BAULKHAM HILLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2010

MATHEMATICS

EXTENSION 2

GENERAL INSTRUCTIONS:

- Reading time − 5 minutes.
- Working time − 3 hours.
- Write using blue or black pen.
- Start each question on a new page.
- Write your Student Number at the top of each page.
- Calculators may be used.
- A table of standard integrals is provided.
- **ALL** necessary working should be shown in every question.

QUESTION 1 (15 marks)

Marks

(a) Find each of the following integrals

(i) $\int x^2(1+2x^3)^{-5}dx$ 2

(ii) $\int tan^4 x dx$ 3

(iii) $\int \frac{dx}{3+2\cos x}$ 4

(b) Find $\int \frac{e^x+e^{2x}}{1+e^{2x}}dx$ 3

(c) (i) Show that $\sin(A+B)+\sin(A-B)=2sinAcosB$ 1

(ii) Hence calculate $\int sin5xcos4xdx$ 2

QUESTION 2 (15 marks)

(a) For $z_1=2-3i$ and $z_2=1+5i$ find, in the form *a*+*ib*, the values of

(i) $z_1+\overline{z}_2$ 2

(ii) $z_1 z_2$ 2

(iii) $\frac{z_1}{z_2}$ 2

(b) (i) Solve $(x+iy)^2=6i$ 2

(ii) Hence or otherwise solve $z^2-(1-i)z-2i=0$ 3

(c) (i) Express $z=1-\sqrt{3}i$ in modulus-argument form 2

(ii) Hence express z^6 in the form $a+ib$ 2

QUESTION 3 (15 marks)

Marks

(a) The hyperbola H has the equation $\frac{x^2}{4} - \frac{y^2}{12} = 1$

Find (i) its eccentricity 2

(ii) the coordinates of its foci 1

(iii) the equations of its directrices 1

(iv) the equations of its asymptotes 1

(b) (i) Show that the gradient of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point P$(a\cos\theta, b\sin\theta)$ is $\frac{-b\cos\theta}{a\sin\theta}$ 2

(ii) Hence show that the equation of the tangent is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ 3

(iii) Show that the x-intercept of this tangent is $(\frac{a}{\cos\theta}, 0)$ 1

(iv) Hence, or otherwise, find the points on the curve $4x^2 + 3y^2 = 12$ whose tangent passes through (2,0) 4

QUESTION 4 (15 marks)

(a) OABC is a square on the Argand diagram and is labeled in an anticlockwise direction. A represents $z = a + ib$ and B represents $4 + 7i$.

(i) Find, in terms of a and b, the complex number represented by C. 2

(ii) Hence evaluate a and b. 2

(b) The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ. Find the equation with roots:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ 2

(ii) α^2, β^2 and γ^2 2

(c) (i) Show that 2 is a root of multiplicity 3 for $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$ 2

(ii) Hence solve $P(x) = 0$ 2

(d) Draw on separate argand diagrams the following loci:

(i) $z\bar{z} = 3$ 1

(ii) $\arg\left(\frac{z}{z-1}\right) = \frac{\pi}{3}$ 2

QUESTION 5 (15 marks)

Marks

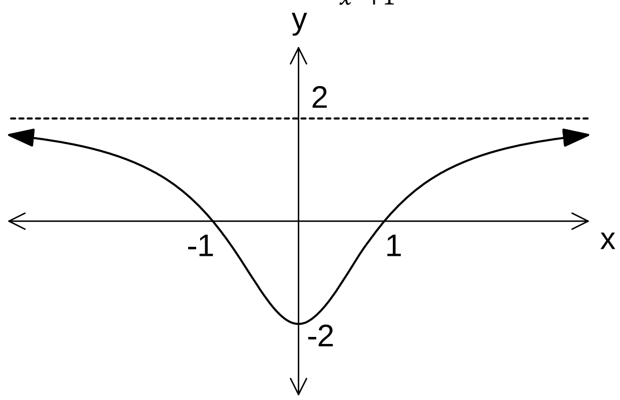
(a) Suppose $x>0, y>0, z>0$

(i) Prove $x^2 + y^2 + z^2 \geq xy + yz + xz$ 2

(ii) Given $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$
prove $x^3 + y^3 + z^3 \geq 3xyz$ 1

(iii) Hence show $a + b + c \geq 3(abc)^{\frac{1}{3}}$ 1

(b) Given below is the graph of $f(x) = 2 - \frac{4}{x^2+1}$.



