2R^3$ cubic units.

Marks

Question 7 (Start a new page)

a) Assume that the moon takes 27 days to revolve around the earth and that the orbit is circular with a radius of 3.85×10^5 km
Calculate the tangential velocity of the moon. Give your answer in km/hr. **2**

b) A particle of mass m kg is projected vertically upwards, with a velocity U ms^{-1} in a medium whose resistance is mkv where k is a positive constant..
Acceleration due to gravity is g ms^{-2}.

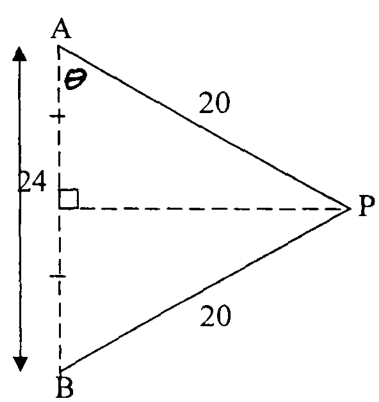
i) Show that the time, t, to reach the highest point is given by **3**

$$t = \frac{1}{k} \ln \left(\frac{g + kU}{g} \right)$$

ii) Show that the greatest height H is given by **3**

$$H = \frac{1}{k^2}\left[kU - g\ln\left(1 + \frac{kU}{g}\right)\right]$$

c. A particle of mass m kg, is attached at P by two strings, each of length 20cm, to two fixed points, A and B which are 24cm apart and lie on a vertical line, as shown in the diagram. The particle moves with constant speed, v ms^{-1} in a horizontal circle about the midpoint of AB so that both pieces of string experience tension . The tension AP is T_1 and tension BP is T_2. The acceleration due to gravity is g ms^{-2}.



i) Copy the diagram and show all forces acting on P. **1**

ii) Resolve the forces on P in both the horizontal and vertical directions. **2**

iii) Find the tension in each part of the string in terms of m, v and g. **2**

iv) Show that $v \geq 4\dfrac{\sqrt{3g}}{15}$. **2**

Marks

Question 8 (Start on a new page)

a) Let $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x\,dx$ where n is a positive integer.

i) Using integration by parts, show that $I_n = \frac{1}{n-1}\left(2^{n-2}\sqrt{3} + (n-2)I_{n-2}\right)$. **4**

(You may use the result $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$).

ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \sec^4 x\,dx$. **3**

b) A particle of mass, m kg, is placed on a rough plane inclined at an angle of θ to the horizontal. The coefficient of friction between the particle and the plane is μ.

A horizontal force X acting on the particle is just sufficient to prevent the particle from sliding down the plane as shown in Fig 1.

When a horizontal force kX acts on the particle, the particle is about to slide up the plane, as shown in Fig 2.

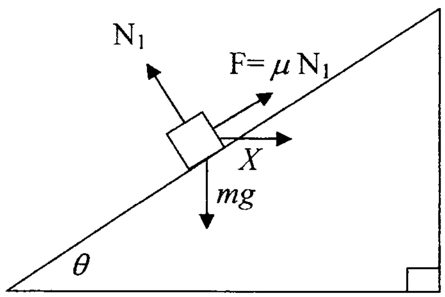


Fig 1

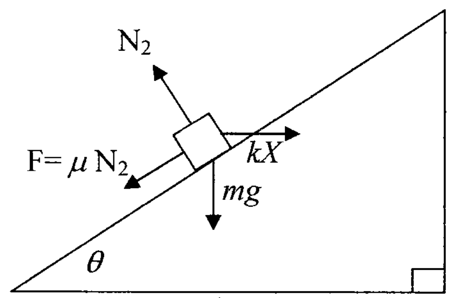


Fig 2

(i) By resolving forces horizontally and vertically, prove that

$(k-1)(1+\mu^2)\sin\theta\cos\theta = \mu(k+1)$ **5**

(ii) Hence show that $k \geq \frac{(1+\mu)^2}{(1-\mu)^2}$ **3**

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad if \quad n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2008 EXTENSION 2 - TRIAL SOLUTIONS

1(a) $\ln\left(x+\sqrt{x^2-49}\right)(+C)$ ✓ ignore

(b) $= \int 2\sin x\cos^3 x\,dx$ ✓

$= \frac{-\cos^4 x}{4}\ (+C)$ ✓

(c) $\int_0^{\pi/3} \frac{dx}{1+\cos x+\sin x}$ let $t=\tan\frac{x}{2}$, $dx=\frac{2dt}{1+t^2}$

when $x=0, t=0$ ✓

when $x=\frac{\pi}{3}, t=\frac{1}{\sqrt3}$

$= \int_0^{1/\sqrt3} \frac{2dt}{(1+t^2)\left(1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}\right)}$ ✓

$= \int_0^{1/\sqrt3} \frac{2dt}{1+t^2+1-t^2+2t}$

$= \int_0^{1/\sqrt3} \frac{\cancel{2}dt}{\cancel{2}(1+t)}$

$= \Big[\ln(1+t)\Big]_0^{1/\sqrt3}$ ✓

$= \ln\left(1+\frac{1}{\sqrt3}\right) - \ln 1$

$= \ln\left(\frac{\sqrt3+1}{\sqrt3}\right)$ ✓

1(d) let $u=x-1$

$du=dx$

when $x=1, u=0$ ✓

when $x=2, u=1$

$\int_0^1 (2-(u+1))\sqrt{u^3}\,du$

$= \int_0^1 (1-u)u^{3/2}\,du$ ✓

$= \int_0^1 u^{3/2}-u^{5/2}\,du$

$= \left[\frac{2}{5}u^{5/2}-\frac{2}{7}u^{7/2}\right]_0^1$ ✓

$= \frac{2}{5}-\frac{2}{7}-0-0$

$= \frac{4}{35}$ as req'd.

