BAULKHAM HILLS HIGH SCHOOL

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 120
- Attempt Questions 1 – 8
- All questions are of equal value
- Start a separate piece of paper for each question.
- Put your student number and the question number at the top of each sheet.

Question 1 (15 marks) - Start on a new page

a) Find the following indefinite integrals

(i) $\int \cos^3 x \, dx$ **2**

(ii) $\int \frac{x-2}{x^2+1} \, dx$ **2**

(iii) $\int x \sin 2x \, dx$ **2**

b) Evaluate $\int_0^1 \frac{dx}{\sqrt{3-2x-x^2}}$ **3**

c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ **3**

d) Find $\int \frac{2x \, dx}{x^3-2x^2+9x-18}$ **3**

Question 2 **(15 marks) - Start on a new page**

a) (i) Express $z_1 = \dfrac{7+4i}{3-2i}$ in the form $a+ib$ where a, b are real **2**

(ii) On an Argand diagram sketch the locus of the point representing the complex number z such that $|z - z_1| = \sqrt{5}$ **1**

(iii) Prove that the locus passes through the origin and find the greatest value of $|z|$ **2**

b) Let $z = 2+3i$ and $w = 1+i$
Find zw and $\frac{1}{w}$ in the form $x+iy$ **2**

c) (i) Express $(1-\sqrt{3}i)$ in modulus argument form **2**

(ii) Hence write $\left(1-\sqrt{3}i\right)^{10}$ in the form $x+iy$ **2**

d) The complex number $z = x+iy$ when x and y are real, is such that $|z-i| = \operatorname{Im}(z)$

(i) Show that the locus of point P representing z has Cartesian equation
$y = \frac{1}{2}(x^2+1)$ and sketch the locus **2**

(ii) By finding the gradients of the tangents to this curve which pass through the origin, find the set of possible values of $\arg z$
$(-\pi < \arg z \leq \pi)$ **2**

Question 3 **(15 marks) - Start a new page** Marks

a) Sketch the graph **2**

$y = (2x+1)(x+1)$ clearly showing all intercepts on the co-ordinate axes and the co-ordinates of any turning points.

b) Use the graph of part (a) to sketch the graphs below, showing clearly the intercepts on the co-ordinate axes, the co-ordinates of any turning points and the equation of any asymptotes.

(i) $y = \log_e[(2x+1)(x+1)]$ **2**

(ii) $y = \dfrac{1}{(2x+1)(x+1)}$ **2**

c) The region bounded by the curve
$$y = \frac{1}{(2x+1)(x+1)}$$
the co-ordinate axes and the line $x = 4$ is rotated through one complete revolution about the y axis.

(i) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral. **2**

(ii) Evaluate the integral in part (i). **4**

d) When $P(x) = x^4 + ax^3 + b$ is divided by x^2+4, the remainder is $-x+13$. **3**
Find the values of a and b.

Question 4 (15 marks) - Start a new page

a) A hyperbola has Cartesian equation

$$3x^2 - y^2 = 12$$

find (i) its eccentricity **1**

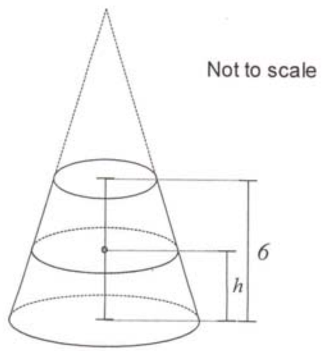
(ii) the co-ordinates of its foci **1**

(iii) the equations of the directrices **1**

(iv) the equations of the asymptotes **1**

hence sketch the hyperbola indicating all the features of your diagram. **1**

b) A right elliptical cone has its top cut off through a plane parallel to its elliptical base. The remaining solid has an ellipse as its base



The remaining solid has the ellipse $\frac{x^2}{9} + \frac{4y^2}{9} = 1$ as its base, and another ellipse $x^2 + 4y^2 = 1$ as its top.

The height of the solid is 6 units.

(i) Given that the area of an ellipse with equation $\frac{x^2}{9} + \frac{4y^2}{9} = 1$ is πab, show that the area of the ellipse at height h units above the base is $A = \frac{\pi(h-9)^2}{18}$ **3**

(ii) Hence find the volume of the soild. **3**

c) A plane curve is defined implicitly by $x^2 + 2xy + y^5 = 4$
This curve has a horizontal tangent at $P(x, y)$ show that $x = \alpha$ is a root of the equation $x^5 + x^2 + 4 = 0$ **4**

