BAULKHAM HILLS HIGH SCHOOL

2009

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value
- Start a separate piece of paper for each question.
- Put your student number and the question number at the top of each sheet.

Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each question on a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.

Question 1 ***(15 marks)*** Use a *separate* piece of paper — ***Marks***

a) Evaluate $\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx$ — *2*

b) Use integration by parts to evaluate $\int_0^1 x\tan^{-1} x\,dx$ — *3*

c) (i) Express $\frac{10+x-x^2}{(x+1)(x^2+3)}$ in the form $\frac{A}{x+1}+\frac{Bx+C}{x^2+3}$ — *3*

(ii) Hence find $\int \frac{10+x-x^2}{(x+1)(x^2+3)}\,dx$ — *2*

d) Find $\int \frac{dx}{\sqrt{7-6x-x^2}}$ — *2*

e) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1-\tan x}{1+\tan x}\,dx$ — *3*

Question 2 ***(15 marks)*** Use a *separate* piece of paper

a) Let $z=1+2i$ and $\omega=3+i$. Find $\frac{1}{z\omega}$ in the form $x+iy$. — *2*

b) Find the real numbers *a* and *b* such that $(a+bi)^2=9+40i$ — *3*

c) (i) Determine the modulus and argument of $-1+i$ — *2*

(ii) Hence find the least positive integer value of *n* for which $(-1+i)^n$ is real. — *1*

d) If $z=x+iy$, describe the locus of *z* if $2|z|=z+\overline{z}+4$ — *2*

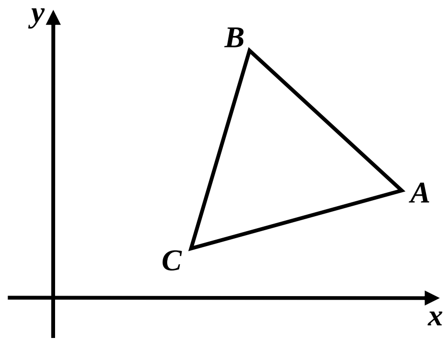
e) On an Argand diagram, sketch the region specified by both the conditions — *3*

$$|z+3-4i|\le 5 \qquad \text{and} \qquad \mathrm{Re}(z)\le 1$$

You must show the intercepts with the axes, but you do not need to find other points of intersection.

Question 2 ***(continued)*** ***Marks***

f) *2*

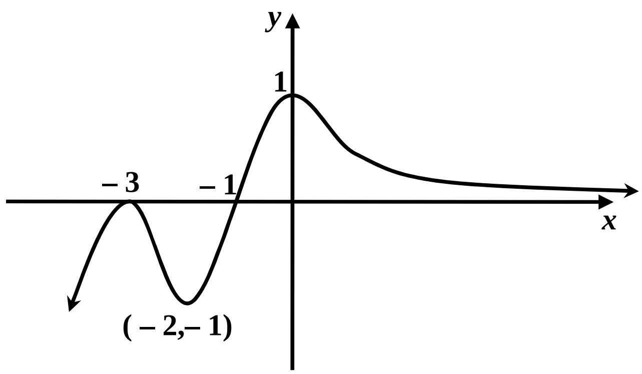


The points A, B and C in the Argand diagram are the vertices of an equilateral triangle described anticlockwise. Show that the complex numbers a, b and c representing A, B and C satisfy;

$$2c = (a+b) + i\sqrt{3}(b-a)$$

<u>**Question 3**</u> ***(15 marks)*** Use a *separate* piece of paper

a) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following;

(i) $y = f(|x|)$ *1*

(ii) $y = f(1-x)$ *1*

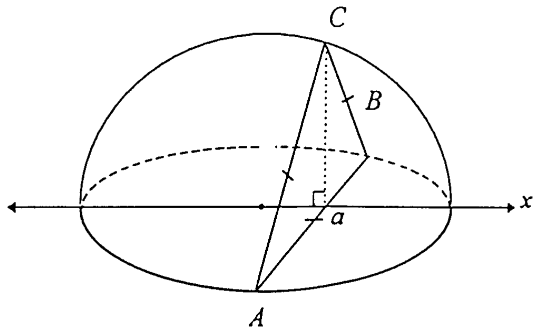
(iii) $y = \dfrac{1}{f(x)}$ *2*

(iv) $y = \sqrt{f(x)}$ *2*

(v) $y = \ln f(x)$ *2*

Question 3*(continued)* ***Marks***

b)



The solid shape above has a circular base of radius 3 units in the horizontal plane. Vertical cross-sections perpendicular to the diameter along the x axis are equilateral triangles as shown in the diagram.

(i) A vertical slice of width Δx is positioned as the point $x = a$. *2*
If its volume is denoted by ΔV, show that $\Delta V = \sqrt{3}\left(9 - x^2\right)\Delta x$

(ii) Hence determine the volume of the solid. *2*

c) If ω is the root of $\omega^5 - 1 = 0$ with the smallest positive argument, find the quadratic equation with roots $\omega + \omega^4$ and $\omega^2 + \omega^3$ *3*

<u>Question 4</u> ***(15 marks)*** Use a *separate* piece of paper

a) Find all roots of the equation $2x^3 - 5x^2 - 4x + 12 = 0$ given that two of the roots are equal. *3*

b) Consider the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b > 0$.

(i) Show that the tangent to the ellipse at the point $P(a\cos\theta, b\sin\theta)$ has the equation $bx\cos\theta + ay\sin\theta - ab = 0$. *2*

(ii) Prove $b^2\cos^2\theta + a^2\sin^2\theta = a^2\left(1 - e^2\cos^2\theta\right)$, where e is the eccentricity. *2*

(iii) R and R' are the feet of the perpendiculars from the foci S and S' on to the tangent at P. Show that $SR \cdot S'R' = b^2$. *3*

c) (i) Prove that $\tan^{-1} n - \tan^{-1}(n-1) = \tan^{-1}\dfrac{1}{n^2 - n + 1}$, where n is a positive integer. *2*

(ii) Hence evaluate $\tan^{-1}1 + \tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{7} + \ldots + \tan^{-1}\dfrac{1}{n^2 - n + 1}$ *2*

(iii) Hence find $\displaystyle\sum_{n=1}^{\infty} \tan^{-1}\dfrac{1}{n^2 - n + 1}$ *1*

<u>Question 5</u> ***(15 marks)*** Use a *separate* piece of paper **_Marks_**

a) The zeros of $x^3 - 3x^2 - 2x + 4$ are α, β and γ

(i) Find a cubic polynomial whose zeros are α^2, β^2 and γ^2 *2*

(ii) Hence, or otherwise, find the value of $\alpha^2 + \beta^2 + \gamma^2$ *1*

(iii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$ *2*

b) A particle of unit mass is thrown vertically downwards with an initial velocity of *u*. It experiences a resistive force of magnitude kv^2 where *v* is its velocity.

