BAULKHAM HILLS HIGH SCHOOL

2012
YEAR 12 TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11 – 16
- Marks may be deducted for careless or badly arranged work

Total marks – 100

Section I Pages 2 – 5

10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 6 – 13

90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1; x \neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x>0$

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

1 The polynomial equation $P(z) = 0$ has one complex coefficient. Three of the roots of this equation are $z = 3 + i$, $z = 2 - i$ and $z = 0$.
The **minimum** degree of $P(z)$ is

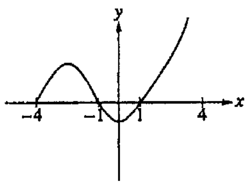
(A) 2

(B) 3

(C) 4

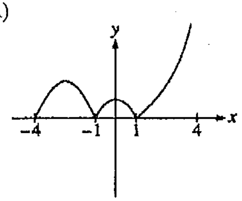
(D) 5

2

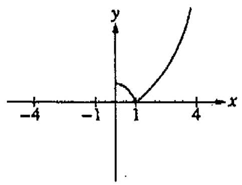


The graph of $y = f(x)$ is shown above. Which of the following could be the graph of $y = f(|x|)$?

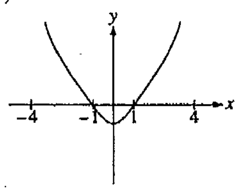
(A)



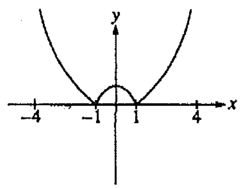
(B)



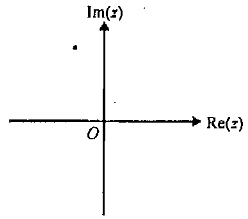
(C)



(D)

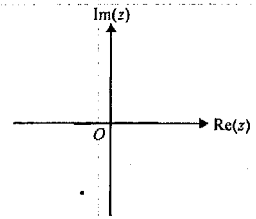


3 A particular complex number z is represented by the point on the following Argand diagram.

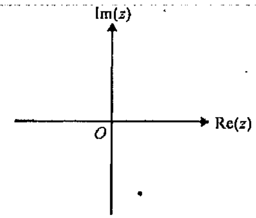


All axes below have the same scale as those in the diagram above.
The complex number $i\bar{z}$ is best represented by

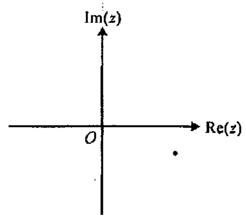
(A)



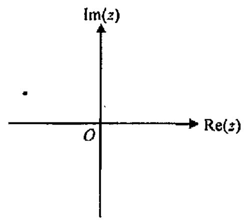
(B)



(C)



(D)



4 If $\int_1^4 f(x)\,dx = 6$, what is the value of $\int_1^4 f(5-x)\,dx$?

(A) 6

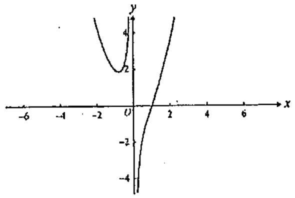
(B) 3

(C) – 1

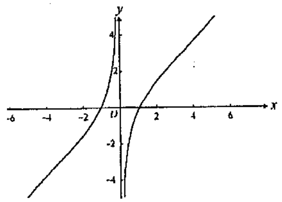
(D) – 6

5 Let $f(x) = \dfrac{x^k + a}{x}$, where k and a are real constants.
If k is an odd integer which is greater than 1 and $a < 0$, a possible graph of $y = f(x)$ could be

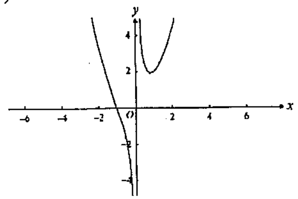
(A)



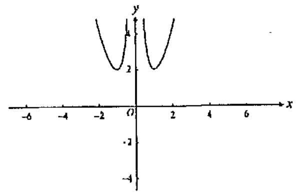
(B)



(C)



(D)



6 If ω is an imaginary cube root of unity, then $\left(1 + \omega - \omega^2\right)^7$ is equal to

(A) 128ω

(B) -128ω

(C) $128\omega^2$

(D) $-128\omega^2$

7 If $z = x + iy$, the locus of points that lie on a circle of radius 2 centred at the origin on the Argand diagram can be represented by the equation

(A) $z\bar{z} = 2$

(B) $(z + \bar{z})^2 - (z - \bar{z})^2 = 16$

(C) $\mathrm{Re}\left(z^2\right) + \mathrm{Im}\left(z^2\right) = 4$

(D) $\{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2 = 16$

8 Let R be the region in the first quadrant enclosed by the graph of $y = (x + 1)^{\frac{1}{3}}$, the line $x = 7$, the x-axis, and the y-axis. The volume of the solid generated when R is revolved about the **y-axis** is given by

(A) $\pi \displaystyle\int_0^7 (x + 1)^{\frac{2}{3}}\, dx$

(B) $\pi \displaystyle\int_0^2 (y^3 - 1)^2\, dy$

(C) $2\pi \displaystyle\int_0^7 x(x + 1)^{\frac{1}{3}}\, dx$

(D) $2\pi \displaystyle\int_0^2 x(x + 1)^{\frac{1}{3}}\, dx$

9 $\displaystyle\int \frac{x}{\sqrt{x + 5}}\, dx =$

(A) $2\sqrt{x + 5} + c$

(B) $\frac{2}{3}\sqrt{(x + 5)^3} + c$

(C) $\frac{2}{3}\left\{\sqrt{(x + 5)^3} - 10\sqrt{x + 5}\right\} + c$

(D) $\frac{2}{3}(x - 10)\sqrt{x + 5} + c$

10 A particle of mass m moves in a straight line under the action of a resultant force F where $F = F(x)$. Given that the velocity v is v_0 where the position x is x_0, and that v is v_1 where x is x_1, it follows that $v_1 =$

(A) $\sqrt{\dfrac{2}{m}} \displaystyle\int_{x_0}^{x_1} \sqrt{F(x)}\, dx + v_0$

(B) $\sqrt{2} \displaystyle\int_{\sqrt{x_0}}^{\sqrt{x_1}} F(x)\, dx + v_0$

(C) $\sqrt{\dfrac{2}{m} \displaystyle\int_{x_0}^{x_1} F(x)\, dx + (v_0)^2}$

(D) $\sqrt{\dfrac{2}{m} \displaystyle\int_{x_0}^{x_1} \left(F(x) + (v_0)^2\right) dx}$

END OF SECTION I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question on the appropriate answer sheet. Each answer sheet must show your BOS#. Extra paper is available.