1(e)(i) $= \int \frac{3dx}{(x-1)(x^2+x+1)}$

let $\frac{3}{(x-1)(x^2+x+1)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$

$3 = A(x^2+x+1)+(Bx+C)(x-1)$

when $x=1$, $3=3A$

$\therefore A=1$ ✓

when $x=0$, $3=A-C$

$C=1-3$

$C=-2$

when $x=-1$ $3=A+(C-B)(-2)$

$3=1+2(2+B)$

$2B=-2$ ✓

$\therefore B=-1, A=1, C=-2$

(ii) $\int\frac{3dx}{x^3-1} = \int \frac{1}{x-1}+\frac{-x-2}{x^2+x+1}\,dx$

$= \ln(x-1) - \int\frac{\frac12(2x+1)+\frac32}{x^2+x+1}\,dx$ ✓

$= \ln(x-1)-\frac12\ln(x^2+x+1)-\frac32\int\frac{dx}{(x+\frac12)^2+\frac34}$ ✓

$= \ln(x-1)-\frac12\ln(x^2+x+1)-\frac32\cdot\frac{2}{\sqrt3}\tan^{-1}\left(\frac{x+\frac12}{\sqrt3/2}\right)+C$

$= \ln(x-1)-\frac12\ln(x^2+x+1)-\sqrt3\tan^{-1}\left(\frac{2x+1}{\sqrt3}\right)+C$ ✓

2 a(i) $(1-i)-(1-3i) = 2i$ ✓

(ii) $\frac{1+i}{1-3i}\times\frac{1+3i}{1+3i}$ ✓

$= \frac{1+4i-3}{1-9i^2}$

$= \frac{-2+4i}{10}$ ✓

$= -\frac15+\frac{2i}{5}$

2 b(i) $z=-2+2\sqrt3 i$

$|z| = \sqrt{4+12}$ ✓

$\therefore r=4$

$\arg(z) = \tan^{-1}\left(\frac{2\sqrt3}{-2}\right)$ ✓

$= \frac{2\pi}{3}$

$\therefore -2+2\sqrt3 i = 4\,\mathrm{cis}\left(\frac{2\pi}{3}\right)$

2b (ii) cont'd

$$\sqrt[4]{-2+2\sqrt{3}i} = \sqrt[4]{4\operatorname{cis}\left(\frac{2\pi}{3}+2k\pi\right)} \quad \text{for } k=0,1,2,3$$

$$= 4^{1/4}\left[\operatorname{cis}\left(\frac{6k\pi+2\pi}{3}\right)\right]^{1/4}$$

$$= \sqrt{2}\operatorname{cis}\frac{2\pi}{12}, \sqrt{2}\operatorname{cis}\frac{8\pi}{12}, \sqrt{2}\operatorname{cis}\frac{14\pi}{12}, \sqrt{2}\operatorname{cis}\frac{20\pi}{12}$$

$$= \sqrt{2}\operatorname{cis}\frac{\pi}{6}, \sqrt{2}\operatorname{cis}\frac{2\pi}{3}, \sqrt{2}\operatorname{cis}\frac{7\pi}{6}, \sqrt{2}\operatorname{cis}\frac{5\pi}{3}$$

$$= \sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right), \sqrt{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right), \sqrt{2}\left(\frac{-\sqrt{3}}{2}-\frac{i}{2}\right),$$

$$\sqrt{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$

$$= \pm\left(\frac{\sqrt{6}}{2}+\frac{i\sqrt{2}}{2}\right), \pm\left(\frac{-\sqrt{2}}{2}+\frac{\sqrt{6}i}{2}\right)$$

✓ for one other sol'n

2(c) let $z = x+iy$

$$3|x-1+iy|^2 = |x+1+iy|^2$$

$$3\left((x-1)^2+y^2\right) = (x+1)^2+y^2$$

$$3(x^2-2x+1+y^2) = x^2+2x+1+y^2 \quad ✓$$

$$2x^2-8x+2+2y^2=0$$

$$x^2-4x+y^2+1=0$$

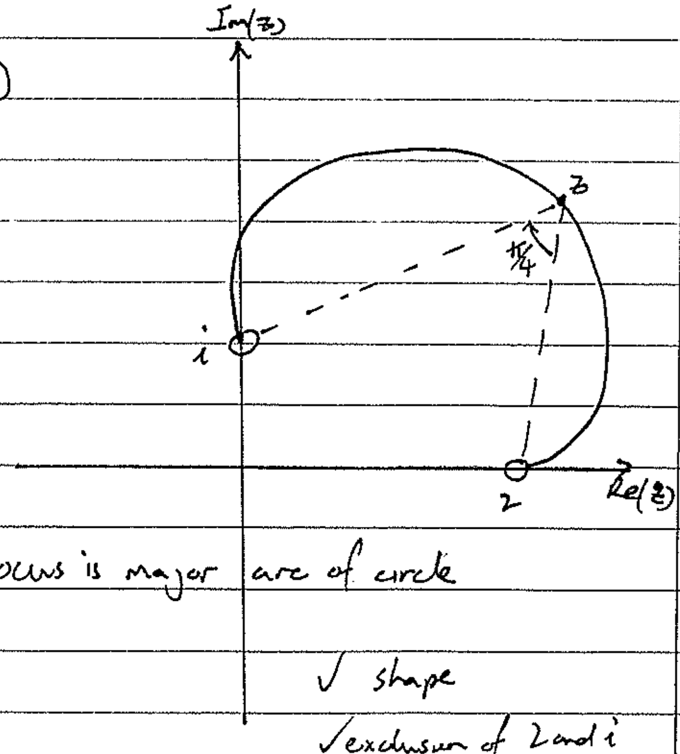
$$x^2-4x+4+y^2=3$$

$$(x-2)^2+y^2=3 \quad ✓$$

$$|x-2+iy|^2=3$$

$$|z-2|^2=3$$

2(d)



locus is major arc of circle

✓ shape

✓ exclusion of 2 and i

2(e) $z = r(\cos\theta + i\sin\theta)$

$$2z^2 = 9+3\sqrt{3}i$$

$$2r^2(\operatorname{cis}\theta)^2 = 9+3\sqrt{3}i$$

$$2r^2\operatorname{cis}(2\theta) = 6\sqrt{3}\operatorname{cis}\frac{\pi}{6} \quad ✓$$

$$\therefore 2r^2 = 6\sqrt{3} \quad \text{and} \quad 2\theta = \frac{\pi}{6}$$

$$r^2 = 3\sqrt{3} = \sqrt{27} \qquad \theta = \frac{\pi}{12}$$

$$r = 27^{1/4} \text{ or } 3^{3/4} \quad ✓ \qquad \theta = \frac{\pi}{12} \quad ✓$$