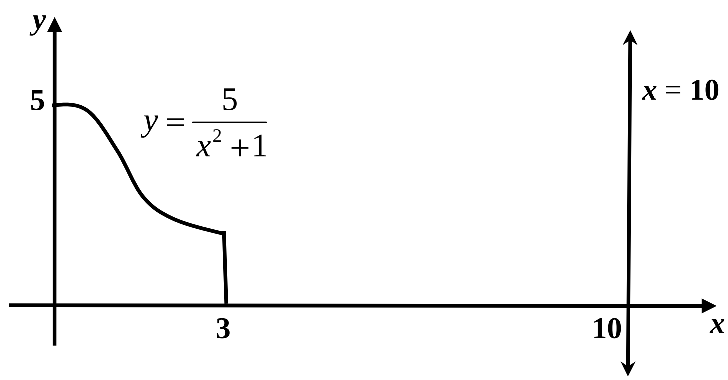
Let *V* be the terminal velocity of the particle.

(i) Show that $V = \sqrt{\dfrac{g}{k}}$, where *g* is the acceleration due to gravity. *2*

(ii) Show that $v^2 = V^2 + \left(u^2 - V^2\right)e^{-2kx}$. *3*

c) A circular flange is formed by rotating the region bounded by the curve $y = \dfrac{5}{x^2+1}$,

the *x* axis and the lines *x* = 0 and *x* = 3 through one complete revolution about the line *x* = 10. All measurements are in centimetres.



(i) Use the method of cylindrical shells to show the volume V cm^3 of the flange is given by $V = 10\pi \int_0^3 \dfrac{10-x}{x^2+1}\,dx$ *3*

(ii) Hence find the volume of the flange to the nearest cm^3. *2*

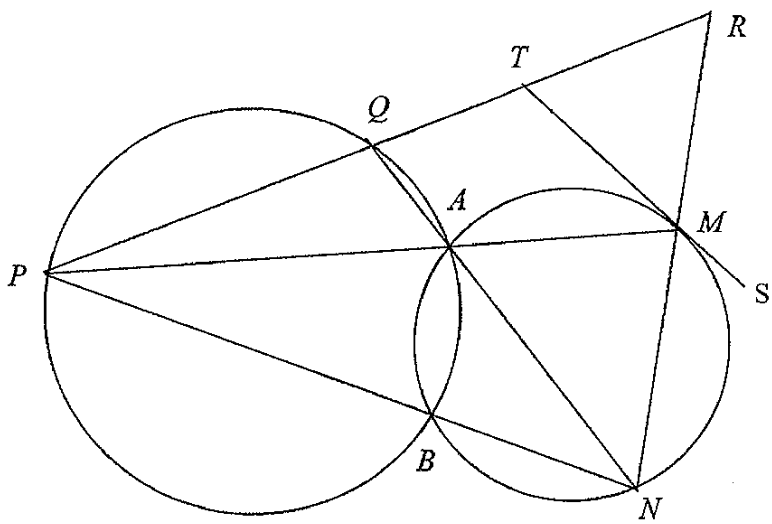
<u>Question 6</u> ***(15 marks)*** Use a *separate* piece of paper **Marks**

a) (i) Show that the tangent to the rectangular hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ has the equation $x + t^2 y = 2ct$. *2*

(ii) The tangents to the rectangular hyperbola $xy = c^2$ at the points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, intersect at *R*. Show that *R* has the coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. *3*

(iii) It is known that *P* and *Q* are variable points on the hyperbola which move so that $pq = 1$. Find the locus of *R* and state any restrictions on the values of *x* for this locus. *2*

b)



In the diagram, the two circles intersect at *A* and *B*. *P* is a point on one circle. *PA* and *PB* produced meet the other circle at *M* and *N* respectively. *NA* produced meets the first circle at *Q*. *PQ* and *NM* produced meet at *R*. The tangent at *M* to the second circle meets *PR* at *T*.

(Note: on page 10 there is a copy of the above diagram that you can use, please detach and include with your solutions)

(i) Show that ***QAMR*** is a cyclic quadrilateral *3*

(ii) Show that $TM = TR$ *3*

c) The continued surd $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\ldots\ldots}}}}} = L$ *2*

Find the exact value of *L*.

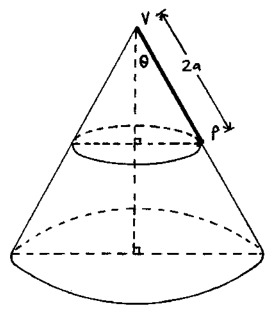
Question 7 ***(15 marks)*** Use a *separate* piece of paper ***Marks***

a) The gradient $\frac{dy}{dx}$ of a curve at a point (x, y) satisfies $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$

(i) By differentiating with respect to x, show that either $\frac{d^2y}{dx^2} = 0$ or $2\frac{dy}{dx} = x$ *2*

(ii) Hence show that the curve is either a straight line or a parabola. *2*

b) A circular cone of semi vertical angle θ is fixed with its vertex upwards as shown. A particle P, of mass $2m$ kg, is attached to the vertex V by a light inextensible string of length $2a$ metres.



The particle P rotates with uniform velocity ω radians/second in a horizontal circle on he outside surface of the cone and in contact with it.