All necessary working should be shown in every question.

Marks

Question 11 *(15 marks)* Use a *separate* answer sheet

(a) Find $\int \frac{dx}{\sqrt{x^2 - 4x + 20}}$ **2**

(b) Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \sin^5 x \, dx$ **2**

(c) (i) Find real numbers a, b and c such that $\frac{10}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$ **2**

(ii) Hence, find $\int \frac{10}{(x+1)(x^2+4)} dx$ **2**

(d) By using the fact that $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x} = \ln 2$, evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}$$

(e) Find $\int \frac{dx}{x^3\sqrt{x^2-4}}$ **4**

Marks

Question 12 *(15 marks)* Use a *separate* answer sheet

(a) Let $z = 1 + i\sqrt{3}$

(i) Write z in modulus-argument form **2**

(ii) Hence, evaluate $z^5 + 16z$ **2**

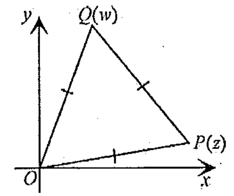
(b) On an Argand diagram, sketch the locus of the points z such that **2**

$$|z-1| = |z+i|$$

(c) Given that $z = x + iy$, find the value of x and the value of y such that **3**

$$z + 3i\bar{z} = -1 + 13i$$

(d) **3**



In the Argand diagram, OPQ is an equilateral triangle. P represents the complex number z and Q represents the complex number w.

Show that $w^3 + z^3 = 0$

(e) Let $w = \frac{3+4i}{5}$ and $z = \frac{5+12i}{13}$, so that $|w| = |z| = 1$.

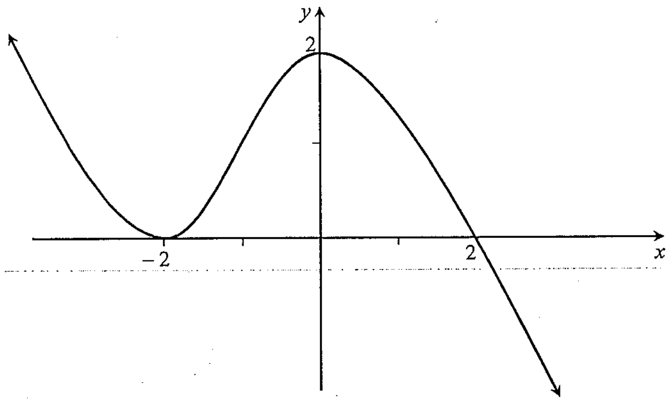
(i) Find wz in the form $x + iy$ **1**

(ii) Hence, or otherwise, find two distinct ways of writing 65^2 as the sum of $a^2 + b^2$, where a and b are integers and $0 < a < b$ **2**

Marks

Question 13 ***(15 marks)*** Use a *separate* answer sheet.

(a) This sketch shows the graph of $y = f(x)$, which has a double root at $x = -2$, and a single root at $x = 2$.



Draw separate one-third page sketches of the following curves, clearly indicating any turning points or asymptotes.

(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y^2 = f(x)$ 2

(iii) $y = \ln f(x)$ 2

(b) The region between the curve $y = x^2 + 2$ and the line $y = x + 8$ is rotated about the x-axis.

(i) By taking slices perpendicular to the x-axis, show that the volume ΔV, of a typical slice with thickness Δx, is given by 2

$$\Delta V = \pi\left(60 + 16x - 3x^2 - x^4\right)\Delta x$$

(ii) Hence, find the total volume of the solid formed. 3

(c) If $f(xy) = f(x) + f(y)$ for all $x,\ y \neq 0$, prove that

(i) $f\left(x^3\right) = 3f(x)$ 1

(ii) $f(1) = f(-1) = 0$ 2

(iii) $f(x)$ is an even function 1

Marks

Question 14 ***(15 marks)*** Use a *separate* answer sheet

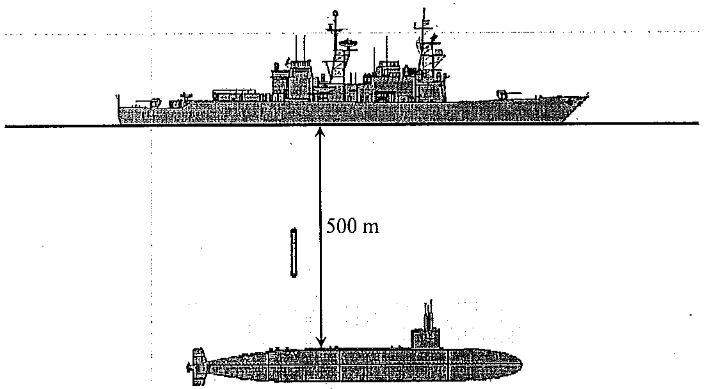
(a) The equation $9x^2 + 16y^2 = 144$ represents an ellipse.

(i) Find the eccentricity e 1

(ii) Find the coordinates of the foci 1

(iii) Find the equations of the directrices 1

(b) A stationary submarine fires a missile of mass 40 kg with a speed of 500 m/s at a ship at rest 500 m above it



The missile is subject to a downward gravitational force of 400 N and a water resistance of $\dfrac{3v^2}{100}$ N, where v is the velocity of the missile.

(i) Show that while the missile is rising, its displacement from the submarine is given by 3

$$x = \frac{2000}{3}\ln\left(\frac{790000}{40000 + 3v^2}\right)$$

(ii) Show that the velocity of the missile at the time of impact with the ship is approximately 333 m/s. 2

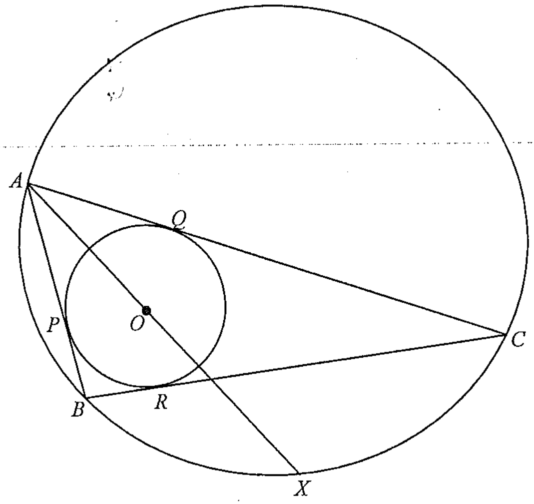
Question 14 continues on page 10

Question 14 (continued)

(c) In the diagram below, ABC is a triangle.