Question 5 (15 marks) - Start a new page

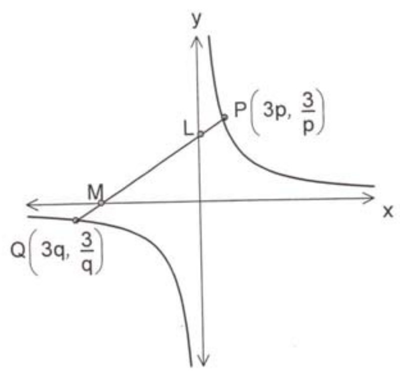
a) The equation $x^3 + px - 1 = 0$ has 3 non zero roots α, β, γ

(i) Find the values of $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^4 + \beta^4 + \gamma^4$ in terms of p and show that p must be negative. **4**

(ii) Find the monic equation with coefficients in terms of p where roots are **2**

$$\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma}, \frac{\gamma}{\alpha\beta}$$

b) A chord PQ of the rectangular hyperbola $xy = 9$ meets the asymptotes at L and M as shown



i) Show that the equation of the chord PQ is $pqy + x = 3(p+q)$ **1**

ii) Find the co-ordinates of N the midpoint of PQ **2**

iii) Show that $PL = MQ$ **2**

iv) If the chord PQ is a tangent to the parabola $y^2 = 3x$ find the locus of N **2**

c) Solve in terms of a **2**
$a^x = e^{2x-1}$ where $a > 0, a \neq \cancel{\frac{1}{e^2}}$ e^2

Question 6 (15 marks) - Start a new page

a) Solve the equation 3
$$x^4 - 6x^3 + 9x^2 + 6x - 20 = 0$$
given $(2 + i)$ is one of it's zeroes.

b) $P(a\cos\theta, b\sin\theta)$ lies on the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad where\ a > b > 0.$$

The tangent and normal at point P cut y-axis at A and B respectively, and S is a focus of the ellipse

i) Show that $\angle ASB = 90°$ 2

ii) Hence show that A, P, S and B are concyclic and state the coordinates of the centre of the circle through A, P, S and B. 3

c) Prove that $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$ 2

Hence evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$ 3

d) Draw a neat sketch of $y = \frac{1}{\sin^{-1} x}$ 2

Question 7 (15 marks) - Start a new page Ma

a) Given that $1, \omega$ and ω^2 are the three cube roots of unity

i) Find the value of $(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega)$ 3

ii) If the equations $x^3 - 1 = 0$ and $px^5 + qx + r = 0$ have a common root, evaluate 3
$$(p + q + r)(p\omega^5 + q\omega + r)(p\omega^{10} + q\omega^2 + r)$$

b) i) Show that
$$(1-\sqrt{x})^{n-1}.\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$$ 1

ii) If $I_n = \int_0^1 (1-\sqrt{x})^n\, dx$ for $n \geq 0$ 3

Show that $I_n = \frac{n}{n+2} I_{n-1}\ for\ n \geq 1$

And hence evaluate I_{100}

c) Prove that the volume, V, the area of the curved surface, S, and the radius of the base, r, of a right circular cone are connected by the equation 2
$$9V^2 = r^2(S^2 - \pi^2 r^4)$$

3

Show that the maximum volume for a given curved surface area S, is
$$\frac{2^{\frac{1}{2}} S^{\frac{3}{2}}}{\pi^{\frac{1}{2}} 3^{\frac{7}{4}}}$$

Question 8 (15 marks) - Start a new page

a) Prove by mathematical induction that $7^n + 3n(7^n) - 1$ is divisible by 9. **3**

b) i) Write the general solution of **1**

$$\tan 4\theta = 1$$

ii) Use De Moivre's Thereom to find $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$ and hence determine the result **4**

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

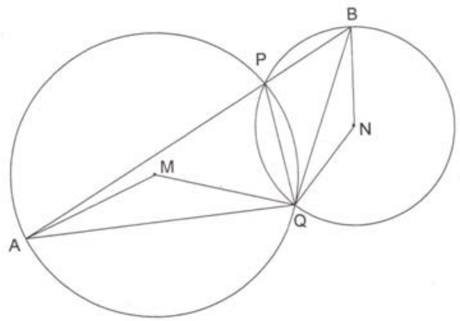
iii) Find the roots of

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ **3**

in the form $x = \tan\theta$ and hence prove that

$$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$$

c) **4**



In the given diagram, two circles whose centres are M and N intersect in P and Q. A line drawn through P meets the two circles in A and B. Prove that $\angle MAQ = \angle NBQ$.