3(a) $y = x(x+3)(x-3)$

$$\frac{dy}{dx} = 3x^2-9$$

when $\frac{dy}{dx}=0$, $3x^2-9=0$

$$x = \pm\sqrt{3}$$

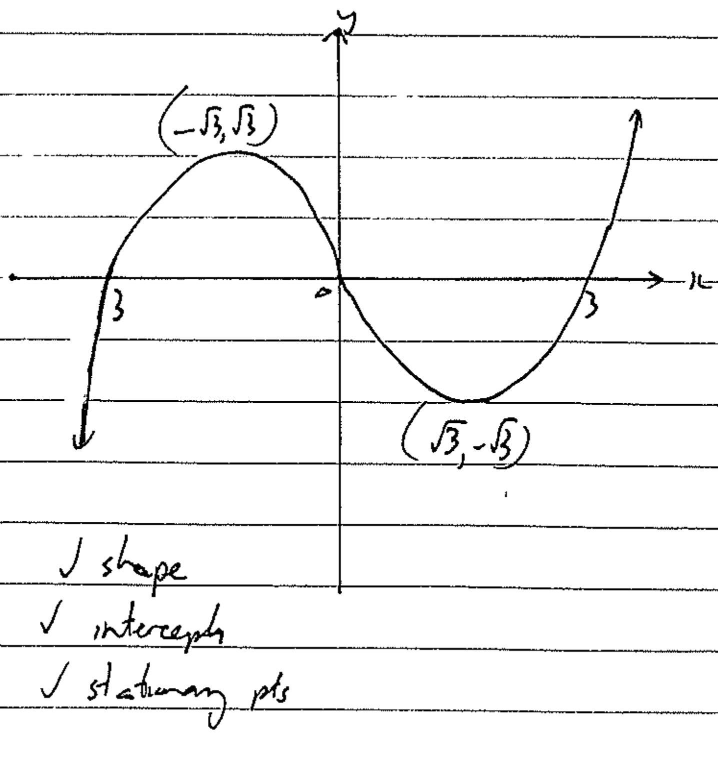
when $x=\sqrt{3}$, $y=-\sqrt{3}$

when $x=-\sqrt{3}$, $y=\sqrt{3}$

$$\frac{d^2y}{dx^2} = 6x$$

when $x=\sqrt{3}$, $\frac{d^2y}{dx^2} = 6\sqrt{3} > 0$ $\therefore$ min turn pt

when $x=-\sqrt{3}$, $\frac{d^2y}{dx^2} = -6\sqrt{3} < 0$ $\therefore$ max turn pt

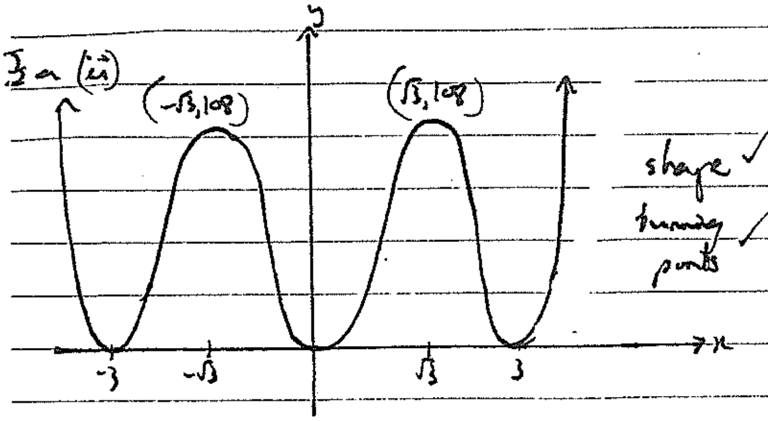


✓ shape

✓ intercepts

✓ stationary pts

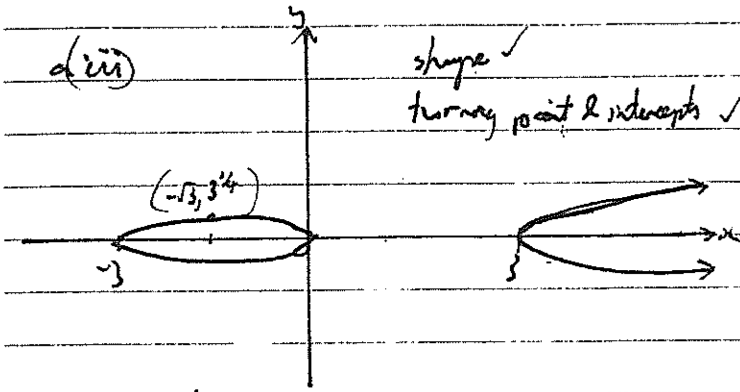
2 a (iii)



shape ✓
turning points ✓

d (iii)

shape ✓
turning point & intercepts ✓



3 (b) (i)
$$\frac{dy}{dx} = \frac{(x^4+1)\,4(x+1)^3 - (x+1)^4 \cdot 4x^3}{(x^4+1)^2}$$
$$= \frac{4(x+1)^3\left(x^4+1 - x^3(x+1)\right)}{(x^4+1)^2}$$
$$\frac{dy}{dx} = \frac{4(x+1)^3(1-x^3)}{(x^4+1)^2} \quad ✓$$