The incircle tangent to all three sides has centre O, and touches the sides AB, AC and BC at P, Q and R respectively.

The circumcircle through A, B and C meets the line AO produced at X.



Copy or trace the diagram onto your answer sheet

(i) Explain why $\angle CBX = \angle CAX$ 1

(ii) Prove that $\angle OBA = \angle OBC$ 2

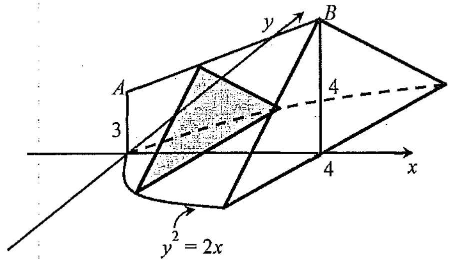
(iii) Prove that ΔXBO is an isosceles triangle 3

(iv) Prove that $BX = XC$ 1

End of Question 14

Question 15 *(15 marks)* Use a *separate* answer sheet

(a)



The base of the above solid is the area enclosed by $y^2 = 2x$ and $x = 4$. Vertical cross-sections of the solid taken parallel to the y-axis are isosceles triangles, and AB is a straight line as shown in the diagram.

(i) Show that the perpendicular height, h, of the similar triangles is given by 1

$$h = \frac{1}{4}x + 3$$

(ii) Hence find the volume of the solid. 2

(b) Daniel and Osborn are playing a match. The match consists of a series of games and each game consists of three points.

Daniel has probability p and Osborn probability of $1 - p$ of winning the first point of any game. In the second and third points of each game, the player who won the previous point has probability p and the player who lost the previous point has probability $1 - p$ of winning the point. If a player wins two consecutive points in a single game, the match ends and that player has won; otherwise the match continues with another game.

(i) Let q be the probability that Osborn wins the match. Show that, for $p \neq 0$ 3

$$q = \frac{1 - p^2}{2 - p}$$

(ii) If Daniel wins the match, Osborn gives him \$1; if Osborn wins the match, Daniel gives him \$$k$. 2

Find the value of k for which the game is fair, that is when each player receives the same amount of money, in the case when $p = \frac{2}{3}$

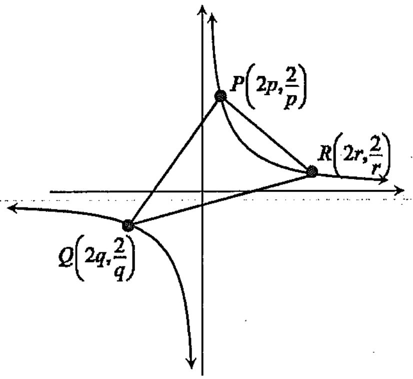
(iii) What happens when $p = 0$? 1

Question 15 continues on page 12

Question 15 (continued)

(c) The diagram shows three points, $P\left(2p, \frac{2}{p}\right)$, $Q\left(2q, \frac{2}{q}\right)$ and $R\left(2r, \frac{2}{r}\right)$.

The rectangular hyperbola $xy = 4$ circumscribes the ΔPQR.



(i) Show that the equation of the line through Q, which is perpendicular to the chord PR is $pqrx - qy = 2\left(pq^2r - 1\right)$. 2

(ii) Write down the equation of the line through R, which is perpendicular to the chord PQ. 1

(iii) Z is the point of intersection of these two lines. 2
Show that Z has the coordinates $\left(-\frac{2}{pqr}, -2pqr\right)$

(iv) Find the locus of Z as P, Q and R move on the rectangular hyperbola. 1

End of Question 15

Question 16 *(15 marks)* Use a *separate* answer sheet

(a) The equation $x^3 + 3x^2 - 4x + 5 = 0$ has roots α, β and γ. 3
Evaluate $\alpha^3 + \beta^3 + \gamma^3$

(b) Find $\int \sin^{-1} x \, dx$ 3

(c) The following result applies to any function f which is continuous, has positive gradient and satisfies $f(0) = 0$

$$ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy \quad (*)$$

where f^{-1} denotes the inverse function of f, and $a \ge 0$ and $b \ge 0$.

(i) By considering the graph of $y = f(x)$, explain briefly why the inequality (*) holds. 2

(ii) By taking $f(x) = x^{p-1}$ in (*), where $p > 1$, show that if $\frac{1}{p} + \frac{1}{q} = 1$ then 2

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

(iii) Show that, for $0 \le a \le \frac{\pi}{2}$ and $0 \le b \le 1$, 3

$$ab \le b\sin^{-1} b + \sqrt{1 - b^2} - \cos a$$

(iv) Deduce that, for $t \ge 1$, 2

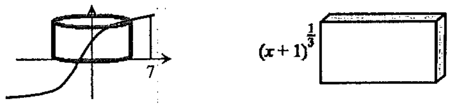
$$\sin^{-1}\left(\frac{1}{t}\right) \ge t - \sqrt{t^2 - 1}$$

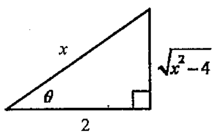
End of paper

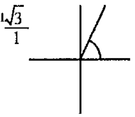
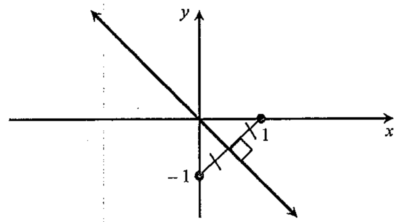
BAULKHAM HILLS HIGH SCHOOL