Use the graph of $y = f(x)$ to sketch, on separate axes, the graphs of

(i) $y = |f(x)|$ 2

(ii) $y = [f(x)]^2$ 2

(iii) $y^2 = f(x)$ 2

(iv) $y = \frac{1}{f(x)}$ 2

(c) For the curve $x^3 + 3x^2y - 2y^3 = 16$

(i) Show that $\frac{dy}{dx} = \frac{x^2+2xy}{2y^2-x^2}$ 1

(ii) Find the coordinates of the stationary points on the curve 2

QUESTION 6 (15 marks)

(a) Find, using slices, the volume generated when the area bounded by $y = x^2$ and the line $y = 3$ is rotated about the line $y = 3$. 4

(b) Find, using cylindrical shells, the volume obtained by revolving about the y-axis the region bounded by the curve $y = sinx$, for $0 \leq x \leq \pi$, and the x-axis. 4

(c) A solid has a semi-circular base whose equation is $= \sqrt{4 - x^2}$. Vertical cross-sections, perpendicular to the diameter, are right-angled triangles whose height is bounded by the parabola $z = 4 - x^2$.

(i) Draw a neat diagram, including a typical slice, representing this information. 1

(ii) By slicing at right angles to the x-axis, show that the volume of the solid is given by $\int_0^2 (4 - x^2)^{\frac{3}{2}} dx$. 3

(iii) Hence calculate this volume. 3

QUESTION 7 (15 marks)

Marks

(a) Use the method of partial fractions to show that $\int_0^1 \frac{6x+4}{(x^2+1)(x+1)} dx = \frac{5\pi}{4} - \frac{1}{2}\log_e 2$ 4

(b) Let $P(z) = z^4 + bz^2 + d$ where b and d are real numbers and $d \neq 0$. *P(z)*has a double zero α.

(i) Prove $P'(z)$ is odd. 2

(ii) Prove that $-\alpha$ is also a double zero of *P(z)*. 2

(c) A mass of 35 kg is dropped from a balloon falling at 30 m/s. The mass experiences air resistance measuring 70*v* Newtons, where *v* m/s is its velocity. Take *g* as 10m/s^2.

(i) Show that the velocity of the mass *t* seconds after being dropped, but before hitting the ground, is given by $v = 5 + 25e^{-2t}$. 3

(ii) Describe what happens to the velocity as $t \to \infty$. 1

(iii) If the mass was dropped from 400m above the ground, how close to the ground will it be after 1 minute? 3

QUESTION 8 (15 marks)

(a) Given $I_n = \int_0^1 x^n e^{-x} dx$

(i) Calculate I_0 2

(ii) Prove $I_n = nI_{n-1} - \frac{1}{e}$ 2

(iii) Hence find $\int_0^1 x^3 e^{-x} dx$ 2

(b) Two particles of equal mass are attached to the ends A and B of a light inextensible string which passes through a small hole at the apex C of a hollow right circular cone fixed with its axis vertical and apex on top. The semi-vertical angle of the cone is θ. The particle at A, where AC is a units, moves in a horizontal circle with constant angular velocity ω on the smooth surface of the cone, while the other particle at B hangs at rest inside the cone.

(i) Represent this information on a diagram showing relevant forces. 1

(ii) Show that $\omega^2 = \frac{g}{a(1+\cos\theta)}$ 2

(iii) Hence, or otherwise, deduce that $\frac{g}{2\omega^2} < a < \frac{g}{\omega^2}$ 2

(c) If $x>0$, prove $x - \frac{1}{3}x^3 < \tan^{-1} x < x - \frac{1}{3}x^3 + \frac{1}{3}x^5$ 4

END OF EXAM

Q1(a)(i) $I=\int x^2(1+2x^3)^{-5}\,dx$

let $u=1+2x^3 \therefore \frac{du}{dx}=6x^2$

$\therefore \frac{du}{6}=x^2dx$

$\therefore I=\frac{1}{6}\int u^{-5}\,du$ — 1

$=\frac{1}{6}\frac{u^{-4}}{-4}+c$

$=-\frac{1}{24}(1+2x^3)^{-4}+c$ — 1 (2)

(ii) $I=\int \tan^4 x\,dx$

$=\int \tan^2 x\tan^2 x\,dx$

$=\int \tan^2 x(\sec^2 x-1)\,dx$ — 1

$=\int \tan^2 x\sec^2 x-\tan^2 x\,dx$ — 1

$=\frac{1}{3}\tan^3 x-\int \sec^2 x-1\,dx$ — 1

$=\frac{1}{3}\tan^3 x-\tan x+x+c$ (3)