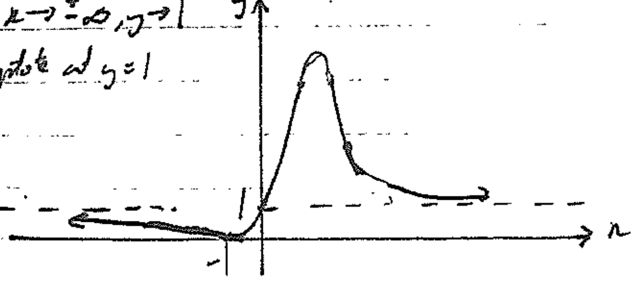
For stat pts $\frac{dy}{dx}=0$

$$0 = 4(x+1)^3(1-x^3)$$
$$\therefore x = -1, 1$$

x	-2	-1	0	1	2
$\frac{dy}{dx}$	$-\frac{36}{17^2}$	0	4	0	$-\frac{468}{289}$

$\therefore$ Min turning point at $(-1,0)$ and max turning point at $(1,8)$ both ✓

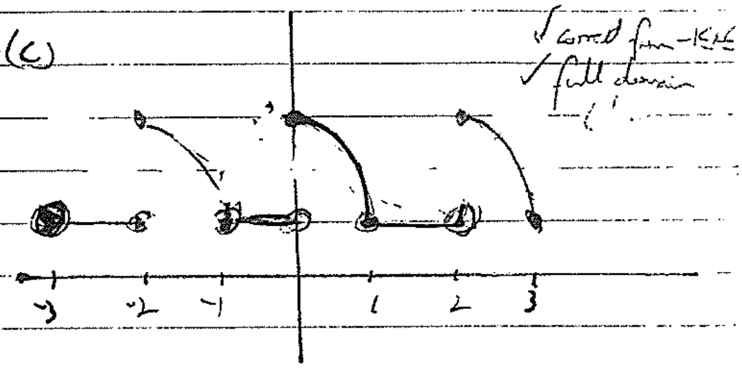
(As $x \to \pm\infty$, $y \to 1$
$\therefore$ asymptote at $y=1$



3 (ii) From graph 2 roots if $0<k<1$ ✓
and $1<k<8$ ✓

3 (c)

✓ correct for $-1<x<1$
✓ full domain

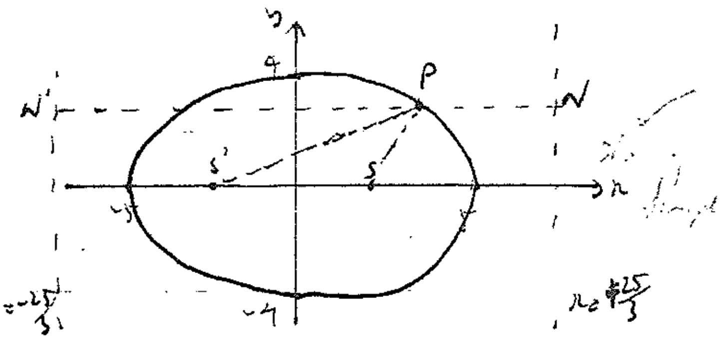


4 (a) $a=5 \quad b=4$
$$b^2 = a^2(1-e^2)$$
$$16 = 25(1-e^2)$$
$$e^2 = 1 - \frac{16}{25}$$
$$e = \frac{3}{5} \quad ✓$$

(b) $S(ae,0) = (3,0)$
$S'(-ae,0) = (-3,0)$ ✓

(c)
$$x = \pm \frac{a}{e}$$
$$= \pm \frac{5}{3/5}$$
$$x = \pm\frac{25}{3} \text{ or } \pm 8\tfrac{1}{3} \quad ✓$$

(d)



(e)
$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$
Sub $x = 5\cos\theta$, $y = 4\sin\theta$ ✓
$$\text{LHS} = \frac{25\cos^2\theta}{25} + \frac{16\sin^2\theta}{16}$$
$$= \cos^2\theta + \sin^2\theta$$
$$= 1 \quad ✓$$
$\therefore$ P can be represented by $(5\cos\theta$

(f) $\frac{PS}{PN} = e$ by definition

$\therefore PS = ePN$

$PN = QN - QP$

$= \frac{25}{3} - 5\cos\theta$

So $PS = \frac{3}{5}\left(\frac{25}{3} - 5\cos\theta\right)$

$= 5 - 3\cos\theta$

And $\frac{PS'}{PN'} = e$

$\therefore PS' = ePN'$

$= \frac{3}{5}\left(\frac{25}{3} + 5\cos\theta\right)$

$= 5 + 3\cos\theta$

$PS + PS' = 5 - 3\cos\theta + 5 + 3\cos\theta$

$= 10$

which is independent of P.

(g) Implicitly differentiating

$\frac{2x}{25} + \frac{2y}{16}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{25} \times \frac{16}{2y}$

$\frac{dy}{dx} = \frac{-16x}{25y}$

At P, $\frac{dy}{dx} = \frac{-16 \times 5\cos\theta}{25 \times 4\sin\theta}$

$= -\frac{4}{5}\frac{\cos\theta}{\sin\theta}$

$\therefore$ Grad of normal at point P is $\frac{5}{4}\frac{\sin\theta}{\cos\theta}$

$y - 4\sin\theta = \frac{5}{4}\frac{\sin\theta}{\cos\theta}(x - 5\cos\theta)$

$4y\cos\theta - 16\sin\theta\cos\theta = 5\sin\theta x - 25\sin\theta\cos\theta$

$\therefore 5\sin\theta x - 4y\cos\theta - 9\sin\theta\cos\theta = 0$

(g) "second"

Meets major axis at L when $y=0$

ie $5x\sin\theta = 9\sin\theta\cos\theta$

$x = \frac{9\cos\theta}{5}$

$\therefore L$ is $\left(\frac{9}{5}\cos\theta, 0\right)$

meets minor axis at M ie $x=0$

ie $4y\cos\theta = -9\sin\theta\cos\theta$

$y = \frac{-9\sin\theta}{4}$

$\therefore M$ is $\left(0, -\frac{9\sin\theta}{4}\right)$

$\frac{PL}{PM} = \frac{\sqrt{(5\cos\theta - \frac{9}{5}\cos\theta)^2 + (4\sin\theta)^2}}{\sqrt{(5\cos\theta)^2 + (4\sin\theta + \frac{9}{4}\sin\theta)^2}}$