End of Exam

2011 EXT 2 TRIAL SOLUTIONS

a i) $\int \cos^3 x\,dx$

1

$= \int \cos x(1-\sin^2 x)\,dx$ 1

$= \int \cos x\,dx - \int \sin^2 x \cos x\,dx$

$= \sin x - \frac{\sin^3 x}{3} + C$ 1

ii) $\int \frac{x-2}{x^2+1}\,dx$

$= \int \frac{x\,dx}{x^2+1} - \int \frac{2}{x^2+1}\,dx$

$= \frac{1}{2}\ln(x^2+1) - 2\tan^{-1}x + C$ 1 1

iii) $\int x \sin 2x\,dx$

let $u = x$ $v' = \sin 2x$

$u' = 1$ $v = -\frac{1}{2}\cos 2x$

$\therefore I = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$ 1

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$ 1

b) $\int_0^1 \frac{dx}{\sqrt{-x^2 - 2x + 3}}$

$= \int_0^1 \frac{dx}{\sqrt{-(x^2+2x+1)+4}}$

$= \int_0^1 \frac{dx}{\sqrt{4-(x+1)^2}}$ 1

$= \left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_0^1$ 1

$= \sin^{-1}1 - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$ 1

c) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}$ 4

let $t = \tan\frac{x}{2}$ $x=\frac{\pi}{2}$ $t=1$

$x=0$ $t=0$

$\therefore I = \int_0^1 \left(\frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\right)\cdot\frac{2dt}{1+t^2}$ 1

$= \int_0^1 \frac{2dt}{1+t^2+1-t^2+2t}$ 1

$= \int_0^1 \frac{dt}{1+t}$

$= \left[\ln(1+t)\right]_0^1$ 1

$= \ln 2 - \ln 1$

$= \ln 2$

d) $\int \frac{2x\,dx}{x^3-2x^2+9x-18}$

$= \int \frac{2x\,dx}{(x^2+9)(x-2)}$ 1

let $\frac{2x}{(x-2)(x^2+9)} \equiv \frac{A}{x-2} + \frac{Bx+C}{x^2+9}$

$A(x^2+9) + (Bx+C)(x-2) \equiv 2x$

$x = 2 \rightarrow 13A = 4$ $\therefore A = \frac{4}{13}$

Const term $9A - 2C = 0$

$2C = 9\times\frac{4}{13}$ $C = \frac{18}{13}$

Coeff x^2 $A + B = 0$ $B = -\frac{4}{13}$ 1

$\therefore I = \frac{1}{13}\left[\int \frac{4}{x-2}\,dx + \int \frac{18-4x}{x^2+9}\,dx\right]$

$= \frac{1}{13}\left[4\ln(x-2) + \frac{18}{3}\tan^{-1}\frac{x}{3} - 2\ln(x^2+9)\right]$

$= \frac{4}{13}\ln(x-2) + \frac{6}{13}\tan^{-1}\frac{x}{3} - \frac{2}{13}\ln(x^2+9) + C$ 1

(1)

b) $z = 2+3i \quad w = (1+i)$

$zw = (2+3i)(1+i)$

$= -1+5i$ 1

$\frac{1}{w} = \frac{1}{1+i} \times \frac{1-i}{1-i}$

$= \frac{1-i}{2}$

$= \frac{1}{2} - \frac{1}{2}i.$ 1

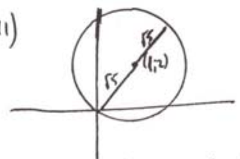
a) i) $z_1 = \frac{7+4i}{3-2i} \times \frac{3+2i}{3+2i}$ 1

$= \frac{21+14i+12i-8}{9+4}$

$= \frac{13+26i}{13}$ 1

$= 1+2i$

ii)



1

eqn of circle is $(x-1)^2+(y-2)^2 = \sqrt{5}^2$

$\therefore x^2-2x+1+y^2-4y+4 = 5$

$x^2+y^2-2x-4y = 0$ 1

$x=0 \; y=0$ satisfies the eqn.

max $|z|$ = diameter of circle

ie $2\sqrt{5}$ 1

c) i) $(1-\sqrt{3}i)$ 1

$= 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)$ 1

or $2\,\text{cis} -\frac{\pi}{3}$

ii) $(1-\sqrt{3}i)^{10}$

$= 2^{10}\,\text{cis} -\frac{10\pi}{3}$ 1

$= 2^{10}\,\text{cis} -\frac{4\pi}{3}$

$= 2^{10}\left(\cos -\frac{4\pi}{3} + i\sin -\frac{4\pi}{3}\right)$

$= 2^{10}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

1

=

d) i) $|z-i| = \text{Im}\, z$

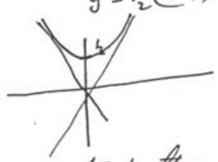
$|x+iy-i| = y$

$x^2+(y-1)^2 = y^2$

$x^2+y^2-2y+1 = y^2$

1

$y = \frac{1}{2}(x^2+1)$



1

ii) equation of tangents thru 0 are

$y = mx.$

$mx = \frac{1}{2}(x^2+1)$

$x^2-2mx+1 = 0$

only one soln

$\therefore \Delta = 0$

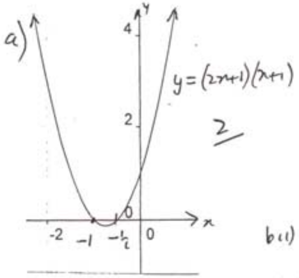
$4m^2-4 = 0$ 1

$m = \pm 1$

$\therefore \frac{\pi}{4} \le \arg z \le \frac{3\pi}{4}$ 1

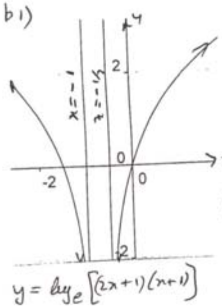
(3)

a)