(iii) $I=\int\frac{dx}{3+2\cos x}$

$=\int\frac{\frac{2dt}{1+t^2}}{3+2\left(\frac{1-t^2}{1+t^2}\right)}$ — 1

$=\int\frac{2dt}{3+3t^2+2-2t^2}$

$=\int\frac{2dt}{5+t^2}$ — 1

$=\frac{2}{\sqrt5}\tan^{-1}\left(\frac{t}{\sqrt5}\right)+c$ — 1

$=\frac{2}{\sqrt5}\tan^{-1}\left(\frac{1}{\sqrt5}\tan\frac{x}{2}\right)+c$ — 1 (4)

b) $I=\int\frac{e^x}{1+e^{2x}}dx+\frac{1}{2}\int\frac{2e^{2x}}{1+e^{2x}}dx$

$=I_1+\frac{1}{2}\ln(1+e^{2x})+c$

In I_1, let $u=e^x \therefore du=e^x dx$

$\therefore I=\int\frac{du}{1+u^2}+\frac{1}{2}\ln(1+e^{2x})+c$ — 1 — 1

$=\tan^{-1}u+\frac{1}{2}\ln(1+e^{2x})+c$ — 1

$=\tan^{-1}e^x+\frac{1}{2}\ln(1+e^{2x})+c$ (3)

c) LHS $=\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B$

$=2\sin A\cos B=$ RHS — 1

$I=\frac{1}{2}\int 2\sin 5x\cos 4x\,dx$ — 1

$=\frac{1}{2}\int \sin 9x+\sin x\,dx$

$=\frac{1}{2}\left(-\frac{1}{9}\cos 9x-\cos x\right)+c$ — 1 (3)

Q2(a)(i) $z_1+\bar{z}_2=2-3i+1-5i$

$=3-8i$ (2)

(ii) $z_1z_2=(2-3i)(1+5i)$

$=2+10i-3i+15$

$=17+7i$ (2)

(iii) $\frac{z_1}{z_2}=\frac{2-3i}{1+5i}\times\frac{1-5i}{1-5i}$

$=\frac{2-10i-3i-15}{1+25}$

$=\frac{-13-13i}{26}$

$=-\frac{1}{2}-\frac{1}{2}i$ (2)

b) (i) $x^2-y^2+2ixy=6i$

$\therefore x^2-y^2=0$ and $2xy=6$ $\therefore y=\frac{3}{x}$

$\therefore x^2-\frac{9}{x^2}=0$

$x^4=9$

$x^2=\pm3$ But $x^2>0$

$\therefore x^2=3$

$x=\pm\sqrt3,\ y=\pm\sqrt3$ (2)

$\therefore z=\pm(\sqrt3+\sqrt3 i)$

(ii) $z=\frac{(1-i)\pm\sqrt{[-(1-i)]^2-4\times1\times-2i}}{2\times1}$

$=\frac{1-i\pm\sqrt{1-2i-1+8i}}{2}$

$=\frac{1-i\pm\sqrt{6i}}{2}$ — 1

$=\frac{1-i\pm(\sqrt3+\sqrt3 i)}{2}$ — 1 — 1

$=\frac{1+\sqrt3+(\sqrt3-1)i}{2},\ \frac{1-\sqrt3+(-\sqrt3-1)i}{2}$ (3)

c) (i) $z=2\left(\frac{1}{2}-\frac{\sqrt3 i}{2}\right)=2\left(\cos-\frac{\pi}{3}+i\sin-\frac{\pi}{3}\right)$ (2)

(ii) $z^6=\left[2\left(\cos-\frac{\pi}{3}+i\sin-\frac{\pi}{3}\right)\right]^6$

$=2^6(\cos-2\pi+i\sin-2\pi)$

$=64(1+0i)$

$=64$ (2)

Q3(a)(i) $a = 2,\ b = \sqrt{12} = 2\sqrt{3}$

15

$b^2 = a^2(e^2 - 1)$ — 1

$12 = 4(e^2 - 1)$

$e^2 - 1 = 3$ ✓1

$e^2 = 4 \quad \underline{e = 2}$ (2)

(ii) $\text{Foci} = (\pm ae, 0) = \underline{(\pm 4, 0)}$ (1)

(iii) Dir. are $x = \pm \frac{a}{e}$ $\therefore \underline{x = \pm 1}$ (1)

(iv) Asym. are $y = \pm \frac{bx}{a}$

$\therefore y = \pm \frac{2\sqrt{3}}{2}x \therefore \underline{y = \pm\sqrt{3}x}$ (1)

(b)(i) Diff. Impl.

$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$

$\frac{2yy'}{b^2} = -\frac{2x}{a^2}$

$y' = \frac{-b^2x}{a^2y}$ ✓1 ✓1

$\therefore m_{a\cos\theta} = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta} = \underline{\frac{-b\cos\theta}{a\sin\theta}}$ (2)

(ii) Tangent is

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ — 1

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$ ✓1

$\frac{bx\cos\theta}{ab} + \frac{ay\sin\theta}{ab} = \frac{ab}{ab}$ — 1

$\underline{\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1}$ (3)

(iii) x-int when $y = 0$

$\therefore \frac{x\cos\theta}{a} = 1$

$x = \frac{a}{\cos\theta}$

$\therefore$ x-int is $\left(\frac{a}{\cos\theta}, 0\right)$ (1)