(i) Show that the tension (T) in the sting is equal to $2m\left(g\cos\theta + r\omega^2\sin\theta\right)$ *2*

(ii) Find the normal force (N) on P sting is equal to $2m\left(g\sin\theta - r\omega^2\cos\theta\right)$ *2*

(iii) Show that, for the particle to remain in uniform circular motion on the surface of the cone, then $\omega < \sqrt{\frac{g}{2a\cos\theta}}$ where g is acceleration due to gravity. *2*

c) (i) Show that $\int_0^{\frac{\pi}{4}} \sec x\,dx = \ln\left(\sqrt{2}+1\right)$ *1*

(ii) Let $I_n = \int_0^{\frac{\pi}{4}} \sec^n\theta\,d\theta$, show that $I_n = \frac{1}{n-1}\left(\left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}\right)$ for $n \geq 2$ *3*

(iii) Hence find I_3 *1*

<u>Question 8</u> ***(15 marks)*** Use a *separate* piece of paper ***Marks***

a) A three digit number has a hundreds digit of *a*, a tens digit of *b* and a units digit of *c*. *2*

If $a + b + c$ is divisible by 3, show that the three digit number is divisible by 3.

b) (i) Use de Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ *1*

(ii) Hence solve $8x^3 - 6x - 1 = 0$ *2*

(iii) Deduce that $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$ *2*

c) The Fibonacci sequence of numbers, $F_1, F_{2,\ldots,}$ is defined by $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

(i) Prove that $F_{2n+3}F_{2n+1} - F_{2n+2}^2 = -F_{2n+2}F_{2n} + F_{2n+1}^2$. *2*

(ii) Prove by induction that $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1$, for all positive integers. *3*

(iii) Hence deduce that $F_{2n}^2 + 1$ is divisible by F_{2n+1} *1*

(iv) Prove that $F_{2n-1}^2 + 1$ is divisible by F_{2n+1}. *2*

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \ n \neq -1; x \neq 0, \text{ if } \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \ a > 0, \ -a < x < a$$

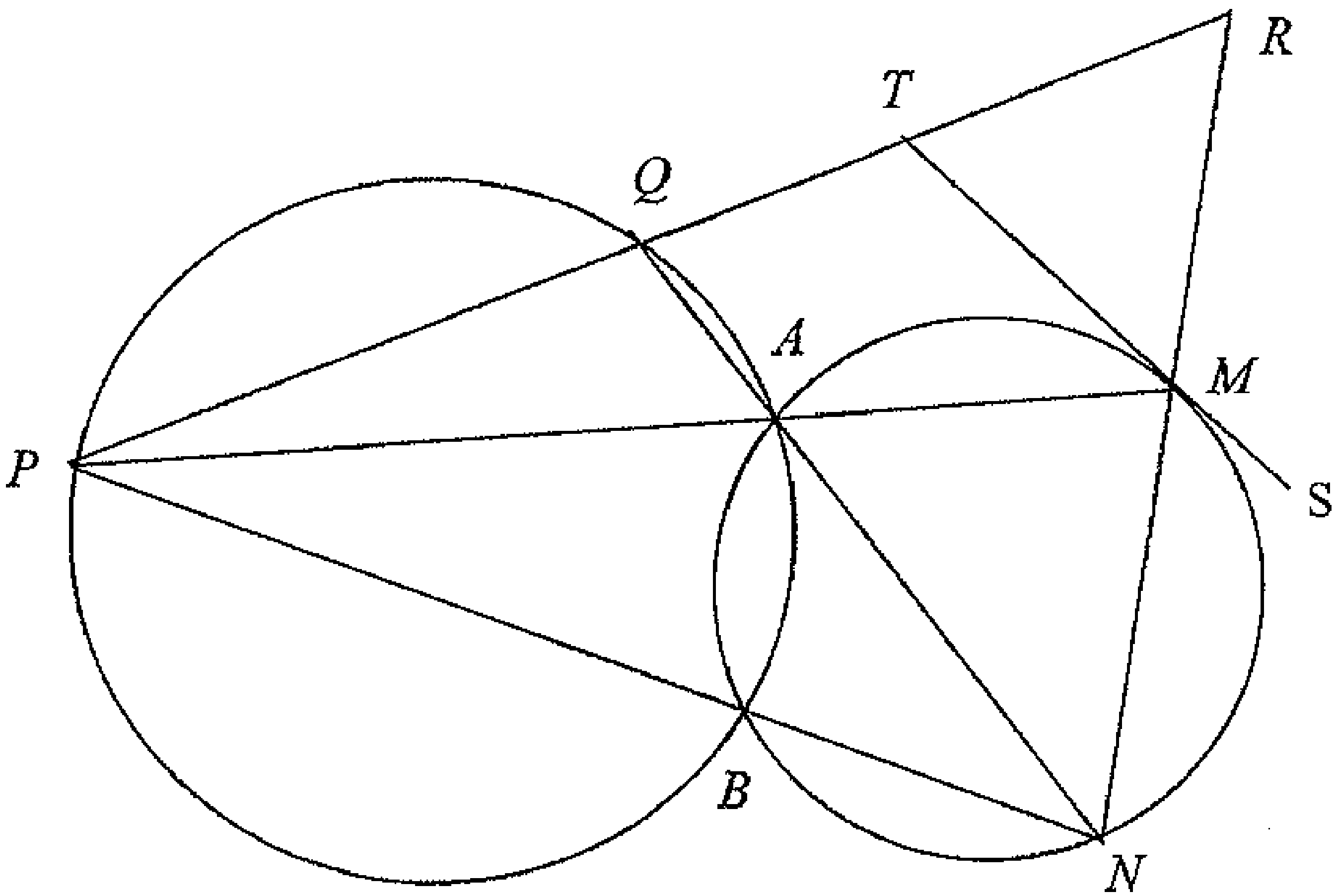
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log x, \quad x > 0$

Question 6 b)

Please detach and include with your solutions.