2

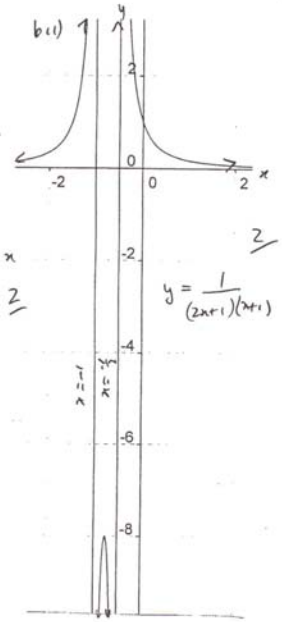
b) i)



$y = \log_e[(2x+1)(x+1)]$

2

b) ii)

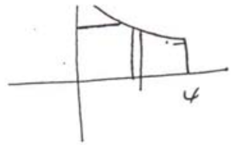


$y = \frac{1}{(2x+1)(x+1)}$

2

(3)

c



take thickness shell be δx.

$\delta V = 2\pi x \cdot y \cdot \delta x$ 1

$= 2\pi \frac{x}{(2x+1)(x+1)} \delta x$

$V = \sum_{x=0}^{x=4} \delta V$

$= 2\pi \int_0^4 \frac{x\,dx}{(2x+1)(x+1)}$ 1

let $\frac{x}{(2x+1)(x+1)} = \frac{a}{2x+1} + \frac{b}{x+1}$

$a = -1 \quad b = 1$ 1

$\therefore V = 2\pi \int_0^4 \left(\frac{1}{x+1} - \frac{1}{2x+1}\right) dx$

$= 2\pi \left[\ln(x+1) - \frac{1}{2}\ln(2x+1)\right]_0^4$ 1

$= 2\pi\left[\ln 5 - \ln 1 - \frac{1}{2}\ln 9 + \frac{1}{2}\ln 1\right]$

$= 2\pi\left(\ln 5 - \frac{1}{2}\ln 9\right)$

$= 2\pi \ln \frac{5}{3} \quad u^3.$ 1

3

d) $x^4 + ax^3 + b = (x^2+4)\,Q(x) - x + 13$

let $x = 2i \quad 16 - 8ai + b = -2i + 13$ 1

$\therefore a = \frac{1}{4}$ 1

$b = -3$ 1

(4)

a) $3x^2 - y^2 = 12$

$\frac{x^2}{4} - \frac{y^2}{12} = 1$

$a = 2 \quad b = 2\sqrt{3}$

$b^2 = a^2(e^2 - 1)$

$12 = 4(e^2 - 1)$

$e^2 - 1 = 3$

$e = 2 \quad — (1) \qquad \underline{1}$

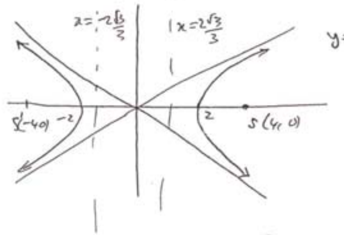
$S(4, 0) \quad S'(-4, 0) \quad — (11) \qquad \underline{1}$

Directrices

$x = \mp \frac{a}{e}$

$x = \mp \frac{2}{\sqrt{3}}$

or $x = \mp \frac{2\sqrt{3}}{3} \qquad \underline{1}$



$y = \mp\sqrt{3}\,x \quad \underline{1}$ (asymptotes)

4c) $x^2 + 2xy + y^5 = 4$

$2x + 2y + 2xy' + 5y^4y' = 0$

$y'(2x + 5y^4) = -(2x + 2y)$

$y' = \frac{-2(x+y)}{2x + 5y^4} \qquad \underline{1}$

$= 0$ wh $\quad y = -x \qquad \underline{1}$

subst $y = -x$ in eqn

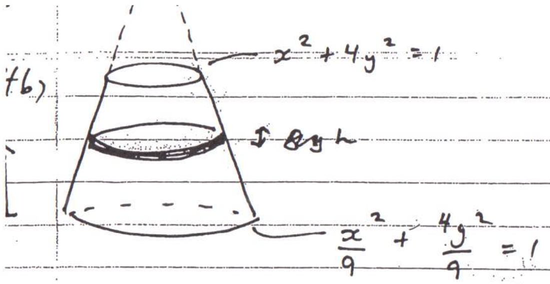
$\therefore x^2 + 2x(-x) + (-x)^5 = 4$

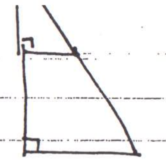
$x^2 - 2x^2 - x^5 = 4 \qquad \underline{1}$