(iv) $4x^2 + 3y^2 = 12$

$\frac{x^2}{3} + \frac{y^2}{4} = 1$ — 1

$\therefore a = \sqrt{3},\ b = 2$

since $a\cos\theta = 2$, $\cos\theta = \frac{2}{\sqrt{3}}$

$\therefore \theta = \frac{\pi}{6}$ or $\frac{-\pi}{6}$ in Q_1 and Q_4 ✓1

$\therefore$ Points of contact are:

$P(a\cos\theta, b\sin\theta)$

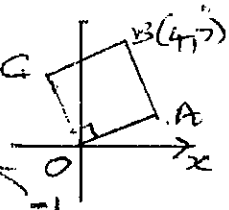
$\therefore P_1 = \left(\sqrt{3}\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}\right) = \underline{\left(1\tfrac{1}{2}, 1\right)}$ ✓1

$P_2 = \left(\sqrt{3}\cos\frac{-\pi}{6}, 2\sin\frac{-\pi}{6}\right) = \underline{\left(1\tfrac{1}{2}, -1\right)}$ ✓1

(4)

Q4(a)(i) $w_c = (a + ib) \times i$ — 1

$\underline{= -b + ai}$ — 1



(ii) $-b + ai + a + ib = 4 + 7i$

$(a - b) + (b + a)i = 4 + 7i$ — 1

$\therefore a - b = 4$

$b + a = 7$

$\therefore 2a = 11$ and $2b = 3$

$\underline{a = 5\tfrac{1}{2}} \quad \underline{b = 1\tfrac{1}{2}}$ — 1 (4)

b)(i) Eqn is of the form

$\left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right) - 1 = 0$ — 1

$1 + 2x^2 - x^3 = 0$

or $\underline{x^3 - 2x^2 - 1 = 0}$ — 1 (2)

(ii) Eqn is of the form

$(\sqrt{x})^3 + 2\sqrt{x} - 1 = 0$ — 1

$\sqrt{x}(x + 2) = 1$

Sq. both sides

$x(x + 2)^2 = 1$

$x(x^2 + 4x + 4) = 1$ ✓1 (2)

$\underline{x^3 + 4x^2 + 4x - 1 = 0}$

c)(i) $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$

$P'(x) = 4x^3 - 9x^2 - 12x + 28$

$P''(x) = 12x^2 - 18x - 12$

$= 6(2x^2 - 3x - 2)$

$= 6(2x + 1)(x - 2)$ ✓1

$= 0$ when $x = -\frac{1}{2}$ or 2

$P\left(-\frac{1}{2}\right) = \frac{1}{16} + \frac{3}{8} - \frac{3}{2} - 14 - 24 = -39\frac{1}{16} \neq 0$

$P(2) = 16 - 24 - 24 + 56 - 24 = 0$ — 1 (2)

$\therefore \underline{x = 2 \text{ is a root of multiplicity } 3}$

(ii) $P(x) = (x - 2)^3(ax + b)$ — 1

Coeff of $x^4 = 1$ $\therefore a = 1$

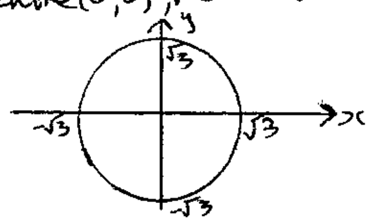
Const. term $= -24$ $\therefore -8b = -24$

$b = 3$ ✓1

$\therefore P(x) = (x - 2)^3(x + 3)$ $\therefore \underline{x = 2, -3}$ (2)

d)(i) $z\bar{z} = 3$ $\therefore x^2 + y^2 = 3$

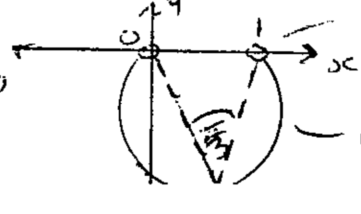
Circle, centre $(0,0)$, radius $\sqrt{3}$



(1)

(ii) $\operatorname{Arg} z - \operatorname{Arg}(z - 1) = \frac{\pi}{3}$

Major Arc, excluding $(0,0)$, $(1,0)$



— 1 (2)

Q5 (a) (i) $(x-y)^2 \geq 0$ for all x, y

$x^2 - 2xy + y^2 \geq 0$

$x^2 + y^2 \geq 2xy$ — 1

Also $y^2 + z^2 \geq 2yz$

$y^2 + x^2 \geq 2zx$ — 1

Adding: $2(x^2+y^2+z^2) \geq 2(xy+yz+xz)$

$\therefore x^2+y^2+z^2 \geq xy+yz+xz$

(ii)