Baulkham Hills Extension 2 Trial Solutions 2009

Question 1

a) $\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int_0^{\sqrt{3}} \frac{2x}{\sqrt{x^2+1}}\,dx$

$= \left[\sqrt{x^2+1}\right]_0^{\sqrt{3}}$

$= \sqrt{4}-\sqrt{1}$

$= \underline{\underline{1}}$ (2)

b) $\displaystyle\int_0^1 x\tan^{-1}x\,dx \qquad u=\tan^{-1}x \qquad v=\tfrac{1}{2}x^2$

$\qquad du=\dfrac{dx}{1+x^2} \qquad dv=x\,dx$

$= \left[\tfrac{1}{2}x^2\tan^{-1}x\right]_0^1 - \tfrac{1}{2}\displaystyle\int_0^1 \frac{x^2}{1+x^2}\,dx$

$= \dfrac{\pi}{8} - \tfrac{1}{2}\displaystyle\int_0^1 \left(1-\frac{1}{1+x^2}\right)dx$

$= \dfrac{\pi}{8} - \tfrac{1}{2}\left[x-\tan^{-1}x\right]_0^1$

$= \dfrac{\pi}{8} - \dfrac{1}{2} + \dfrac{\pi}{8}$

$= \underline{\underline{\dfrac{\pi}{4}-\dfrac{1}{2}}} \quad \left(=\dfrac{\pi-2}{4}\right)$ (3)

c)(i) $A(x^2+3)+(Bx+C)(x+1) = 10+x-x^2$

$x=-1$: $4A=8$, $A=2$

$x=0$: $3A+C=10$, $C=4$

$x=1$: $4A+2B+2C=10$, $2B=-6$, $B=-3$

$\therefore \dfrac{10+x-x^2}{(x+1)(x^2+3)} = \underline{\dfrac{2}{x+1}+\dfrac{4-3x}{x^2+3}}$ (3)

(ii) $\displaystyle\int \frac{10+x-x^2}{(x+1)(x^2+3)}\,dx = \int\left[\frac{2}{x+1}+\frac{4-3x}{x^2+3}\right]dx$

$= \displaystyle\int\left[\frac{2}{x+1}+\frac{4}{x^2+3}-\frac{3}{2}\cdot\frac{2x}{x^2+3}\right]dx$

$= \underline{2\log(x+1)+\dfrac{4}{\sqrt{3}}\tan^{-1}\dfrac{x}{\sqrt{3}}-\dfrac{3}{2}\log(x^2+3)+c}$ (2)

d) $\displaystyle\int \frac{dx}{\sqrt{7-6x-x^2}} = \int \frac{dx}{\sqrt{16-(x+3)^2}}$

$= \underline{\sin^{-1}\left(\dfrac{x+3}{4}\right)+c}$ (2)

e) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1-\tan x}{1+\tan x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x-\sin x}{\cos x+\sin x}\,dx$

$= \left[\log(\cos x+\sin x)\right]_0^{\frac{\pi}{2}}$

$= \log 1 - \log 1$

$= \underline{\underline{0}}$ (3)

Question 2

a) $zw = (1+2i)(3+i)$

$= 1+7i$

$\dfrac{i}{zw} = \dfrac{\overline{zw}}{|zw|^2}$

$= \underline{\underline{\dfrac{1-7i}{50}}} \quad \left(=\dfrac{1}{50}-\dfrac{7}{50}i\right)$ (2)

b) $a^2-b^2=9 \qquad 2ab=40$

$a^2-\dfrac{400}{a^2}=9 \qquad b=\dfrac{20}{a}$

$a^4-9a^2-400=0$

$(a^2-25)(a^2+16)=0$

$a^2=25$ or $a^2=-16$

$a=\pm 5$ — no real solutions

$\therefore \underline{\underline{a=5, b=4 \text{ or } a=-5, b=-4}}$ (3)

OR

$a=\pm\sqrt{\dfrac{9+|9+40i|}{2}} \qquad b=\dfrac{40}{2(5)}$

$=\pm\sqrt{\dfrac{9+41}{2}} \qquad = 4$

$=\pm 5$

$\therefore \underline{\underline{a=5, b=4 \text{ or } a=-5, b=-4}}$

c)(i) $|-1+i|$

$= \sqrt{1^2+1^2}$

$= \underline{\underline{\sqrt{2}}}$

$\arg(-1+i)$

$= \tan^{-1}\left(\dfrac{1}{-1}\right)$

$= \underline{\underline{\dfrac{3\pi}{4}}}$ (2)

(ii) $(-1+i)^n = \sqrt{2}^n \operatorname{cis}\dfrac{3\pi n}{4}$

for a real number argument must be an integer multiple of π

ie $\dfrac{3\pi n}{4} = k\pi \qquad k$ integer

$\dfrac{3}{4}n = k$

$\therefore \underline{\underline{n=4}}$ (1)

d) $2|z| = z+\bar{z}+4$

$2\sqrt{x^2+y^2} = 2x+4$

$\sqrt{x^2+y^2} = x+2$

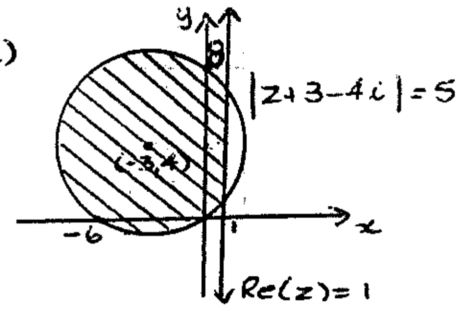
$x^2+y^2 = x^2+4x+4$

$y^2 = 4x+4$

$y^2 = 4(x+1)$

$\therefore$ \underline{locus is the parabola $y^2=4(x+1)$} (2)

e)



③

f) $c-a = \text{cis}\frac{\pi}{3}(b-a)$

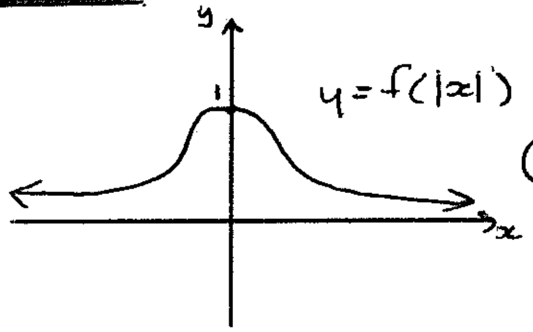
$= \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(b-a)$

$2c-2a = (b-a)+i\sqrt{3}(b-a)$

$\underline{2c = (a+b)+i\sqrt{3}(b-a)}$ ②

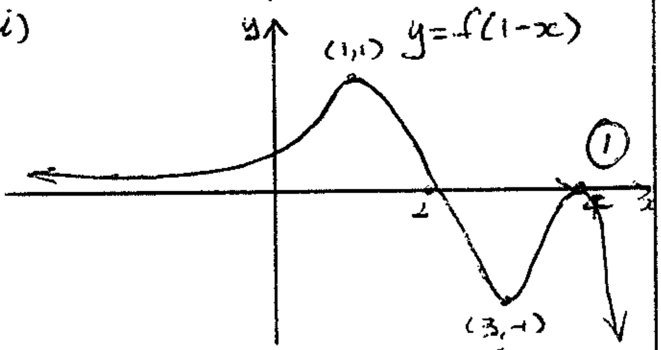
Question 3

a) (i)