$= \frac{\sqrt{(\frac{16}{5}\cos\theta)^2 + 16\sin^2\theta}}{\sqrt{25\cos^2\theta + (\frac{25}{4}\sin\theta)^2}}$

$= \frac{\frac{4}{5}\sqrt{16\cos^2\theta + 25\sin^2\theta}}{\frac{5}{4}\sqrt{16\cos^2\theta + 25\sin^2\theta}}$

$\therefore \frac{PL}{PM} = \frac{16}{25}$

(h)

Gradient of PS $m = \frac{4\sin\theta - 0}{5\cos\theta + 3}$

$\tan\angle SPM = \left|\frac{\frac{5\sin\theta}{4\cos\theta} - \frac{4\sin\theta}{5\cos\theta - 3}}{1 + \frac{5\sin\theta}{4\cos\theta} \times \frac{4\sin\theta}{5\cos\theta - 3}}\right|$

$= \left|\frac{25\sin\theta\cos\theta - 15\sin\theta - 16\sin\theta\cos\theta}{20\cos^2\theta - 12\cos\theta + 20\sin^2\theta}\right|$

$= \left|\frac{9\sin\theta\cos\theta - 15\sin\theta}{20 - 12\cos\theta}\right|$

$= \left|\frac{-3\sin\theta(5 - 3\cos\theta)}{4(5 - 3\cos\theta)}\right|$

4(b) (cont)

$= \frac{3\sin\theta}{4}$

Gradient of $PS' = \frac{4\sin\theta}{5\cos\theta + 3}$

$\tan\angle S'PM = \left| \frac{\frac{5\sin\theta}{4\cos\theta} - \frac{4\sin\theta}{5\cos\theta+3}}{1 + \frac{5\sin\theta}{4\cos\theta} \times \frac{4\sin\theta}{5\cos\theta+3}} \right|$

$= \left| \frac{25\sin\theta\cos\theta + 15\sin\theta - 16\sin\theta\cos\theta}{20\cos^2\theta + 12\cos\theta + 20\sin^2\theta} \right|$

$= \left| \frac{9\sin\theta\cos\theta + 15\sin\theta}{20 + 12\cos\theta} \right|$

$= \left| \frac{3\sin\theta(3\cos\theta + 5)}{4(5 + 3\cos\theta)} \right|$

$= \frac{3\sin\theta}{4}$ ✓

$\therefore \angle SPM = \angle S'PM$

5(a)(i) Because coefficients of $P(z)$ are all real then roots occur in complex conjugates ✓

$\overline{3-i} = 3+i$

(ii) $(3-i)$ and $(3+i)$ are roots

$\therefore (z-(3-i))(z-(3+i))$ is a factor

$z^2 - (3-i)z - (3+i)z + (3-i)(3+i)$ is a factor ✓

$z^2 - 6z + 10$ is a factor.

By inspection

$z^3 + rz + 60 = (z^2 - 6z + 10)(z + 6)$ ✓

5 b (i) If $x-a$ is a common factor to $P(x)$ and $Q(x)$

let $P(x) = (x-a)A(x)$ ✓

$Q(x) = (x-a)B(x)$

$R(x) = P(x) - Q(x)$

$= (x-a)A(x) - (x-a)B(x)$ ✓

$= (x-a)(A(x) - B(x))$

$\therefore (x-a)$ is a factor of $R(x)$.

(ii) $R(x) = P(x) - Q(x)$

$= (6x^3 + 7x^2 - x - 2) - (6x^3 - 5x^2 - 3x + 2)$ ✓

$= 12x^2 + 2x - 4$

$= 2(6x^2 + x - 2)$

$= 2(3x+2)(2x-1)$ ✓

$\therefore x = -\frac{2}{3}$ and $x = \frac{1}{2}$ are common zeros

5(c) (i) $\alpha + \beta + \gamma = -\frac{4}{3}$ ✓

$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{2}{3}$ ✓

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \gamma\alpha + \beta\gamma)$

$= \frac{16}{9} - 2 \times -\frac{2}{3}$

$= \frac{28}{9}$ ✓

(iii) Since α, β, γ are roots,

$3\alpha^3 + 4\alpha^2 - 2\alpha - 1 = 0$ ①

$3\beta^3 + 4\beta^2 - 2\beta - 1 = 0$ ② ✓

$3\gamma^3 + 4\gamma^2 - 2\gamma - 1 = 0$ ③

①+②+③

$3(\alpha^3 + \beta^3 + \gamma^3) + 4(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha + \beta + \gamma) - 3 = 0$

$3(\alpha^3 + \beta^3 + \gamma^3) + 4 \times \frac{28}{9} - 2\left(-\frac{4}{3}\right) - 3 = 0$

$3(\alpha^3 + \beta^3 + \gamma^3) = -12\frac{1}{9}$ ✓

$\alpha^3 + \beta^3 + \gamma^3 = -4\frac{1}{27}$ or $-\frac{109}{27}$

5(d) $P'(x) = 5x^4 + 8x^3 - 6x^2 - 16x - 7$ has root of mult.

$P''(x) = 20x^3 + 24x^2 - 12x - 16$ " "

$P'''(x) = 60x^2 + 48x - 12$ " " ✓

$P'''(x) = 12(5x^2 + 4x - 1)$

$= 12(5x-1)(x+1)$ ✓

$\therefore$ Root of multiplicity 4 is $x = -1$ or $x = \frac{1}{5}$

$P(-1) = -1 + 2 + 2 - 8 + 7 - 2$

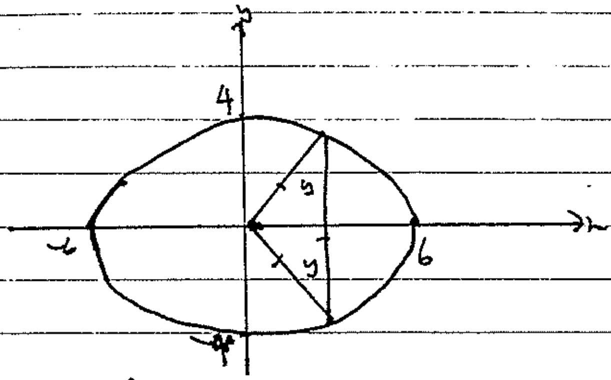
$= 0$ ✓

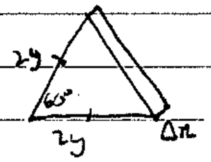
$\therefore x = -1$ is root of multiplicity 4.