$x^3+y^3+z^3-3xyz = (x+y+z) \times (x^2+y^2+z^2-xy-yz-zx)$

$\geq (x+y+z)(xy+yz+xz-xy-yz-zx)$

(from part (i)) — 1

$\geq (x+y+z) \times 0$

≥ 0

$\therefore x^3+y^3+z^3 \geq 3xyz$

(iii) In (ii)

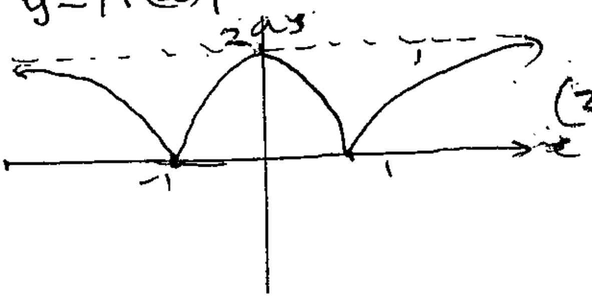
let $x^3 = a$, $y^3 = b$, $z^3 = c$

$\therefore a+b+c \geq 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$ — 1

$\geq 3(abc)^{\frac{1}{3}}$

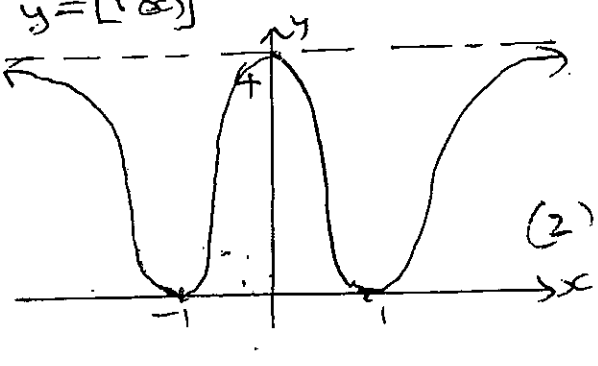
(4)

b) (i) $y = |f(x)|$



(2)

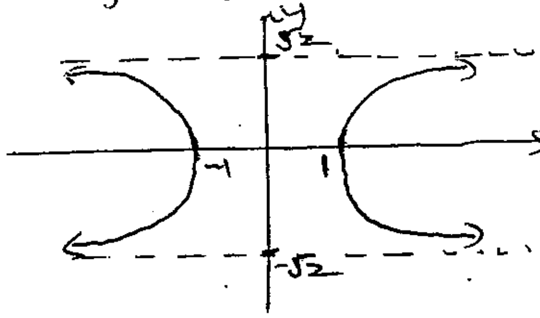
(ii) $y = [f(x)]^2$



(2)

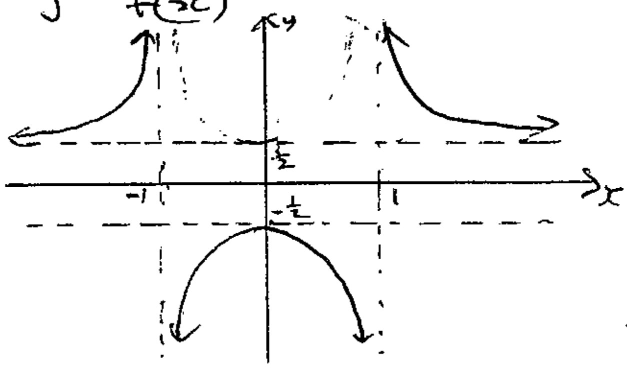
(iii) $y^2 = f(x)$

$\therefore y = \pm\sqrt{f(x)}$



(2)

(iv) $y = \frac{1}{f(x)}$



c) (i) $x^3 + 3x^2y - 2y^3 = 16$

Diff. imp.

$3x^2 + 6xy + 3x^2y' - 6y^2 \cdot y' = 0$ 1

$y'(6y^2 - 3x^2) = 3x^2 + 6xy$

$\therefore y' = \frac{3(x^2+2xy)}{3(2y^2-x^2)}$

$= \frac{x^2+2xy}{2y^2-x^2}$ (1)

(ii) St pts when $y' = 0$

$x^2 + 2xy = 0$

$x(x+2y) = 0$

$\therefore x = 0, y = -2$ 1

or $x = -2y$

$\therefore (-2y)^3 + 3(-2y)^2y - 2y^3 = 16$

$-8y^3 + 12y^3 - 2y^3 = 16$

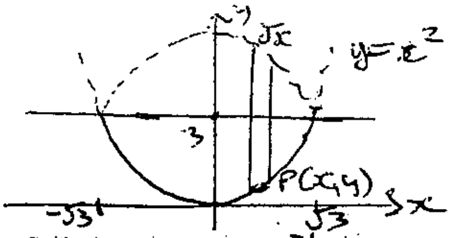
$2y^3 = 16$

$y = 2$ 1

$\therefore x = -4$

$\therefore$ St. pts are $(-4, 2)$ and $(0, -2)$ (2)