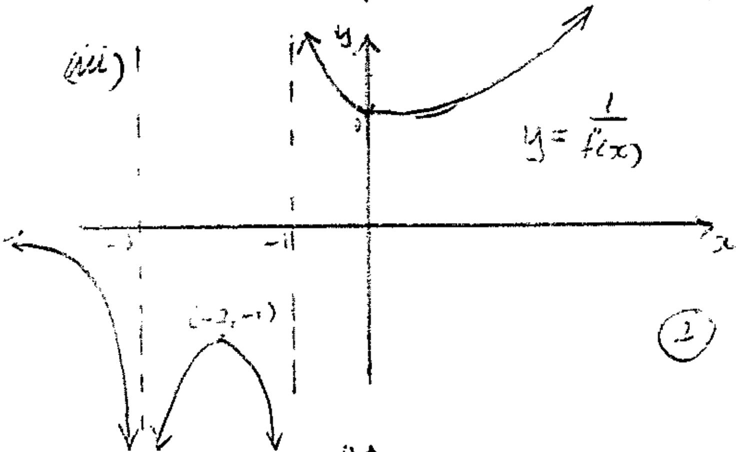
①

(ii)



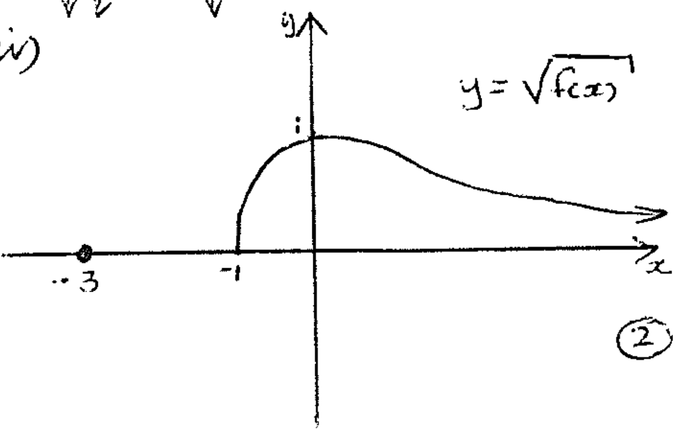
①

(iii)



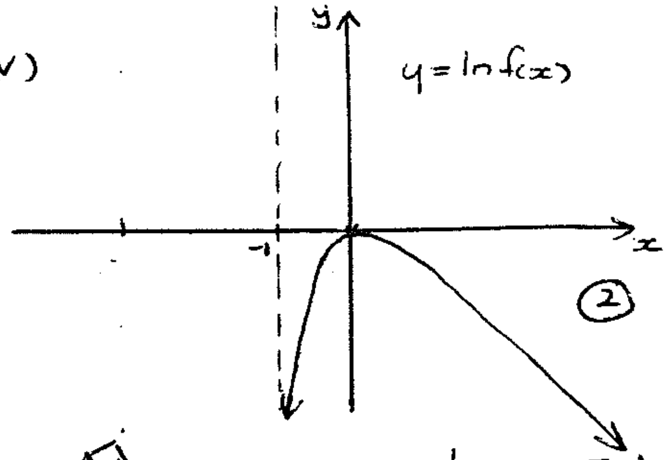
②

(iv)



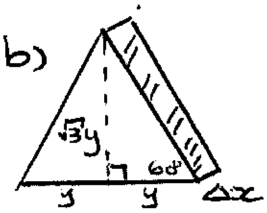
②

(v)



②

b)



$A(x) = \frac{1}{2}(2y)(\sqrt{3}y)$

$= \sqrt{3}y^2$

$= \sqrt{3}(9-x^2)$ ②

$\underline{\Delta V = \sqrt{3}(9-x^2)\Delta x}$

(ii) $V = \lim_{\Delta x\to 0}\sum_{x=-3}^{3}\sqrt{3}(9-x^2)\Delta x$

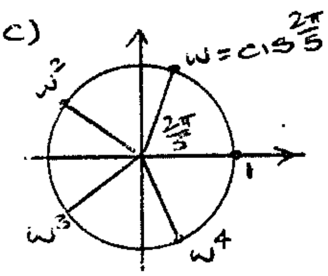
$= 2\sqrt{3}\int_0^3 (9-x^2)\,dx$

$= 2\sqrt{3}\left[9x-\frac{1}{3}x^3\right]_0^3$

$= 2\sqrt{3}(18)$

$= \underline{36\sqrt{3}\text{ units}^3}$ ②

c)



$z^5-1=0$

$1+w+w^2+w^3+w^4=0$

(sum of roots)

$\alpha+\beta = w+w^4+w^2+w^3$

$= -1$

$\alpha\beta = (w+w^4)(w^2+w^3)$

$= w^3+w^4+w^6+w^7$

$= w^3+w^4+w+w^4 \quad (\because w^5=1)$

$= -1$

$x^2-(\alpha+\beta)x+\alpha\beta=0$ ③

$\underline{x^2+x-1=0}$

Question 4

a) $P(x)=2x^3-5x^2-4x+12$

$P'(x) = 6x^2-10x-4$

$= 2(3x+1)(x-2)$

$\therefore$ double root is either $x=-\frac{1}{3}$ or $x=2$

$P(2)=0 \quad \therefore x=2$ is double root

$2x^3-5x^2-4x+12$

$=(x-2)^2(2x-3)$

$\therefore$ roots are $2, 2$ and $\frac{3}{2}$ (3)

b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$

at P, $\frac{dy}{dx}=\frac{-ab^2\cos\theta}{a^2b\sin\theta}$

$=\frac{-b\cos\theta}{a\sin\theta}$

$y-b\sin\theta=\frac{-b\cos\theta}{a\sin\theta}(x-a\cos\theta)$

$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$

$bx\cos\theta+ay\sin\theta-ab(\sin^2\theta+\cos^2\theta)=0$

$bx\cos\theta+ay\sin\theta-ab=0$ (2)

(ii) $b^2\cos^2\theta+a^2\sin^2\theta$

$=a^2(1-e^2)\cos^2\theta+a^2\sin^2\theta$

$=a^2(\cos^2\theta-e^2\cos^2\theta+\sin^2\theta)$

$=a^2(1-e^2\cos^2\theta)$ (2)