$-x^2 - x^5 = 4 \qquad \underline{1}$

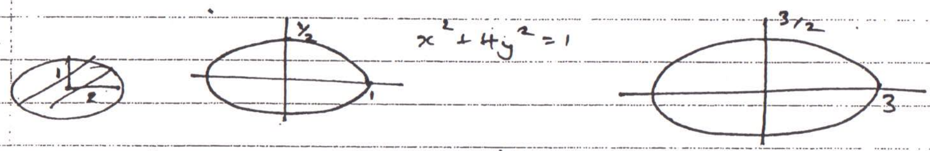
$x^5 + x^2 + 4 = 0$

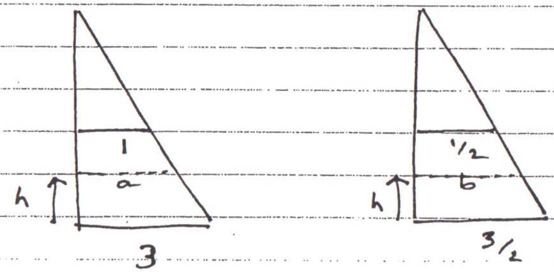
16)





1)





$a = mh + b$

If $h=0$, $a=3$

$3 = 0 + b \therefore b = 3$

If $h=6$, $a=1$

$1 = 6m + 3$

$-2 = 6m$

$m = -1/3$

$\therefore a = -\frac{1}{3}h + 3$ 1

$\therefore b = mh + c$

If $h=0$, $b = 3/2$

$\frac{3}{2} = 0 + c$

$c = 3/2$

If $h=6$, $b = 1/2$

$\frac{1}{2} = 6m + \frac{3}{2}$

$-1 = 6m$

$m = -1/6$

$\therefore b = -\frac{1}{6}h + \frac{3}{2}$ 1

$\therefore$ Area of ellipse at height h

$= \pi\left(-\frac{1}{3}h+3\right)\left(-1/6h+3/2\right)$

$\therefore$ ~~$\Delta V = \pi$~~

$= \pi \cdot -\frac{1}{3}(h-9) \cdot -\frac{1}{6}(h-9)$

$= \frac{1}{18}\pi\,(h-9)^2$ 1

$x \,\square = \frac{3}{2}$

$\square = \frac{3}{2} \times 6$

$\therefore \Delta V = \frac{1}{18}\pi \cdot (h-9)^2 \cdot \Delta h$

$V = \frac{\pi}{18}\int_0^6 (h-9)^2 \cdot dh$ 1

$= \frac{\pi}{18} \cdot \left[\frac{(h-9)^3}{3}\right]_0^6$ 1

$= \frac{\pi}{18}\left(\frac{(-3)^3}{3} - \frac{(-9)^3}{3}\right)$ $\frac{729}{3}$

$= \frac{\pi}{18}\left(-9 + 243\right)$

$= \frac{234\pi}{18}$

$= 13\pi$ 1

~~Given that the area of an ellipse is πab units^2,~~

(i) ~~Given that~~ Show that the area

5

a) i) $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

$\sum\alpha = 0 \quad \sum\alpha\beta = p$ 1/2

$\therefore \sum\alpha^2 = 0 - 2p = -2p$ 1/2

$x^3 = -px + 1$

$x^4 = -px^2 + x$

$\alpha^4 = -p\alpha^2 + \alpha$ etc 1/2

$\therefore \sum\alpha^4 = -p\sum\alpha^2 + \sum\alpha$

$= -p \cdot -2p + 0$

$= 2p^2$. 1/2

$\sum\alpha^2 > 0$ if root are non zero real

$\therefore -2p > 0 \therefore p < 0$

roots are $\frac{\alpha^2}{\alpha\beta\gamma}, \frac{\beta^2}{\alpha\beta\gamma}, \frac{\gamma^2}{\alpha\beta\gamma}$