By inspection $P(x) = (x+1)^4(x-2)$

$\therefore$ Roots are -1 and 2 ✓

3(a)





Area of $\Delta = \frac{1}{2} ab\sin C$

$= \frac{1}{2}\, 4y^2 \sin 60°$

$= y^2\sqrt{3}$

$= \frac{4}{9}(36-x^2)\sqrt{3}$

Volume of slice $= \frac{4\sqrt{3}}{9}(36-x^2)\,\Delta x$ ✓

Volume $= \lim_{\Delta x\to 0} \sum_{-6}^{6} \frac{4\sqrt{3}}{9}(36-x^2)\,\Delta x$

$= \frac{4\sqrt{3}}{9}\int_{-6}^{6} 36-x^2\, dx$ ✓

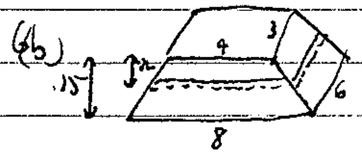
$= \frac{4\sqrt{3}}{9}\left[36x - \frac{x^3}{3}\right]_{-6}^{6}$ ✓

$= \frac{8\sqrt{3}}{9}\left[36x - \frac{x^3}{3}\right]_{0}^{6}$

$= \frac{8\sqrt{3}}{9}(216-72)$ ✓

$= 128\sqrt{3}$ units^3

(6b)



Since y and w are linear functions of x

$y = ax+b$ and $w = cx+d$

when $x=0, y=4$ when $x=0$, $w=3$

$4=b$ $3=d$

$y = ax+4$ $\therefore w = cx+3$

when $x=15, y=8$ when $x=15$ $w=6$

$8 = 15a+4$ $6 = 15c+3$

$15a = 4$ $3 = 15c$

$a = \frac{4}{15}$ $c = \frac{1}{5}$

$\therefore y = \frac{4x}{15}+4$ ✓ $\therefore w = \frac{x}{5}+3$

$\therefore$ Area of slice $= \left(\frac{4x}{15}+4\right)\left(\frac{x}{5}+3\right)$ ✓

$= \frac{4x^2}{75} + \frac{12x}{15} + \frac{4x}{5} + 12$ ✓

Area $= \frac{4x^2}{75} + \frac{24x}{15} + 12$

Volume of slice $= \left(\frac{4x^2}{75} + \frac{8x}{5} + 12\right)\Delta x$

Volume $\lim_{\Delta x\to 0} \sum_{x=0}^{15} \left(\frac{4x^2}{75} + \frac{8x}{5} + 12\right)\Delta x$

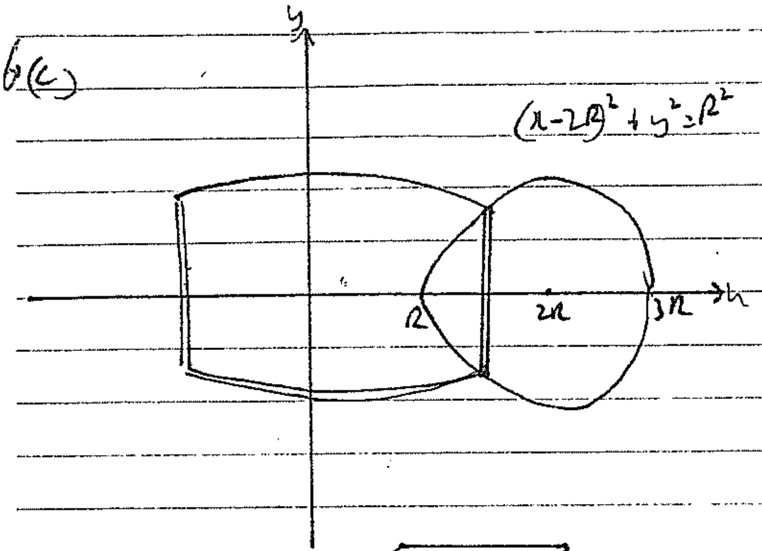
$= \int_0^{15} \frac{4x^2}{75} + \frac{8x}{5} + 12\, dx$

$= \left[\frac{4x^3}{225} + \frac{4x^2}{5} + 12x\right]_0^{15}$ ✓

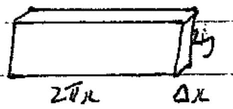
$= 60 + 180 + 180$

$= 420\, m^3$ ✓

6(c)



Each shell



Area of slice $= 2\pi x 2y$

$= 4\pi x\sqrt{R^2-(x-2R)^2}$

Volume of slice $= 4\pi x\sqrt{R^2-(x-2R)^2}\,\Delta x$

Total volume $= \displaystyle\int_R^{3R} 4\pi x\sqrt{R^2-(x-2R)^2}\,dx$ ✓

let $x-2R = R\sin\theta$ ✓

$x = 2R+R\sin\theta$

$dx = R\cos\theta\, d\theta$

when $x=R$, $-R = R\sin\theta$, $\theta = -\frac{\pi}{2}$

when $x=3R$, $R=R\sin\theta$, $\theta=\frac{\pi}{2}$ ✓

$V = 4\pi\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2R+R\sin\theta)\sqrt{R^2-R^2\sin^2\theta}\,R\cos\theta\, d\theta$ ✓

$= 4\pi R^3\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2+\sin\theta)\cos^2\theta\, d\theta$

$= 4\pi R^3\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta+\sin\theta\cos^2\theta\, d\theta$ ✓