(iii) $SR=\frac{|abe\cos\theta+0-ab|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$

$=\frac{|abe\cos\theta-ab|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$

$S'R'=\frac{|-abe\cos\theta-ab|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$

$SR\cdot S'R'=\frac{|a^2b^2-a^2b^2e^2\cos^2\theta|}{b^2\cos^2\theta+a^2\sin^2\theta}$

$=\frac{a^2b^2|1-e^2\cos^2\theta|}{a^2(1-e^2\cos^2\theta)}$

$=b^2$ (3)

NOTE: $e^2<1$, $\cos^2\theta<1$

$\therefore 1-e^2\cos^2\theta>0$.

c)(i) let $\alpha=\tan^{-1}n$

$\beta=\tan^{-1}(n-1)$

$\tan(\alpha+\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$=\frac{n-(n-1)}{1+n(n-1)}$

$=\frac{1}{n^2-n+1}$

$\therefore \alpha-\beta=\tan^{-1}\left(\frac{1}{n^2-n+1}\right)$ (2)

ie $\tan^{-1}n-\tan^{-1}(n-1)=\tan^{-1}\left(\frac{1}{n^2-n+1}\right)$

(ii) $\tan^{-1}1+\tan^{-1}\frac{1}{3}+\ldots+\tan^{-1}\frac{1}{n^2-n+1}$

$n=1;\ \tan^{-1}1=\tan^{-1}1-\tan^{-1}0$

$n=2;\ \tan^{-1}\frac{1}{3}=\tan^{-1}2-\tan^{-1}1$

$n=3;\ \tan^{-1}\frac{1}{7}=\tan^{-1}3-\tan^{-1}2$

$\tan^{-1}1+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\ldots+\tan^{-1}\frac{1}{n^2-n+1}$

$=\tan^{-1}1-\tan^{-1}0+\tan^{-1}2-\tan^{-1}1+\tan^{-1}3-\tan^{-1}2+\ldots+\tan^{-1}n-\tan^{-1}(n-1)$

$=\tan^{-1}n-\tan^{-1}0$

$=\tan^{-1}n$ (2)

(iii) $\sum_{n=1}^{\infty}\tan^{-1}\frac{1}{n^2-n+1}$

$=\lim_{n\to\infty}\tan^{-1}n$

$=\frac{\pi}{2}$ (1)

Question 5

a) x^3-3x^2-2x+4

let $y=x^2$

$x=y^{\frac{1}{2}}$

$y^{\frac{3}{2}}-3y-2y^{\frac{1}{2}}+4=0$

$y^{\frac{1}{2}}(y-2)=3y-4$

$y(y^2-4y+4)=9y^2-24y+16$

$y^3-4y^2+4y=9y^2-24y+16$

$y^3-13y^2+28y-16=0$ (2)

$\therefore$ polynomial is $y^3-13y^2+28y-16$

(ii) $\alpha^2+\beta^2+\gamma^2=13$ (1)

(iii) $\sum\alpha^3-3\sum\alpha^2-2\sum\alpha+12=0$

$\sum\alpha^3=3\sum\alpha^2+2\sum\alpha-12$

$=3(13)+2(3)-12$

$=33$ (2)

b)

$m\ddot{x} = mg - kv^2$

$m=1$

$\ddot{x} = g - kv^2$

terminal velocity occurs when $\ddot{x} = 0$

$\therefore 0 = g - kV^2$

$V^2 = \frac{g}{k}$

$\underline{\underline{V = \sqrt{\frac{g}{k}}}}$ ②

(iii) $v\frac{dv}{dx} = g - kv^2$

$\frac{dv}{dx} = \frac{g - kv^2}{v}$

$\frac{dx}{dv} = \frac{v}{g - kv^2}$

$\int_0^x dx = -\frac{1}{2k}\int_u^v \frac{-2kv}{g - kv^2}$

$x = -\frac{1}{2k}\left[\log(g - kv^2)\right]_u^v$

$-2kx = \log(g - kv^2) - \log(g - ku^2)$

$= \log\left(\frac{g - kv^2}{g - ku^2}\right)$

$e^{-2kx} = \frac{g - kv^2}{g - ku^2}$

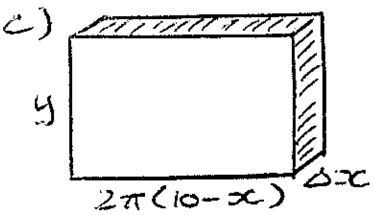
$g - kv^2 = (g - ku^2)e^{-2kx}$

$kv^2 = g + (ku^2 - g)e^{-2kx}$

$v^2 = \frac{g}{k} + \left(u^2 - \frac{g}{k}\right)e^{-2kx}$ ③

$\underline{v^2 = V^2 + (u^2 - V^2)e^{-2kx}}$

c)



$A(x) = 2\pi(10-x)y$

$= \frac{10\pi(10-x)}{x^2+1}$

$\Delta V = \frac{10\pi(10-x)}{x^2+1}\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=0}^{3} \frac{10\pi(10-x)}{x^2+1}\Delta x$

$\underline{\underline{= 10\pi\int_0^3 \frac{10-x}{x^2+1}\,dx}}$ ③

(ii) $V = 10\pi\int_0^3\left[\frac{10}{x^2+1} - \frac{1}{2}\cdot\frac{2x}{x^2+1}\right]dx$

$= 10\pi\left[10\tan^{-1}x - \frac{1}{2}\log(x^2+1)\right]_0^3$

$= 10\pi\left(10\tan^{-1}3 - \frac{1}{2}\log 10 - 0\right)$

$= 356.2303802$

$\underline{\underline{= 356\text{cm}^3}}$ (to nearest cm^3)