$\alpha\beta\gamma = 1$

$\therefore$ root are $\alpha^2, \beta^2, \gamma^2$ 1/2

at $y = x^2$

$\therefore x = \sqrt{y}$

$\therefore y^{3/2} + py^{1/2} - 1 = 0$

$y^{1/2}(y+p) = 1$

$y(y^2 + 2py + p^2) = \cancel{0}\ 1$ 1/2

$\therefore y^3 + 2py^2 + p^2y - 1 = 0$

b) eq of chord is

$y - \frac{3}{p} = \frac{\frac{3}{p} - \frac{3}{q}}{3p - 3q}(x - 3p)$

$y - \frac{3}{p} = \frac{-1}{pq}(x - 3p)$ 1/2

$\therefore pqy + x = 3(p+q)$

mid pt of PQ is $x = \frac{3p+3q}{2} = \frac{3}{2}(p+q)$ 1/2 1/2

$y = \frac{\left(\frac{3}{p} + \frac{3}{q}\right)}{2} = \frac{3(p+q)}{2pq}$

L is pt $x = 0 \quad y = \frac{3(p+q)}{pq}$

M is pt $y = 0 \quad x = 3(p+q)$

$\therefore$ MP of LM is ~~$\frac{3(p+q)}{2}$~~ ~~$3p$~~

$\frac{3(p+q)}{2}, \frac{3(p+q)}{2pq}$

$\therefore$ MP of LM is same as MP of QP 1/2

$\therefore LP = MQ$.

$pqy + \cancel{x} = 3(p+q) \quad -(1)$

$y^2 = 3x \quad -(2)$

sub $x = \frac{y^2}{3} \implies (1)$

$\therefore pqy + \frac{y^2}{3} = 3(p+q)$

$\therefore y^2 + 3pqy - 9(p+q) = 0$

this only has one soln 1/2

$\therefore \Delta = 0$

$\therefore 9p^2q^2 = -36(p+q)$

$\therefore (pq)^2 = -4(p+q)$

N is pt $x = \frac{3}{2} \cdot -\frac{(pq)^2}{4} = -\frac{3}{8}(pq)^2$

$y = \frac{3}{2pq}(p+q) = \frac{3}{2pq} \cdot \frac{(pq)^2}{-4}$

$= -\frac{3}{8}(pq)$

$\implies \quad x = -\frac{8}{3}y^2$ 1/2

but $pq < 0$

$\therefore y > 0 \quad x < 0$

3

c) $a^x = e^{2x-1}$

$\ln a^x = \ln e^{2x-1}$ 1/2

$x\ln a = (2x-1)\ln e$

$x\ln a = 2x - 1$

$2x - x\ln a = 1$

$x(2 - \ln a) = 1$

$x = \frac{1}{2 - \ln a}$ 1/2

$a > 0 \quad \ln a \neq 2$

(6)

a) $x^4-6x^2+9x^2+6x-20=0$

if $(2+i)$ is a zero

$2-i$ is also a zero 1

$(x-(2+i))(x-(2-i))$

$= x^2-4x+5$

$(x^4-6x^2+9x^2+6x-20) \div (x^2-4x+5)$

$= x^2-2x-4$ 1

$x^2-2x-4 \Rightarrow x = \frac{2 \mp \sqrt{4+16}}{2}$

$= 1 \mp \frac{\sqrt{20}}{2}$

$= 1 \mp \sqrt{5}$

$\therefore$ roots are $2 \mp i,\ 1 \mp \sqrt{5}$. 1

b) $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ —T

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2-b^2$ —N

sub $x=0$ in T $\therefore \frac{y\sin\theta}{b} = 1$

$y = \frac{b}{\sin\theta}$

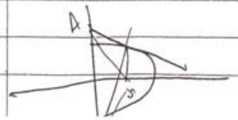
A is pt $\left(0, \frac{b}{\sin\theta}\right)$

sub $x=0$ in N

$\frac{-by}{\sin\theta} = a^2-b^2$

$y = \frac{(b^2-a^2)\sin\theta}{b}$ 1

B is pt $\left(0, \frac{b^2-a^2\sin\theta}{b}\right)$



$m_{AS} \cdot m_{PS}$

$= \frac{\frac{-b}{\sin\theta}}{ae} \times \frac{\frac{a^2-b^2}{b}\sin\theta}{ae}$

$= \frac{a^2-b^2}{a^2e^2}$

$= \frac{a^2-a^2(1-e^2)}{a^2e^2}$

$= \frac{a^2-a^2+a^2e^2}{a^2e^2}$

$m_{AS} \cdot m_{PS}$

$= \frac{\frac{-b}{\sin\theta}}{ae} \times \frac{\frac{a^2-b^2}{b}\sin\theta}{ae}$

$= \frac{-(a^2-b^2)}{a^2e^2}$

$= -\frac{(a^2-a^2+a^2e^2)}{a^2e^2}$ 1

$= -1$

$\therefore AS \perp PS$.

$\therefore \angle ASB = 90°$. 1

now $APB = 90°$ (TANGENT $\perp$ NORMAL)

$\angle ASB = 90°$

$\therefore$ AB subtends same angle at S and P 1

$\therefore$ APSB are cyclic

AB is diameter.