$= 4\pi R^3\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+\cos 2\theta + \sin\theta\cos^2\theta\, d\theta$

$= 4\pi R^3\left[\theta + \frac{\sin 2\theta}{2} - \frac{\cos^3\theta}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 4\pi R^3\left(\left(\frac{\pi}{2}+0-0\right)-\left(-\frac{\pi}{2}+0-0\right)\right)$ ✓

$= 4\pi^2R^3$

7(a) 1 rev in 27 days

2π rad / 27 days

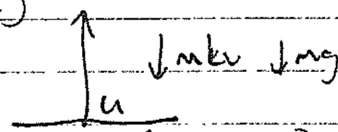
$\frac{2\pi}{27\times 24}$ rad in 1 hr ✓

$v = r\omega$

$= 3.85\times 10^5 \times \frac{2\pi}{27\times 24}$ ✓

$= 3733$ km/hr ✓

7(b)(i)



$m\ddot{x} = -(gm+mkv)$

$\ddot{x} = -(g+kv)$ ✓

$\frac{dv}{dt} = -(g+kv)$

$\frac{dt}{dv} = \frac{-1}{g+kv}$

$t = -\frac{1}{k}\ln(g+vk) + c$ ✓

when $t=0$, $v=u$

$0 = \frac{-1}{k}\ln(g+uk)+c$

$c = \frac{1}{k}\ln(g+uk)$

$\therefore t = \frac{1}{k}\left(\ln(g+uk)-\ln(g+vk)\right)$

$= \frac{1}{k}\ln\left(\frac{g+uk}{g+vk}\right)$ ✓

At highest point $v=0$

$\therefore t = \frac{1}{k}\ln\left(\frac{g+uk}{g}\right)$ ✓

7 (b) (ii)

$$\ddot{x} = -(g+kv)$$

$$v\frac{dv}{dx} = -(g+kv)$$

$$\frac{dv}{dx} = -\left(\frac{g+kv}{v}\right)$$

$$\frac{dx}{dv} = \frac{-v}{g+kv}$$

$$\frac{dx}{dv} = \frac{-\frac{1}{k}(g+kv) + \frac{g}{k}}{g+kv}$$ ✓

$$\frac{dx}{dv} = -\frac{1}{k} + \frac{g}{k^2}\left(\frac{k}{g+kv}\right)$$

$$\left[x\right]_0^H = \left[\frac{-v}{k} + \frac{g}{k^2}\ln(g+kv)\right]_u^0$$ ✓

$$H - 0 = \left(0+\frac{g}{k^2}\ln(g)\right) - \left(-\frac{u}{k} + \frac{g}{k^2}\ln(g+ku)\right)$$

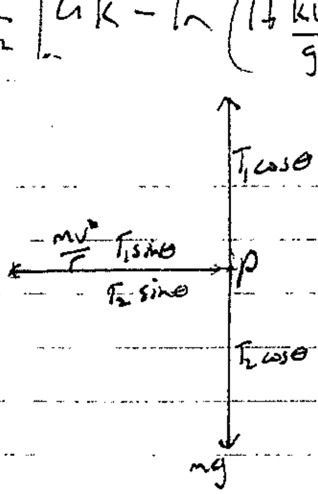
$$= \frac{g}{k^2}\ln g + \frac{u}{k} - \frac{g}{k^2}\ln(g+ku)$$

$$= \frac{1}{k^2}\left(uk + \ln\left(\frac{g}{g+ku}\right)\right)$$

$$= \frac{1}{k^2}\left(uk - g\ln\left(\frac{g+ku}{g}\right)\right)$$ ✓

$$\therefore H = \frac{1}{k^2}\left[uk - \ln\left(1+\frac{ku}{g}\right)\right]$$

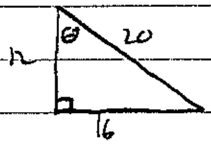
7 (c) (i)



✓

(ii) Resolve vertically

$$T_1\cos\theta - T_2\cos\theta = mg$$



$\cos\theta = 3/5$

$\sin\theta = 4/5$

OR ✓

$$\text{or } \frac{3}{5}T_1 - \frac{3}{5}T_2 = mg \quad (1)$$

Resolve horizontally

$$T_1\sin\theta + T_2\sin\theta = \frac{mv^2}{r}$$ OR ✓

$$\frac{4}{5}T_1 + \frac{4}{5}T_2 = \frac{mv^2}{0.16} \quad (2)$$

(iii) From (2) $T_1 + T_2 = \frac{500mv^2}{64}$ (3)

From (1) $T_1 - T_2 = \frac{5mg}{3}$ (4)

(3) + (4)

$$2T_1 = \frac{500mv^2}{64} + \frac{5mg}{3}$$

$$T_1 = \frac{5m}{2}\left(\frac{100v^2}{64} + \frac{g}{3}\right)$$ ✓

(3) − (4) $2T_2 = 5m\left(\frac{100v^2}{64} - \frac{g}{3}\right)$

$$\therefore T_2 = \frac{5m}{2}\left(\frac{25v^2}{16} - \frac{g}{3}\right)$$ ✓

and $T_1 = \frac{5m}{2}\left(\frac{25v^2}{16} + \frac{g}{3}\right)$

(iv) Since $T_2 \geq 0$

$$\frac{25v^2}{16} - \frac{g}{3} \geq 0$$ ✓

$$\frac{25v^2}{16} \geq \frac{g}{3}$$

$$v^2 \geq \frac{16g}{75}$$

$$v \geq \frac{4\sqrt{g}}{5\sqrt{3}}$$

$$v \geq 4\sqrt{3g}/15$$ ✓

8(a)

(i) $I_n = \int_{\pi/6}^{\pi/2} \operatorname{cosec}^n x \, dx$

$= \int_{\pi/6}^{\pi/2} \operatorname{cosec}^2 x \operatorname{cosec}^{n-2} x \, dx$ $\qquad u = \operatorname{cosec}^{n-2} x \qquad v' = \operatorname{cosec}^2 x$