Question 6

a) $xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

when $x = ct$, $\frac{dy}{dx} = \frac{-c^2}{c^2t^2}$

$= -\frac{1}{t^2}$

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2y - ct = -x + ct$

$\underline{\underline{x + t^2y = 2ct}}$ ②

(ii) $x + p^2y = 2cp$

$x + q^2y = 2cq$

$(p^2 - q^2)y = 2c(p-q)$

$y = \frac{2c(p-q)}{(p-q)(p+q)}$

$y = \frac{2c}{p+q}$

$x + \frac{2cp^2}{p+q} = 2cp$

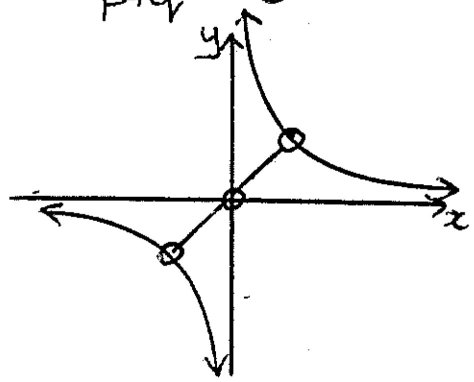
$x = \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$

$= \frac{2cpq}{p+q}$

$\therefore$ $\underline{R \text{ is } \left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)}$ ③

(iii) $pq = 1$

$x = \frac{2c}{p+q} = y$

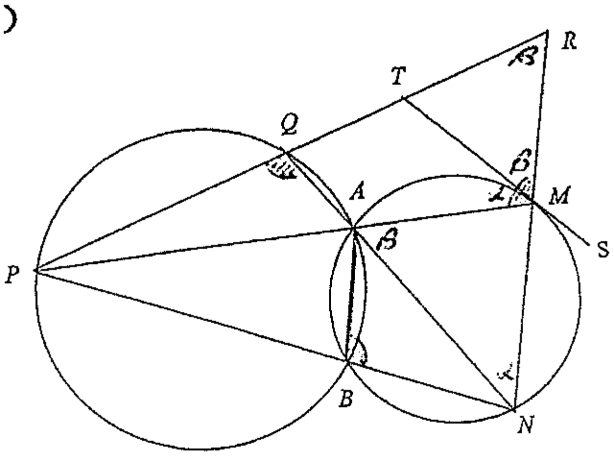


but if $pq = 1$, then P and Q must be on the same branch of the hyperbola.
Thus the tangents can only meet in the same quadrant as the branch they lie on.

Also tangents cannot meet inside hyperbola.

$\therefore$ locus is the line $y = x$ in the domain $-c < x < 0$ and $0 < x < c$ (2)

b)



(i) $\angle RMA = \angle ABN$ (exterior $\angle$ cyclic quadrilateral MABN)

$\angle ABN = \angle PQN$ (exterior $\angle$ cyclic quadrilateral BAQP)

$\therefore \angle PQN = \angle RMA$

$\therefore$ QAMR is cyclic quadrilateral as exterior $\angle$ = opposite interior $\angle$. (3)

(ii) let $\angle TMA = \alpha$
$\angle MNA = \alpha$ (alternate segment thm)

let $\angle RMT = \beta$
$\therefore \angle RMA = \alpha + \beta$ (common $\angle$)
$\angle RMA = \angle MNA + \angle MAN$ (exterior $\angle$, ΔMAN)
$\alpha + \beta = \alpha + \angle MAN$
$\angle MAN = \beta$

$\angle MAN = \angle QRM$ (exterior $\angle$ cyclic quadrilateral AQRM)

$\therefore \angle QRM = \beta = \angle RMT$

ΔRMT is isosceles (2 = $\angle$'s)

$\therefore TM = TR$ (= sides in isosceles ΔRMT) (3)

c) $L = \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\dots}}}}}$

$L^2 = 6 + \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\dots}}}}$

$L^2 = 6 + L$
$L^2 - L - 6 = 0$
$(L-3)(L+2) = 0$
$L = 3$ or $L = -2$
But $L > 0$

$\therefore L = 3$ (2)

Question 7

a)(i) $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$

$2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - (x)\left(\frac{d^2y}{dx^2}\right) - \left(\frac{dy}{dx}\right)(1) + \frac{dy}{dx} = 0$

$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - x\frac{d^2y}{dx^2} = 0$

$\frac{d^2y}{dx^2}\left(2\frac{dy}{dx} - x\right) = 0$

$\frac{d^2y}{dx^2} = 0$ or $2\frac{dy}{dx} = x$ (2)

(ii) $\frac{d^2y}{dx^2} = 0$ OR $\frac{dy}{dx} = \frac{x}{2}$

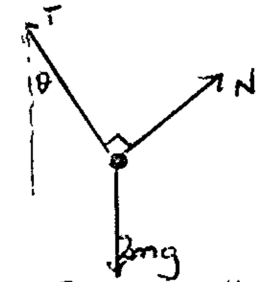
$\frac{dy}{dx} = c$

$y = \frac{1}{4}x^2 + c$

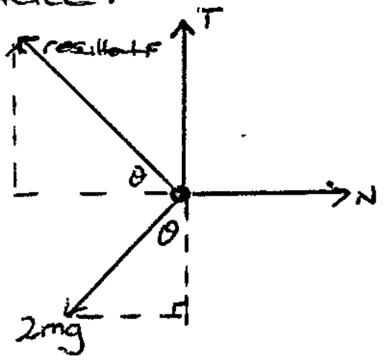
$y = cx + k$

$\therefore$ the curve is either a straight line or a parabola (2)

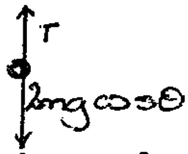
b) (i)



look at forces // and ⊥ to surface.



F // to surface $= 2mr\omega^2 \sin\theta$

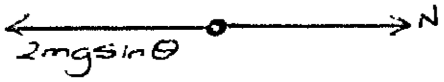


$T + 2mg\cos\theta = 2mr\omega^2 \sin\theta$

$T = 2mg\cos\theta + 2mr\omega^2 \sin\theta$

$\underline{T = 2m(g\cos\theta + r\omega^2\sin\theta)}$ ②

(ii) F ⊥ to surface $= 2mr\omega^2\cos\theta$



$2mg\sin\theta - N = 2mr\omega^2\cos\theta$

$N = 2mg\sin\theta - 2mr\omega^2\cos\theta$

$\underline{N = 2m(g\sin\theta - r\omega^2\cos\theta)}$ ②

(iii) particle will lose contact with surface when $N = 0$

ie

$2m(g\sin\theta - r\omega^2\cos\theta) = 0$

$g\sin\theta - r\omega^2\cos\theta = 0$

$\omega^2 = \dfrac{g\sin\theta}{r\cos\theta}$

but $\dfrac{r}{2a} = \sin\theta$

$r = 2a\sin\theta$

$\omega^2 = \dfrac{g}{2a\cos\theta}$

$\omega = \sqrt{\dfrac{g}{2a\cos\theta}}$

∴ particle will remain in contact when

$\underline{\omega < \sqrt{\dfrac{g}{2a\cos\theta}}}$ ②

c) (i) $\displaystyle\int_0^{\frac{\pi}{4}} \sec x \, dx$

$= \left[\log(\sec x + \tan x)\right]_0^{\frac{\pi}{4}}$

$= \log(\sqrt{2} + 1) - 0$

$= \underline{\log(\sqrt{2} + 1)}$ ①

(ii) $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \sec^n\theta \, d\theta$