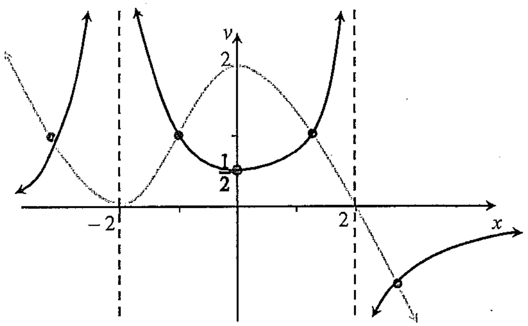
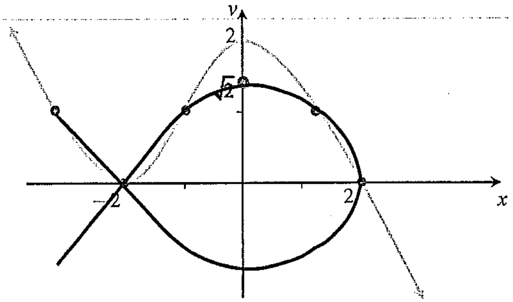
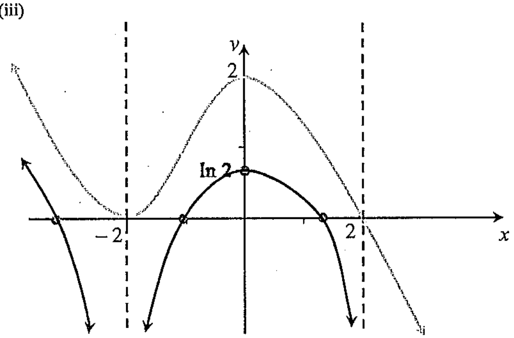
YEAR 12 EXTENSION 2 TRIAL HSC 2012 SOLUTIONS

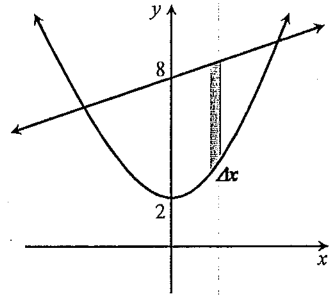
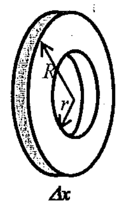
Solution	Marks	Comments
SECTION I		
1. B - $\sum \alpha = 3 + i + 2 - i + 0 + \ldots = 5 + \ldots$ $\sum \alpha\beta = (3+i)(2-i) + (3+i)(0) + (2-i)(0) + \ldots = 7 - i + \ldots$ $\sum \alpha\beta\gamma = (3+i)(2-i)(0) + \ldots = 0 + \ldots$ $\therefore$ The three roots satisfy the condition of one complex coefficient and as a polynomial's degree cannot be lower than the number of roots minimum degree = 3	1	
2. C – the part of the original graph where $f(x) < 0$ (i.e. left of y-axis) disappears and is replaced with the reflection of the part of the original graph where $f(x) > 0$ (i.e. right of y-axis) **OR** right of y-axis is reflected in y-axis.	1	
3. C - ① $\bar{z}$ is a reflection of z in the real axis ② $\times\, i$ is an anti-clockwise rotation of 90°	1	
4. A - $\int_1^4 f(5-x)\,dx = -\int_4^1 f(u)\,du$ $u = 5-x$, $du = -dx$ $= \int_1^4 f(u)\,du$ when $x = 1$, $u = 4$; when $x = 4$, $u = 1$ $= \int_1^4 f(x)\,dx$ $= 6$	1	
5. A - $f(x) = \dfrac{x^k + a}{x}$ $= x^{k-1} + \dfrac{a}{x}$ $\lim\limits_{x \to \pm\infty} f(x) = x^{k-1}$; as $k-1$ is even; $\lim\limits_{x \to \pm\infty} f(x) = \infty$ as $a < 0$: $\lim\limits_{x \to 0^+} \dfrac{a}{x} = -\infty$ as $a < 0$: $\lim\limits_{x \to 0^-} \dfrac{a}{x} = \infty$	1	
6. D - If ω is an imaginary cube root of unity, then $1 + \omega + \omega^2 = 0$ $\left(1 + \omega - w^2\right)^7 = \left(1 + \omega + w^2 - 2w^2\right)^7$ $= \left(-2\omega^2\right)^7$ $= -128\omega^{14}$ $= -128\left(w^3\right)^4 \times w^2$ $= -128w^2$	1	
7. B - $(z + \bar{z})^2 - (z - \bar{z})^2 = 16$ $(2x)^2 - (2iy)^2 = 16$ $4x^2 + 4y^2 = 16$ $x^2 + y^2 = 4$	1	

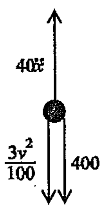
Solution	Marks	Comments
8. C –  $\Delta V = 2\pi x(x+1)^{\frac{1}{3}}\,\Delta x$ $V = 2\pi \lim\limits_{\Delta x \to 0} \sum\limits_{x=0}^{7} x(x+1)^{\frac{1}{3}}\,\Delta x$ $= 2\pi \int_0^7 x(x+1)^{\frac{1}{3}}\,dx$	1	
9. D - $\int \dfrac{x}{\sqrt{x+5}}\,dx = \int \left(\dfrac{x+5}{\sqrt{x+5}} - \dfrac{5}{\sqrt{x+5}}\right)dx$ $= \int (x+5)^{\frac{1}{2}} - 5(x+5)^{-\frac{1}{2}}\,dx$ $= \frac{2}{3}(x+5)^{\frac{3}{2}} - 10(x+5)^{\frac{1}{2}} + c$ $= \frac{2}{3}(x+5)\sqrt{x+5} - 10\sqrt{x+5} + c$ $= \frac{2}{3}\left\{(x+5)\sqrt{x+5} - 15\sqrt{x+5}\right\} + c$ $= \frac{2}{3}(x-10)\sqrt{x+5} + c$	1	
10. C - $m\ddot{x} = F(x)$ $\ddot{x} = \dfrac{F(x)}{m}$ $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = \dfrac{F(x)}{m}$ $\left[\frac{1}{2}v^2\right]_{v_0}^{v_1} = \dfrac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$ $\frac{1}{2}\left((v_1)^2 - (v_0)^2\right) = \dfrac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$ $(v_1)^2 - (v_0)^2 = \dfrac{2}{m}\int_{x_0}^{x_1} F(x)\,dx$ $(v_1)^2 = \dfrac{2}{m}\int_{x_0}^{x_1} F(x)\,dx + (v_0)^2$ $v_1 = \sqrt{\dfrac{2}{m}\int_{x_0}^{x_1} F(x)\,dx + (v_0)^2}$	1	
SECTION II		
QUESTION 11		
11(a) $\int \dfrac{dx}{\sqrt{x^2 - 4x + 20}} = \int \dfrac{dx}{\sqrt{(x-2)^2 + 16}}$ $= \ln\left\| x - 2 + \sqrt{x^2 - 4x + 20}\right\| + c$	2	**2 marks** • Correct answer **1 mark** • Completes the square in the denominator • Correctly uses standard integral for their denominator, after completing the square