Q6.(a) 15



—1

Vol. of slice $\delta V = \pi r^2 h$

$= \pi(3-y)^2\delta x$

$= \pi(3-x^2)^2\delta x$ —1

Total Vol., $V = \lim\limits_{\delta x \to 0} \sum\limits_{x=-\sqrt{3}}^{x=\sqrt{3}} \pi(3-x^2)^2\delta x$

$= 2\pi\int_0^{\sqrt{3}}(3-x^2)^2 dx$ —1

$= 2\pi\int_0^{\sqrt{3}} 9-6x^2+x^4\,dx$

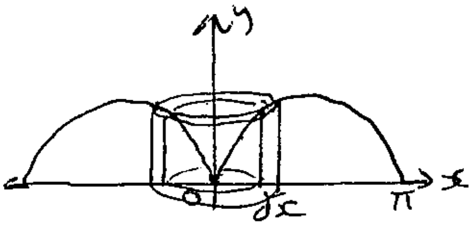
$= 2\pi\left[9x-2x^3+\frac{x^5}{5}\right]_0^{\sqrt{3}}$

$= 2\pi\left[9\sqrt{3}-6\sqrt{3}+\frac{9\sqrt{3}}{5}\right]$

$= \frac{2\pi}{5}\left[45\sqrt{3}-30\sqrt{3}+9\sqrt{3}\right]$

$= \frac{48\pi\sqrt{3}}{5}\ u^3$ —1 (4)

b)



Vol_{shell} = in. circ. × height × thick.

$= 2\pi x y\,\delta x$ —1

Vol. $= \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{x=\pi} 2\pi x y\,\delta x$

$= 2\pi\int_0^{\pi} x\sin x\,dx$ —1

$= 2\pi\left[uv - \int vu'\right]$

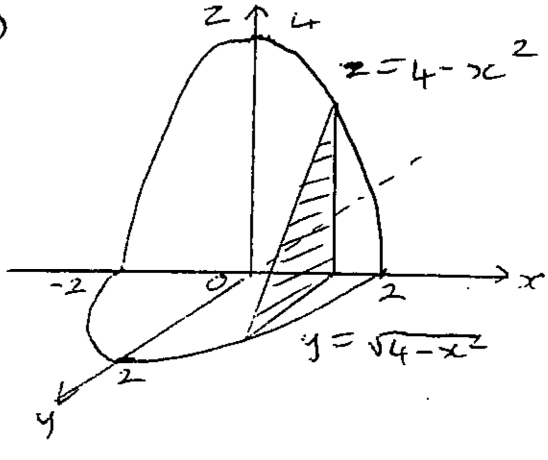
$= 2\pi\left\{\left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} -\cos x\,dx\right\}$ —1

$= 2\pi\left\{-\pi\cos\pi + \left[\sin x\right]_0^{\pi}\right\}$

$= 2\pi\left[\pi - 0\right]$

$= 2\pi^2\ u^3$ —1 (4)

c) (i)



—(1)

(ii) $\text{Vol}_{slice} = \left(\frac{1}{2}bh\right)H$

$= \frac{1}{2}yz\,\delta x$ —1

$= \frac{1}{2}\sqrt{4-x^2}(4-x^2)\delta x$

$= \frac{1}{2}(4-x^2)^{\frac{3}{2}}\delta x$ —1

Vol. $= \lim\limits_{\delta x \to 0} \sum\limits_{x=-2}^{x=2} \frac{1}{2}(4-x^2)^{\frac{3}{2}}\delta x$

$= 2\int_0^2 \frac{1}{2}(4-x^2)^{\frac{3}{2}} dx$ —1

$= \int_0^2 (4-x^2)^{\frac{3}{2}} dx$ (3)

(iii) $V = \int_0^2 (4-x^2)^{\frac{3}{2}} dx$

Let $x = 2\sin\theta$ $\therefore dx = 2\cos\theta\,d\theta$

When $x = 0$, $\theta = 0$

When $x = 2$, $\theta = \frac{\pi}{2}$ —1

$\therefore V = \int_0^{\frac{\pi}{2}} (4-4\sin^2\theta)^{\frac{3}{2}}\cdot 2\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} 8(1-\sin^2\theta)^{\frac{3}{2}}\cdot 2\cos\theta\,d\theta$

$= 16\int_0^{\frac{\pi}{2}} \cos^3\theta\cdot\cos\theta\,d\theta$

$= 16\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$ —1

$= 16\int_0^{\frac{\pi}{2}} \left[\frac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$

$= 4\int_0^{\frac{\pi}{2}} 1+2\cos 2\theta+\cos^2 2\theta\,d\theta$

$= 4\left[\theta+\sin 2\theta\right]_0^{\frac{\pi}{2}} + 4\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 4\theta)\,d\theta$