$u = \sec^{n-2}\theta$ $\qquad v = \tan\theta$

$du = (n-2)\sec^{n-3}\theta \cdot \sec\theta\tan\theta \, d\theta$ $\qquad dv = \sec^2\theta \, d\theta$

$= (n-2)\sec^{n-2}\theta \tan\theta \, d\theta$

$I_n = \left[\sec^{n-2}\theta\tan\theta\right]_0^{\frac{\pi}{4}} - (n-2)\displaystyle\int_0^{\frac{\pi}{4}} \sec^{n-2}\theta\tan^2\theta \, d\theta$

$= (\sqrt{2})^{n-2} - (n-2)\displaystyle\int_0^{\frac{\pi}{4}} \sec^{n-2}\theta(\sec^2\theta - 1)\, d\theta$

$= (\sqrt{2})^{n-2} - (n-2)\displaystyle\int_0^{\frac{\pi}{4}} (\sec^n\theta - \sec^{n-2}\theta)\, d\theta$

$= (\sqrt{2})^{n-2} - (n-2)I_n + (n-2)I_{n-2}$

$(n-1)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$

$\underline{I_n = \dfrac{1}{n-1}\left((\sqrt{2})^{n-2} + (n-2)I_{n-2}\right)}$ ③

(iii) $I_3 = \frac{1}{2}(\sqrt{2} + I_1)$

$= \dfrac{\sqrt{2}}{2} + \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$

$= \underline{\frac{1}{2}(\sqrt{2} + \log(\sqrt{2} + 1))}$ ①

Question 8

a) $100a + 10b + c$
$= 99a + 9b + a+b+c$
$= 3(33a+3b) + (a+b+c)$
but $a+b+c$ is divisible by 3.
$\therefore$ number is divisible by 3 (2)

b) $(\text{cis}\,\theta)^3 = c^3 + 3ic^2s - 3cs^2 - ic^3$
$\text{cis}\,3\theta =$
equating real parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$
$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$
$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$
$= 4\cos^3\theta - 3\cos\theta$ (1)

(ii) $8x^3 - 6x - 1 = 0$
$2(4x^3 - 3x) - 1 = 0$
let $x = \cos\theta$
$2\cos 3\theta - 1 = 0$
$\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$
$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$\therefore x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$ (2)

(iii) $\cos\frac{\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = 0$ (Sum of roots)
$\cos\frac{\pi}{9} = -\cos\frac{5\pi}{9} - \cos\frac{7\pi}{9}$
but $\cos\frac{5\pi}{9} = \cos(\pi - \frac{4\pi}{9})$
$= -\cos\frac{4\pi}{9}$
$\cos\frac{7\pi}{9} = \cos(\pi - \frac{2\pi}{9})$
$= -\cos\frac{2\pi}{9}$
$\cos\frac{\pi}{9} = \cos\frac{4\pi}{9} + \cos\frac{2\pi}{9}$ (2)

c) (i) $F_{2n+3}F_{2n+1} - F_{2n+2}^2$
$= (F_{2n+2} + F_{2n+1})F_{2n+1} - F_{2n+2}^2$
$= F_{2n+1}F_{2n+2} + F_{2n+1}^2 - F_{2n+2}^2$
$= F_{2n+1}^2 - F_{2n+2}(-F_{2n+1} + F_{2n+2})$
$= F_{2n+1}^2 - F_{2n+2}(-F_{2n+1} + F_{2n+1} + F_{2n})$
$= F_{2n+1}^2 - F_{2n+2}F_{2n}$ (2)

(ii) Prove true for $n = 1$

$LHS = F_3F_1 - F_2^2$
$= 2(1) - 1^2$
$= 1$
Hence the result is true for $n=1$

Assume the result is true for $n=k$ where k is a positive integer

ie $F_{2k+1}F_{2k-1} - F_{2k}^2 = 1$

Prove true for $n=k+1$
ie Prove
$F_{2k+3}F_{2k+1} - F_{2k+2}^2 = 1$

Proof:

$F_{2k+3}F_{2k+1} - F_{2k+2}^2$
$= (F_{2k+2} + F_{2k+1})F_{2k+1} - F_{2k+2}^2$
$= (F_{2k+1} - F_{2k+2})F_{2k+2} + F_{2k+1}^2$
$= (F_{2k+1} - F_{2k+1} - F_{2k})(F_{2k+1} + F_{2k}) + F_{2k+1}^2$
$= F_{2k+1}^2 - F_{2k}F_{2k+1} - F_{2k}^2$
$= F_{2k+1}(F_{2k+1} - F_{2k}) - F_{2k}^2$
$= F_{2k+1}(F_{2k} + F_{2k-1} - F_{2k}) - F_{2k}^2$
$= F_{2k+1}F_{2k-1} - F_{2k}^2$
$= 1$

Hence the result is true for $n=k+1$ if it is true for $n=k$

Since the result is true for $n=1$, then it is true for all positive integral values of n, by induction (3)

(iii) $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1$

$\therefore F_{2n}^2 + 1 = F_{2n+1}F_{2n-1}$ (1)
which is divisible by F_{2n+1}

(iv) $F_{2n-1}^2 + 1$

$= (F_{2n+1} - F_{2n})^2 + 1$

$= F_{2n+1}^2 - 2F_{2n+1}F_{2n} + F_{2n}^2 + 1$

$= F_{2n+1}^2 - 2F_{2n+1}F_{2n} + F_{2n+1}F_{2n+3}$

$= F_{2n+1}(F_{2n+1} - 2F_{2n} + F_{2n+3})$

which is divisible by F_{2n+1} (2)