Solution	Marks	Comments
11(b) $\cos^4 x$ = even function , $\sin^5 x$ = odd function $\therefore \cos^4 x \times \sin^5 x$ is an odd function $\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \sin^5 x \, dx = 0$	2	**2 marks** • Correct answer with justification/working **1 mark** • Bald answer
11(c) (i) $a(x^2+4)+(bx+c)(x+1)=10$ $x=-1$: $5a=10$, $a=2$ $x=0$: $4a+c=10$, $8+c=10$, $c=2$ coefficient of x^2: $a+b=0$, $2+b=0$, $b=-2$ $\therefore a=2$, $b=-2$, $c=2$	2	**2 marks** • Correct answer **1 mark** • Makes progress towards finding values using correct methods
11(c) (ii) $\int \frac{10}{(x+1)(x^2+4)}dx = \int \left(\frac{2}{x+1}+\frac{-2x+2}{x^2+4}\right)dx$ $= \int \left(\frac{2}{x+1}-\frac{2x}{x^2+4}+\frac{2}{x^2+4}\right)dx$ $= 2\ln\lvert x+1\rvert - \ln(x^2+4)+\tan^{-1}\frac{x}{2}+c$	2	**2 marks** • Correct solution *using their values from (i)* **1 mark** • Finds two correct primitives obtained from their integrand
11(d) $\int_0^{\frac{\pi}{2}} \frac{xdx}{1+\cos x+\sin x} = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{1+\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}dx$ $= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{1+\sin x+\cos x}dx$ $\therefore 2\int_0^{\frac{\pi}{2}} \frac{xdx}{1+\cos x+\sin x} = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}$ $2\int_0^{\frac{\pi}{2}} \frac{xdx}{1+\cos x+\sin x} = \frac{\pi}{2}\times \ln 2$ $\int_0^{\frac{\pi}{2}} \frac{xdx}{1+\cos x+\sin x} = \frac{\pi \ln 2}{4}$	3	**3 marks** • Correct solution **2 marks** • Significant progress towards the correct solution. **1 mark** • Makes use of $\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$ • Attempts to use given relationship validly
11(e) $\int \frac{dx}{x^3\sqrt{x^2-4}}$ $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$ $= \int \frac{2\sec\theta\tan\theta\,d\theta}{8\sec^3\theta\times 2\tan\theta}d\theta$ $= \frac{1}{8}\int \frac{d\theta}{\sec^2\theta}$ $= \frac{1}{8}\int \cos^2\theta\,d\theta$ $= \frac{1}{16}\int (1+\cos 2\theta)\,d\theta$ $= \frac{1}{16}\left(\theta+\frac{1}{2}\sin 2\theta\right)+c$ $= \frac{1}{16}(\theta+\sin\theta\cos\theta)+c$ $= \frac{1}{16}\left(\sec^{-1}\frac{x}{2}+\frac{2\sqrt{x^2-4}}{x^2}\right)+c$ 	4	**4 marks** • Correct solution **3 marks** • Obtains the correct primitive in terms of the substituted variable **2 marks** • obtains $\frac{1}{8}\int \cos^2\theta\,d\theta$ or equivalent merit **1 mark** • Makes a valid substitution

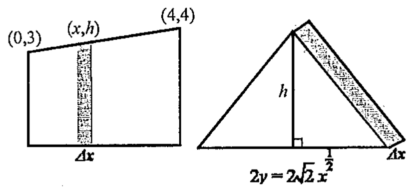
Solution	Marks	Comments
QUESTION 12		
12(a) (i) $\lvert 1+i\sqrt{3}\rvert = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$ $\arg(1+i\sqrt{3}) = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$  $1+i\sqrt{3} = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$	2	**2 marks** • Correctly substitutes both the modulus and the argument into the required form. **1 mark** • Finds either the modulus or the argument • *Note: argument should be quoted* $-\pi < \arg z \le \pi$
12(a) (ii) $z^5 = 2^5\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$ $= 32\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$ $= 16\bar{z}$ $\therefore z^5+16z = 16\bar{z}+16z = 16(z+\bar{z}) = 16\times 2\times 2\cos\frac{\pi}{3} = 64\times\frac{1}{2} = 32$	2	**2 marks** • Correct solution **1 mark** • Bald answer • Uses De Moivre's to find z^5
12(b) 	2	**2 marks** • Correct sketch. **1 mark** • Recognises locus is the perpendicular bisector of two points • States the locus without sketching *Note: not required to find equation of locus* $(y=-x)$
12(c) $z+3i\bar{z} = -1+13i$ $x+iy+3i(x-iy) = -1+13i$ $x+iy+3ix+3y = -1+13i$ $x+3y=-1 \Rightarrow 3x+9y=-3$ $3x+y=13$ $3x+y=13$ $8y=-16$ $y=-2$ $\therefore x=5$, $y=-2$	3	**3 marks** • Correct solution **2 marks** • Finds a pair of equations by equating real and imaginary parts **1 mark** • $\bar{z}=x-iy$ • Attempts to equate real and imaginary parts
12(d) $w = z\times \text{cis}\frac{\pi}{3}$ $w^3 = z^3\times\text{cis}(-\pi)$ $w^3 = z^3\times -1$ $w^3 = -z^3$ $w^3+z^3=0$	3	**3 marks** • Correct solution **2 marks** • Evalutes w^3 in terms of z **1 mark** • Recognises that rotating a vector 60° anticlockwise is equivalent to multiplying by $\text{cis}\frac{\pi}{3}$
12(e) (i) $\frac{3+4i}{5}\times\frac{5+12i}{13} = \frac{15+36i+20i-48}{65}$ $= -\frac{33}{65}+\frac{56}{65}i$	1	**1 mark** • Correct answer
12(e) (ii) $\lvert wz\rvert^2 = 1$ $\left(-\frac{33}{65}\right)^2+\left(\frac{56}{65}\right)^2 = 1$ $33^2+56^2=65^2$ Similarly; $\lvert w\bar{z}\rvert^2=1$ $w\bar{z} = \frac{3+4i}{5}\times\frac{5-12i}{13} = \frac{63-16i}{65}$ $\therefore 33^2+56^2 = 16^2+63^2 = 65^2$	2	**2 marks** • Finds two different ways **1 mark** • Finds one way