$= 4\times\frac{\pi}{2} + 2\left[\theta+\frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$

$= 2\pi + \pi - 0$

$= 3\pi\ u^3$ —1 (3)

Q7 (a) let $\frac{6x+4}{(x^2+1)(x+1)} = \frac{ax+b}{x^2+1} + \frac{c}{x+1}$

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$\therefore 6x+4 = (ax+b)(x+1) + c(x^2+1)$ -1

If $x=-1$, $-2 = c\times 2$ $\therefore c = -1$

If $x=0$, $4 = b+c$ $\therefore b = 5$

Eq. coeff of x^2

$a+c = 0$ $\therefore a = 1$ -1

$\therefore I = \int_0^1 \frac{x+5}{x^2+1} - \frac{1}{x+1}\,dx$ -1

$= \int_0^1 \frac{x}{x^2+1}\,dx + \int_0^1 \frac{5}{x^2+1}\,dx - \int_0^1 \frac{dx}{x+1}$

$= \frac{1}{2}\left[\ln(x^2+1)\right]_0^1 + 5\left[\tan^{-1}x\right]_0^1 - \left[\ln(x+1)\right]_0^1$

$= \frac{1}{2}(\ln 2 - \ln 1) + 5(\tan^{-1}1 - \tan^{-1}0)$ -1

$- (\ln 2 - \ln 1)$

$= \frac{1}{2}\ln 2 - \ln 2 + 5\times\frac{\pi}{4}$

$= \frac{5\pi}{4} - \frac{1}{2}\ln 2$ (4)

b)(i) $P(z) = z^4 + bz^2 + d$

$P'(z) = 4z^3 + 2bz$ -1

$P'(-z) = (-z)^3 + 2b(-z)$

$= -4z^3 - 2bz$

$= -(4z^3 + 2bz)$

$= -P'(z)$ -1

$\therefore P'(z)$ is odd. (2)

ii) $P(-\alpha) = (-\alpha)^4 + b(-\alpha)^2 + d$

$= \alpha^4 + b\alpha^2 + d$

$= P(\alpha)$

$= 0$ -1

$\therefore -\alpha$ is also a zero of $P(z)$

let β = the fourth root.

$\Sigma x = \alpha + \alpha + -\alpha + \beta = -\frac{b}{a} = 0$ -1

$\therefore \beta = -\alpha$

$\therefore -\alpha$ is a double root of $P(z)$ (2)

(c)(i) Result. Force, $ma = mg - mkv$

$35a = 350 - 70v$ -1

$a = 10 - 2v$

$\therefore \frac{dt}{dv} = \frac{1}{10-2v}$

$t = -\frac{1}{2}\ln(10-2v) + c$ -1

$2t = -\ln(10-2v) + 2c$

$\ln(10-2v) = 2c - 2t$

$10 - 2v = e^{2c-2t} = Ae^{-2t}$

$2v = 10 - Ae^{-2t}$

$v = 5 - \frac{A}{2}e^{-2t}$ -1

When $t=0$, $v=30$

$\therefore 30 = 5 - \frac{A}{2}$

$A = -50$

$\therefore v = 5 + 25e^{-2t}$ (3)

(ii) As $t \to \infty$, $e^{-2t} \to 0$

$\therefore v \to 5\ ms^{-1}$ (1)

(iii) $\frac{dx}{dt} = 5 + 25e^{-2t}$

$x = 5t - \frac{25}{2}e^{-2t} + c$ -1

When $t=0$, $x=0$ $\therefore c = \frac{25}{2}$

$\therefore x = 5t - \frac{25}{2}e^{-2t} + \frac{25}{2}$

When $t = 60$

$x = 300 - \frac{25}{2}e^{-120} + \frac{25}{2}$

$= 312\frac{1}{2} - \frac{25}{2}e^{-120}$ -1

$\doteqdot 312.5$

$\therefore$ Height $= 400 - 312.5$ -1

$= 87.5m$ (3)

Q8 (a) (i) $I_n=\int_0^1 x^n e^{-x}\,dx$

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$I_0=\int_0^1 x^0 e^{-x}\,dx$

$=\int_0^1 e^{-x}\,dx$ — 1

$=[-e^{-x}]_0^1$

$=-e^{-1}--e^{0}$

$=1-e^{-1}$ — 1 (2)

(ii) $I_n=\int_0^1 x^n e^{-x}\,dx$

Let $u=x^n \therefore u'=nx^{n-1}$

let $v'=e^{-x} \therefore v=-e^{-x}$ — 1

$I_n=uv-\int vu'$

$=[-x^n e^{-x}]_0^1+n\int_0^1 x^{n-1}e^{-x}\,dx$ — 1

$=-e^{-1}+nI_{n-1}$

$I_n=nI_{n-1}-\frac{1}{e}$ (2)