$\therefore$ centre is $\left(0, \left(\frac{b}{\sin\theta} + \frac{b^2-a^2}{b}\sin\theta\right)/2\right)$

(7)

7

a) i) $1+\omega+\omega^2=0$ 1

$(1+2\omega+3\omega^2)(1+2\omega^2+3\omega)$

$= (\omega+2\omega^2)(\omega^2+2\omega)$ 1

$= \omega^3+2\omega^2+2\omega^4+4\omega^3$

$= 1+2\omega^2+2\omega+4$

$= (5+2(\omega^2+\omega+1)-2)$

$= 5-2 = 3.$ 1

ii) common root must be $1, \omega$ or ω^2 1

if $x=1$ $p+q+r=0$

$x=\omega$ $p\omega^5+q\omega+r=0$ 1

$x=\omega^2$ $p\omega^{10}+q\omega^2+r=0$

$\therefore (p+q+r)(p\omega^5+q\omega+r)(p\omega^{10}+q\omega^2+r)=0$ 1

b) i) $(1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$

$= (1-\sqrt{x})^{n-1}(1-(1-\sqrt{x}))$

$= (1-\sqrt{x})^{n-1}(1-1+\sqrt{x})$

$= \sqrt{x}(1-\sqrt{x})^{n-1}$ 1

ii) $I_n = \int_0^1 (1-\sqrt{x})^n dx$

let $u = (1-\sqrt{x})^n$ $v'=1$

$u' = n(1-\sqrt{x})^{n-1} \cdot \frac{-1}{2\sqrt{x}}$ $v = x$

$\therefore I_n = \left[x(1-\sqrt{x})^n\right]_0^1 + \frac{n}{2}\int \frac{(1-\sqrt{x})^{n-1}}{\sqrt{x}} \cdot x\, dx$ 1

$= 0 + \frac{n}{2}\int (1-\sqrt{x})^{n-1}\sqrt{x}\, dx$

$= \frac{n}{2}\int (1-x)^{n-1} dx - \frac{n}{2}\int (1-\sqrt{x})^n dx$

$= \frac{n}{2} I_{n-1} - \frac{n}{2} I_n$

$\therefore \left(1+\frac{n}{2}\right) I_n = \frac{n}{2} I_{n-1}$ 1

$\therefore I_n = \frac{n}{n+2} I_{n-1}$

$\therefore I_1 = \frac{1}{3} I_0$

but $I_0 = \int_0^1 dx = [x]_0^1 = 1$

$\therefore I_1 = \frac{1}{3}$

$I_2 = \frac{2}{4} \cdot \frac{1}{3}$

$I_3 = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3}$

$\therefore I_{100} = \frac{100}{102} \cdot \frac{99}{101} \cdot \frac{98}{100} \cdot \frac{97}{99} \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3}$

$= \frac{2}{102 \cdot 101}$ 1

7(c)

$V = \frac{1}{3}\pi r^2 h$

$9V^2 = \pi^2 r^4 h^2$

now $S = \pi r l$

where $l^2 = h^2 + r^2$ 1

$\therefore S = \pi r\sqrt{h^2+r^2}$

$h^2 + r^2 = \frac{S^2}{\pi^2 r^2}$

$h^2 = \frac{S^2}{\pi^2 r^2} - r^2$

$h^2 = \frac{S^2 - \pi^2 r^4}{\pi^2 r^2}$ 1

$\therefore 9V^2 = \pi^2 r^4\left(\frac{S^2-\pi^2 r^4}{\pi^2 r^2}\right)$ 1

$9V^2 = r^2(S^2 - \pi^2 r^4)$ *

max Volume when $9V^2$ is max

$\therefore \frac{d\,9V^2}{dr} = 2rS^2 - 6\pi^2 r^5$

$= 0$ when

$r = 0$ or $r^4 = \frac{S^2}{3\pi^2}$

or $r^2 = \frac{S}{3^{1/2}\pi}$ 1

now $V = \frac{1}{3}\pi r^2 h$ 1

$= \frac{1}{3}\pi \cdot \frac{S}{3^{1/2}\pi}\sqrt{\frac{S^2 - \pi^2 r^4}{\pi^2 r^2}}$

$= \frac{S}{3^{3/2}}\sqrt{\frac{S^2 - \pi^2\frac{S^2}{3\pi^2}}{\pi^2\cdot\frac{S}{3^{1/2}\pi}}}$