Solution	Marks	Comments
QUESTION 13		
13(a) (i) 	2	**2 marks** • Correct graph **1 mark** • Basic shape correct with most of key features **Key Features** • x-intercepts become asymptotes. • y-intercept becomes $\frac{1}{2}$ • y-value of stays± 1 same • turning point stays with same x-value • $y < 1 \Rightarrow y > 1$ and visa-versa
13(a) (ii) 	2	**2 marks** • Correct graph **1 mark** • Basic shape correct with most of key features • Correct graph of $y = \sqrt{f(x)}$ **Key Features** • x-intercepts become critical points. • y-intercept becomes $\sqrt{2}$ • y-value of $+1$ stays same • turning point stays with same x-value • double root becomes single root • single root becomes vertical tangent • symmetric in x-axis
13(a) (iii) 	2	**2 marks** • Correct graph **1 mark** • Basic shape correct with most of key features **Key Features** • $y < 0$, becomes undefined • y-intercept becomes $\ln 2$ • y-value of $+1$ becomes x-intercept • turning point stays with same x-value

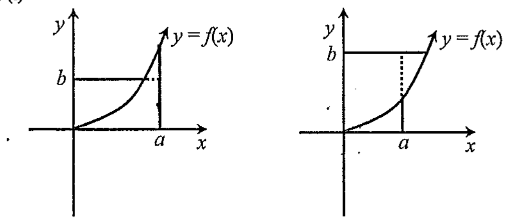
Solution	Marks	Comments
13(b) (i) Point of intersection $x^2 + 2 = x + 8$ $x^2 - x - 6 = 0$ $(x-3)(x+2) = 0$ $x = -2$ or $x = 3$ $A(x) = \pi[(x+8)^2 - (x^2+2)^2]$ $= \pi(x^2 + 16x + 64 - x^4 - 4x^2 - 4)$ $= \pi(60 + 16x - 3x^2 - x^4)$ $\Delta V = \pi(60 + 16x - 3x^2 - x^4)\,\Delta x$	2	**2 marks** • Correct solution **1 mark** • Substitutes functions into $\pi(R^2 - r^2)$ correctly
13(b) (ii) $V = \lim_{\Delta x \to 0} \sum_{x=-2}^{3} \pi(60 + 16x - 3x^2 - x^4)\,\Delta x$ $= \pi\int_{-2}^{3} (60 + 16x - 3x^2 - x^4)\,dx$ $= \pi\left[60x + 8x^2 - x^3 - \frac{1}{5}x^5\right]_{-2}^{3}$ $= \pi\left\{\left(180 + 72 - 27 - \frac{243}{5}\right) - \left(-120 + 32 + 8 + \frac{32}{5}\right)\right\}$ $= 250\pi$ units^3	3	**3 marks** • Correct solution **2 marks** • Correct primitive • Substitutes correct limits into their integrand involving at least four terms **1 mark** • Evaluates correct limits • Finds primitive of their integrand • Expresses as a limit of a sum
13(c) (i) $f(x^3) = f(x \times x^2)$ $= f(x) + f(x^2)$ $= f(x) + f(x \times x)$ $= f(x) + f(x) + f(x)$ $= 3f(x)$	1	**1 mark** • Correct solution
13(c) (ii) $f(1) = f(1 \times 1)$ $= f(1) + f(1)$ $f(1) = 0$ $f(-1) = f((-1)^3)$ $= 3f(-1)$ $2f(-1) = 0$ $f(-1) = 0$ $\therefore f(1) = f(-1) = 0$	2	**2 marks** • Evaluates both correctly **1 mark** • Evaluates either $f(1)$ or $f(-1)$
13(c) (iii) $f(-x) = f(-1 \times x)$ $= f(-1) + f(x)$ $= 0 + f(x)$ $= f(x)$ $\therefore$ even function	1	**1 mark** • Correct solution

Solution	Marks	Comments
QUESTION 14		
14(a) (i) $9x^2+16y^2=144 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$ $a^2=16$ $b^2=a^2(1-e^2)$ $9=16(1-e^2)$ $1-e^2=\frac{9}{16}$ $e^2=\frac{7}{16}$ $e=\frac{\sqrt{7}}{4}$	1	**1 mark** • Correct solution
14(a) (ii) foci $=\pm(ae,0)$ $=\pm\left(4\times\frac{\sqrt{7}}{4},0\right)$ $=\pm(\sqrt{7},0)$	1	**1 mark** • Correct solution *Do not penalise for lack of* $\pm$
14(a) (iii) $x=\pm\frac{a}{e}=\pm\frac{4}{1}\times\frac{4}{\sqrt{7}}$ $=\pm\frac{16}{\sqrt{7}}$	1	**1 mark** • Correct solution • *Do not penalise for lack of* $\pm$
14(b) (i) $40\ddot{x}=-400-\frac{3v^2}{100}$ $\ddot{x}=-10-\frac{3v^2}{4000}$ $v\frac{dv}{dx}=\frac{-40000-3v^2}{4000}$ $\frac{dv}{dx}=\frac{-40000-3v^2}{4000v}$ $\int_0^x dx=-\frac{2000}{3}\int_{500}^{v}\frac{6v}{40000+3v^2}\,dv$ $x=-\frac{2000}{3}\left[\ln(40000+3v^2)\right]_{500}^{v}$ $=-\frac{2000}{3}\ln\left(\frac{40000+3v^2}{40000+750000}\right)$ $=\frac{2000}{3}\ln\left(\frac{790000}{40000+3v^2}\right)$ 	3	**3 marks** • Correct solution **2 marks** • Correct integrand in terms of v *Note: do not penalise for limits of integration* **1 mark** • Correct force equation
14(b) (ii) when $x=500$, $500=\frac{2000}{3}\ln\left(\frac{790000}{40000+3v^2}\right)$ $0.75=\ln\left(\frac{790000}{40000+3v^2}\right)$ $e^{0.75}=\frac{790000}{40000+3v^2}$ $e^{0.75}(40000+3v^2)=790000$ $3e^{0.75}v^2=790000-40000e^{0.75}$ $v^2=\frac{790000-40000e^{0.75}}{3e^{0.75}}$ $v=\sqrt{\frac{790000-40000e^{0.75}}{3e^{0.75}}}$ $v=333.2514449\ldots$ $v=333$ m/s	2	**2 marks** • Correct solution **1 mark** • Attempts to make v the subject of the formula