(iii) $I_3=\int_0^1 x^3 e^{-x}\,dx$

$=3I_2-\frac{1}{e}$

$=3[2I_1-\frac{1}{e}]-\frac{1}{e}$ — 1

$=6I_1-\frac{4}{e}$

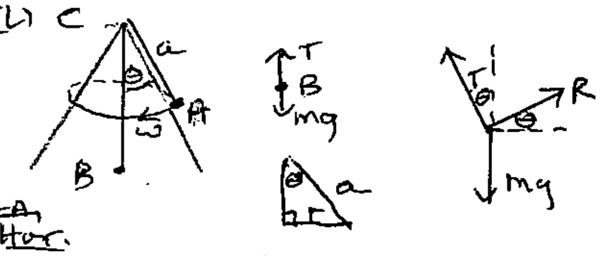
$=6[I_0-\frac{1}{e}]-\frac{4}{e}$

$=6I_0-\frac{10}{e}$

$=6[1-e^{-1}]-\frac{10}{e}$

$=6-\frac{16}{e}$ — 1 (2)

b) (i)



— (1)

ii) At A

Hor.

Cent. Force, $mr\omega^2=T\sin\theta-R\cos\theta$ --(1)

Ver. $mg=T\cos\theta+R\sin\theta$ --(2)

At B $mg=T$ — 1

$\therefore$ In (2) $mg=mg\cos\theta+R\sin\theta$

$R\sin\theta=mg(1-\cos\theta)$

$R=\frac{mg(1-\cos\theta)}{\sin\theta}$ — 1

In (1) $ma\sin\theta\,\omega^2=mg\sin\theta-R\cos\theta$

$ma\omega^2\sin\theta=mg\sin\theta-\frac{mg(1-\cos\theta)}{\sin\theta}.\cos\theta$

$a\omega^2\sin^2\theta=g\sin^2\theta-g(1-\cos\theta)\cos\theta$

$\sin^2\theta(g-a\omega^2)=g\cos\theta(1-\cos\theta)$

$g-a\omega^2=\frac{g\cos\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$

$a\omega^2=g-\frac{g\cos\theta}{1+\cos\theta}$

$=\frac{g+g\cos\theta-g\cos\theta}{1+\cos\theta}$ /2 $\Rightarrow \therefore \omega^2=\frac{g}{1+\cos\theta}$

(iii) When $R>0$

$\frac{mg(1-\cos\theta)}{\sin\theta}>0$

$1-\cos\theta>0$

$\cos\theta<1$

$\therefore 0^\circ<\theta<90^\circ$

When $\theta=0^\circ$, $\omega^2=\frac{g}{a(1+1)}=\frac{g}{2a}$

$\therefore a=\frac{g}{2\omega^2}$ — 1

When $\theta=90^\circ$, $\omega^2=\frac{g}{a(1+0)}=\frac{g}{a}$

$\therefore a=\frac{g}{\omega^2}$ — 1

$\therefore \frac{g}{2\omega^2}<a<\frac{g}{\omega^2}$ (2)

c) $\frac{d}{dx}(x-\frac{1}{3}x^3)=1-x^2$

$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$

$\frac{d}{dx}(x-\frac{1}{3}x^3+\frac{1}{3}x^5)=1-x^2+\frac{5}{3}x^4$ — 1

Now, $\frac{1}{1+x^2}-(1-x^2)=\frac{1-(1-x^4)}{1+x^2}=\frac{x^4}{1+x^2}$

>0 for $x>0$

$\therefore 1-x^2<\frac{1}{1+x^2}$ — 1

Also

$1-x^2+\frac{5}{3}x^4-\frac{1}{1+x^2}=\frac{(1+x^2)(1-x^2+\frac{5}{3}x^4)-1}{1+x^2}$

$=\frac{1-x^2+\frac{5}{3}x^4+x^2-x^4+\frac{5}{3}x^6-1}{1+x^2}$

$=\frac{\frac{2}{3}x^4+\frac{5}{3}x^6}{1+x^2}>0$ for $x>0$

$\therefore \frac{1}{1+x^2}<1-x^2+\frac{5}{3}x^4$ — 1

$\therefore 1-x^2<\frac{1}{1+x^2}<1-x^2+\frac{5}{3}x^4$

Integrating each exp. gives: — 1

$\int_0^x 1-x^2\,dx<\int_0^x \frac{1}{1+x^2}\,dx<\int_0^x 1-x^2+\frac{5}{3}x^4\,dx$

$[x-\frac{x^3}{3}]_0^x<[\tan^{-1}x]_0^x<[x-\frac{1}{3}x^3+\frac{1}{3}x^5]_0^x$

ie.

$x-\frac{x^3}{3}<\tan^{-1}x<x-\frac{1}{3}x^3+\frac{1}{3}x^5$ (4)