$= \frac{S}{3^{3/2}}\sqrt{\frac{2S^2}{3}\cdot\frac{3^{1/2}}{S\pi}}$

$= \frac{S}{3^{3/2}}\sqrt{\frac{2S}{3^{1/2}\pi}}$

$= \frac{S}{3^{3/2}}\cdot\frac{2^{1/2}S^{1/2}}{3^{1/4}\pi^{1/2}}$ 1

$= \frac{2^{1/2}S^{3/2}}{3^{7/4}\pi^{1/2}}$

8 a) when $n=1$

$7^n + 3n(7^n) - 1$

$= 7 + 3\cdot 7 - 1$

$= 7 + 21 - 1 = 27$

which is ÷ by 9

$\therefore$ true for $n=1$ 1

assume true for some value k

i.e. $7^k + 3k(7^k) - 1 = 9N$

where N is an integer

must prove $7^{k+1} + 3(k+1)(7^{k+1}) - 1$

is ÷ by 9

$\therefore = 7.7^k + 3k.7.7^k + 3.7.7^k - 1$

$= 7(7^k + 3k.7^k - 1) + 7 + 21.7^k - 1$

$= 7.9N + 6 + 21.7^k$

$= 9(7N) + 3(2 + 7.7^k)$ 1

which is ÷ by 9 if $2 + 7.7^k$ is ÷ by 3

if $k=1$ $2 + 7.7^k = 2 + 49 = 51$ ÷ by 3

assume $2 + 7^{k+1} = 3P$ (P is an integer)

must prove $2 + 7^{k+2}$ is ÷ by 3

$2 + 7^{k+2} = 2 + 7.7^{k+1}$

$= 7(2 + 7.7^{k+1}) - 12$

$= 21P - 12$

$= 3(7P - 4)$ which is ÷ by 3 1

$7^{k+1} + 3(k+1)7^{k+1} - 1$ is ÷ by 9

$\therefore$ as true for $n=1$ must be true for $n=2$

true for $n=3$ - - $n=4$

and so on for all n

$\therefore 7^n + 3n(7^n) - 1$ is ÷ by 9.

b) $\tan 4\theta = 1$

$4\theta = \frac{\pi}{4} + 2k\pi$ $k = 0, \pm 1, \ldots$

or $\frac{5\pi}{4} + 2k\pi$ $k = 0, \pm 1, \ldots$

or $4\theta = \frac{\pi}{4} + k\pi$ $k = 0 \pm 1 \ldots$

$\therefore \theta = \frac{\pi}{16} + \frac{k}{4}\pi$ $k = 0 \pm 1 \ldots$ 1

$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

$\therefore \cos 4\theta + i\sin 4\theta = \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta$

$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ 1

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ 1

$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

÷ top + bottom by $\cos^4\theta$

$\therefore \tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ 1

let $\tan 4\theta = 1$ and $\tan\theta = x$

$\therefore \frac{4x - 4x^3}{1 - 6x^2 + x^4} = 1$ 1

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

when $x = \tan\theta$.

$\therefore x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$. 1

$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$ 1

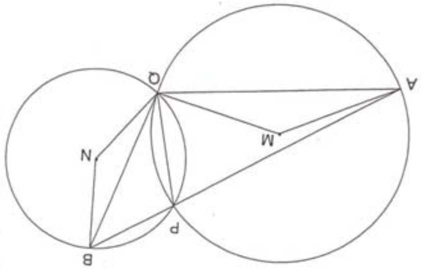
$= (-4)^2 - 6.-2 = 16 + 12 = 28$

$\tan^2\frac{\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{9\pi}{16} + \tan^2\frac{13\pi}{16} = 28$

but $\tan^2\frac{9\pi}{16} = \tan^2\frac{7\pi}{16}$ and $\tan^2\frac{13\pi}{16} = \tan^2\frac{3\pi}{16}$

$\therefore \tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$

8 c



let $\angle MAQ = x^\circ$

$AM = MQ$ (equal radii)

$\therefore \triangle AMQ$ is isosceles

$\therefore \angle MAQ = \angle MQA = x$

$\angle AMQ = 180 - 2\angle MAQ$ (angle sum of triangle)

$\therefore \angle AMQ = 180 - 2x$.

$\angle APQ = \frac{1}{2}\angle AMQ$ (ANGLE AT CENTRE IS TWICE ANGLE AT CIRCUMFERENCE

$\therefore \angle APQ = 90 - x$

$\angle APQ + \angle BPQ = 180^\circ$ (st line)

$\therefore \angle BPQ = 90 + x$.

reflex $\angle BNQ = 2\angle BPQ$ (" . .)

reflex $\angle BNQ = 180 + 2x$.

$\therefore$ in $\triangle BNQ$ $\angle BNQ = 180 - 2x$

but $\angle NBQ = \angle NQB$ ($NB = NQ$ equal radii)

$\therefore \angle NBQ = \frac{180 - (180 - 2x)}{2}$

$= x$

$= \angle MAQ$

$\therefore \angle NBQ = \angle MAQ$.

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