Solution	Marks	Comments
14(c) (i) $\angle$'s in the same segment are =	1	**1 mark** • Correct explanation
14(c) (ii) $\angle OPB=\angle ORB=90^\circ$ (radius $\perp$ tangent) $OP=OR$ (= radii) OB is common side $\therefore\ \Delta OPB\equiv\Delta ORB$ (RHS) $\angle OBA=\angle OBC$ (matching $\angle$'s in $\equiv$ Δ's)	2	**2 marks** • Correct solution **1 mark** • Significant progress towards correct solution
14(c) (iii) Part (ii) proves that tangents drawn from an external point are bisected by the line joining the external point and the centre of the circle. thus $\angle BAO=\angle CAO$ $\angle CBX=\angle CAX$ (proven in (i)) $\therefore\angle BAO=\angle CBX$ $\angle XOB=\angle BAO+\angle OBA$ (exterior $\angle$, ΔOAB) $\angle OBX=\angle CBX+\angle OBC$ (common $\angle$) $\therefore\ \angle OBX=\angle XOB$ ($\angle OBA=\angle OBC$, proven in(ii)) ΔXBO is isosceles (2 = $\angle$'s)	3	**3 marks** • Correct solution **2 marks** • Correct solution with poor reasoning • Significant progress towards a correct solution **1 mark** • Progress towards a correct solution involving some relevant logic
14(c) (iv) $\angle BAX=\angle BCX$ ($\angle$'s in same segment are =) $\angle BAX=\angle XBC=\angle CBX$ (proven in previous parts) $\therefore\angle XBC=\angle BCX$ Thus $BX=XC$ (sides opposite = $\angle$'s in a Δ are =)	1	**1 mark** • Correct explanation

Solution	Marks	Comments
QUESTION 15		
15(a) (i) $m=\frac{4-3}{4-0}$ $=\frac{1}{4}$ $h-3=\frac{1}{4}(x-0)$ $h-3=\frac{1}{4}x$ $h=\frac{1}{4}x+3$ 	1	**1 mark** • Correct explanation
15(a) (ii) $A(x)=\frac{1}{2}\times 2\sqrt{2}x^{\frac{1}{2}}\times\left(\frac{1}{4}x+3\right)$ $=\sqrt{2}\left(\frac{1}{4}x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)$ $\Delta V=\sqrt{2}\left(\frac{1}{4}x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)\Delta x$ $V=\lim_{\Delta x\to 0}\sum_{x=0}^{4}\sqrt{2}\left(\frac{1}{4}x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)\Delta x$ $=\sqrt{2}\int_0^4\left(\frac{1}{4}x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)dx$ $=\sqrt{2}\left[\frac{1}{10}x^{\frac{5}{2}}+2x^{\frac{3}{2}}\right]_0^4$ $=\sqrt{2}\left(\frac{32}{10}+16\right)$ $=\frac{96\sqrt{2}}{5}$ units^3	2	**2 marks** • Correct solution **1 mark** • Establishes correct integrand from the sum of the slices • Correct answer obtained without reference to the slice
15(b) (i) **Case 1: Osborn wins directly** $P(O\text{ wins direct})=P(OO)+P(DOO)$ $=(1-p)\times p+p\times(1-p)\times p$ $=p(1-p)+p^2(1-p)$ $=p(1-p)(1+p)$ $=p(1-p^2)$ **Case 2: Osborn wins a rematch** $P(\text{rematch})=P(ODO)+P(DOD)$ $=(1-p)\times(1-p)\times(1-p)+p\times(1-p)\times(1-p)$ $=(1-p)^3+p(1-p)^2$ $=(1-p)^2(1-p+p)$ $=(1-p)^2$ The proportion of matches that Osborn wins would be q $\therefore P(\text{Osborn wins a rematch})=q(1-p)^2$ **Osborn wins overall** $P(\text{Osborn wins})=P(O\text{ wins direct})+P(O\text{ wins a rematch})$ $q=p(1-p^2)+q(1-p)^2$ $q\left(1-(1-p)^2\right)=p(1-p^2)$ $q=\frac{p(1-p^2)}{(1-(1-p))(1+(1-p))}$ $q=\frac{p(1-p^2)}{p(2-p)}$ $q=\frac{1-p^2}{2-p}$	3	**3 marks** • Correct solution **2 marks** • Considers multiple cases and correctly finds the probability of one case **1 mark** • Breaks the problem into logical cases • Finds the probability of one case