$\qquad u' = (n-2)\operatorname{cosec}^{n-3} x \cdot -\cot x \operatorname{cosec} x \qquad v = -\cot x$

$= \left[-\cot x \operatorname{cosec}^{n-2} x\right]_{\pi/6}^{\pi/2} - \int +(n-2)\cot x \operatorname{cosec}^{n-2} x \cot x \, dx$ $\qquad u' = -(n-2)\cot x \operatorname{cosec}^{n-2} x \qquad v = \cot x$

$= -(0 \times 1) + \left(\sqrt{3} \times \frac{1}{(\frac{1}{2})^{n-2}}\right) - (n-2)\int \cot^2 x \operatorname{cosec}^{n-2} x \, dx$

$I_n = \sqrt{3} \times 2^{n-2} - (n-2) \int_{\pi/2}^{\pi/6} \cos^2 x \operatorname{cosec}^n x \, dx$

$= \sqrt{3} \times 2^{n-2} - (n-2) \int_{\pi/2}^{\pi/6} (1 - \sin^2 x) \operatorname{cosec}^n x \, dx$

$(n-1) I_n = \sqrt{3} \times 2^{n-2} + (n-2) \int_{\pi/2}^{\pi/6} \operatorname{cosec}^{n-2} x \, dx$

$I_n = \frac{1}{n-1}\left(\sqrt{3} \times 2^{n-2} + (n-2) I_{n-2}\right)$

(ii) $\int_0^{\pi/3} \sec^4 x \, dx = \int_0^{\pi/3} \operatorname{cosec}^4 \left(\frac{\pi}{2} - x\right) dx$ $\qquad$ let $y = \frac{\pi}{2} - x$

$\qquad dy = -dx$

$= \int_{\pi/2}^{\pi/6} \operatorname{cosec}^4 y \, . -dy$ $\qquad$ when $x = 0$, $y = \frac{\pi}{2}$

$\qquad$ when $x = \frac{\pi}{3}$, $y = \frac{\pi}{6}$

$= \int_{\pi/6}^{\pi/2} \operatorname{cosec}^4 y \, dy$

$I_4 = \frac{1}{3}\left(\sqrt{3} \times 2^2 + 2 \int_{\pi/2}^{\pi/6} \operatorname{cosec}^2 y\right)$

$= \frac{1}{3}\left(\sqrt{3} \times 2^2 + 2\left(\sqrt{3} \times 2^0 + 0 I_0\right)\right)$

$= \frac{1}{3}\left(4\sqrt{3} + 2\sqrt{3}\right)$

$= 2\sqrt{3}$

8b (i)

Horizontally $N_1 \sin\theta - \mu N_1 \cos\theta = X$ ① ✓ Horizontally $N_2 \sin\theta + \mu N_2 \cos\theta = kX$ ③

Vertically $N_1 \cos\theta + \mu N_1 \sin\theta = mg$ ② Vertically $N_2 \cos\theta - \mu N_2 \sin\theta = mg$ ④ ✓

$$\frac{②}{①}: \quad \frac{mg}{X} = \frac{N_1 \cos\theta + \mu N_1 \sin\theta}{N_1 \sin\theta - \mu N_1 \cos\theta} \qquad \frac{④}{③}: \quad \frac{mg}{kX} = \frac{N_2 \cos\theta - \mu N_2 \sin\theta}{N_2 \sin\theta + \mu N_2 \cos\theta}$$

$$\frac{mg}{X} = \frac{\cos\theta + \mu \sin\theta}{\sin\theta - \mu\cos\theta} \quad ⑤ \qquad \frac{mg}{X} = \frac{k(\cos\theta - \mu\sin\theta)}{\sin\theta + \mu\cos\theta} \quad ⑥ \quad ✓$$

Equating ⑤ and ⑥

$$\frac{\cos\theta + \mu\sin\theta}{\sin\theta - \mu\cos\theta} = \frac{k(\cos\theta - \mu\sin\theta)}{\sin\theta + \mu\cos\theta} \quad ✓$$

$$(\cos\theta + \mu\sin\theta)(\sin\theta + \mu\cos\theta) = k(\cos\theta - \mu\sin\theta)(\sin\theta - \mu\cos\theta)$$

$$\sin\theta\cos\theta + \mu\cos^2\theta + \mu\sin^2\theta + \mu^2\sin\theta\cos\theta = k(\sin\theta\cos\theta - \mu\cos^2\theta - \mu\sin^2\theta + \mu^2\sin\theta\cos\theta)$$

$$\sin\theta\cos\theta(1+\mu^2) + \mu = k(\sin\theta\cos\theta - \mu + \mu^2\sin\theta\cos\theta) \quad ✓$$

$$\mu + \mu k = \sin\theta\cos\theta\left(k + k\mu^2 - (1+\mu^2)\right)$$

$$\mu(k+1) = \sin\theta\cos\theta\left(k(1+\mu^2) - 1(1+\mu^2)\right)$$

$$\therefore (k-1)(1+\mu^2)\sin\theta\cos\theta = \mu(k+1) \quad \text{as req'd.}$$

8b (ii)

$$2(k-1)\sin\theta\cos\theta(1+\mu^2) = 2\mu(k+1) \quad ✓$$

$$(k-1)\sin 2\theta\,(1+\mu^2) = 2\mu(k+1)$$

$$\sin 2\theta = \frac{2\mu(k+1)}{(k-1)(1+\mu^2)}$$

But $\sin 2\theta \le 1$

$$\therefore \frac{2\mu(k+1)}{(k-1)(1+\mu^2)} \le 1 \quad ✓$$

$$2\mu(k+1) \le (k-1)(1+\mu^2)$$

$$2\mu + 1 + \mu^2 \le k(1 + \mu^2 - 2\mu)$$

$$(\mu+1)^2 \le k(\mu-1)^2 \quad ✓$$

$$k \ge \frac{(\mu+1)^2}{(\mu-1)^2} \quad \text{or} \quad \frac{(\mu+1)^2}{(1-\mu)^2}$$