Solution	Marks	Comments
15(b) (ii) If $p=\frac{2}{3}$, $q=\frac{1-\frac{4}{9}}{2-\frac{2}{3}}$ $=\frac{5}{9}\times\frac{3}{4}$ $=\frac{5}{12}$ In 12 games, Osborn wins 5 games and Daniel wins 7 Daniel receives \$7 in 7 games, so Osborn receives \$7 in 5 games $k=\frac{7}{5}$ $k=\$1.40$	2	**2 marks** • Correct solution **1 mark** • Calculates q • Calculates k using $q=\frac{2}{3}$ or $q=\frac{1}{3}$
15(b) (iii) If $p=0$, then $1-p=1$, so the results must go; ODO rematch ODO rematch ODO rematch ………. i.e. the match will never end	1	**1 mark** • Correct explanation
15(c) (i) $m_{PR}=\frac{\frac{2}{r}-\frac{2}{p}}{2r-2p}$ $=\frac{p-r}{pr}\times\frac{1}{r-p}$ $=-\frac{1}{pr}$ $\therefore$ required $m=pr$ $y-\frac{2}{q}=pr(x-2q)$ $qy-2=pqr(x-2q)$ $qy-2=pqrx-2pq^2r$ $pqrx-qy=2pq^2r-2$ $pqrx-qy=2(pq^2r-1)$	2	**2 marks** • Substitutes into point-slope formula and arrives at the required result **1 mark** • Finds the required slope
15(c) (ii) $pqrx-ry=2(pqr^2-1)$	1	**1 mark** • Correct answer
15(c) (iii) $pqrx-qy=2(pq^2r-1)$ $\underline{pqrx-ry=2(pqr^2-1)}$ $(q-r)y=2(pqr^2-pq^2r)$ $y=\frac{2pqr(r-q)}{(q-r)}$ $y=-2pqr$ $pqrx-2pq^2r=2(pq^2r-1)$ $pqrx=-2$ $x=-\frac{2}{pqr}$ $\therefore Z\left(-\frac{2}{pqr},\ -2pqr\right)$	2	**2 marks** • Evaluates the correct coordinates • Substitutes the x and y value into **both** of the lines **1 mark** • Finds either the x or the y value • Substitutes either the x value or the y value into **both** lines • Substitutes the x and y value into **one** of the lines
15(c) (iv) $-\frac{2}{pqr}\times-2pqr=4$ $\therefore$ the locus of Z is the rectangular hyperbola $xy=4$	1	**1 mark** • Correct solution

Solution	Marks	Comments
QUESTION 16		
16(a) $\Sigma\alpha^3 + 3\Sigma\alpha^2 - 4\Sigma\alpha + 15 = 0$ $\Sigma\alpha^3 + 3(17) - 4(-3) + 15 = 0$ $\Sigma\alpha^3 + 78 = 0$ $\Sigma\alpha^3 = -78$ $\Sigma\alpha = -3$ $\Sigma\alpha\beta = -4$ $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$ $= (-3)^2 - 2(-4)$ $= 17$	3	**3 marks** • Correct solution **2 marks** • Establishes a correct equation involving $\Sigma\alpha^3$ • Uses $\Sigma\alpha^3 + 3\Sigma\alpha^2 - 4\Sigma\alpha + 5 = 0$ **1 mark** • Evaluates $\Sigma\alpha^2$
16(b) $\int \sin^{-1}x\,dx$ $u = \sin^{-1}x$ $\quad v = x$ $du = \frac{dx}{\sqrt{1-x^2}}$ $\quad dv = dx$ $= x\sin^{-1}x - \int \frac{x\,dx}{\sqrt{1-x^2}}$ $= x\sin^{-1}x + \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}}$ $= x\sin^{-1}x + \sqrt{1-x^2} + c$	3	**3 marks** • Correct solution **2 marks** • Makes significant progress towards the final solution **1 mark** • Attempts to use integration by parts
16(c) (i)  $\int_0^a f(x)\,dx$ is the area between the curve $y = f(x)$, the x-axis and the line $x = a$ $\int_0^b f^{-1}(y)\,dy$ is the area between the curve $y = f(x)$, the y-axis and the line $y = b$ The sum of these areas is greater than or equal to the area of the rectangle (ab), with equality holding if $f(a) = b$. i.e. $ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy$	2	**2 marks** • Correct explanation using the graph **1 mark** • Refers to the graph in a logical attempt to explain the desired result.
16(c) (ii) If $f(x) = x^{p-1}$ then $f^{-1}(y) = y^{\frac{1}{p-1}}$ $ab \le \int_0^a x^{p-1}\,dx + \int_0^b y^{\frac{1}{p-1}}\,dy$ $ab \le \int_0^a x^{p-1}\,dx + \int_0^b y^{q-1}\,dy$ $ab \le \left[\frac{x^p}{p}\right]_0^a + \left[\frac{y^q}{q}\right]_0^b$ $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ $\frac{1}{p} + \frac{1}{q} = 1$ $\frac{1}{q} = 1 - \frac{1}{p}$ $= \frac{p-1}{p}$ $q = \frac{p}{p-1}$ $q - 1 = \frac{p}{p-1} - \frac{p-1}{p-1}$ $= \frac{1}{p-1}$	2	**2 marks** • Correctly shows desired result **1 mark** • Makes q the subject of $\frac{1}{p} + \frac{1}{q} = 1$. • Correctly identifies the inverse function

Solution	Marks	Comments
16(c) (iii) $y = \sin x$ satisfies the conditions of (*), so let $y = \sin x$ $ab \le \int_0^a \sin x\,dx + \int_0^b \sin^{-1}y\,dy$ $ab \le \left[-\cos x\right]_0^a + \left[y\sin^{-1}y + \sqrt{1-y^2}\right]_0^b$ $ab \le -\cos a + \cos 0 + b\sin^{-1}b + \sqrt{1+b^2} - 0 - \sqrt{1}$ $ab \le -\cos a + 1 + b\sin^{-1}b + \sqrt{1-b^2} - 0 - 1$ $ab \le b\sin^{-1}b + \sqrt{1-b^2} - \cos a$	3	**3 marks** • Correctly shows desired result **2 marks** • Finds the correct primitive of the RHS **1 mark** • Realises that if $y=\sin x$ is used then RHS of expression is $\int_0^a \sin x\,dx + \int_0^b \sin^{-1}y\,dy$
16(c) (iv) Let $a = 0$ and $b = \frac{1}{t}$ (note: if $t \ge 1$, then $0 < \frac{1}{t} \le 1$) Substituting into (iii) $(0)\left(\frac{1}{t}\right) \le \left(\frac{1}{t}\right)\sin^{-1}\left(\frac{1}{t}\right) + \sqrt{1-\left(\frac{1}{t}\right)^2} - \cos(0)$ $0 \le \frac{1}{t}\sin^{-1}\left(\frac{1}{t}\right) + \sqrt{1-\frac{1}{t^2}} - 1$ $\frac{1}{t}\sin^{-1}\left(\frac{1}{t}\right) \ge 1 - \sqrt{1+\frac{1}{t^2}}$ $\sin^{-1}\left(\frac{1}{t}\right) \ge t - t\sqrt{1-\frac{1}{t^2}}$ $\sin^{-1}\left(\frac{1}{t}\right) \ge t - \sqrt{t^2-1}$	2	**2 marks** • Shows the desired result **1 mark** • Realises that $a = 0$ and $b = \frac{1}{t}$ .will give the desired result.